Question 1 (Artin 7.1.1). Does the rule $g \ast x = xg^{-1}$ define an operation of $G$ on $G$?

Solution. Yes. It satisfies both axioms for operations: for $x \in G$, $\text{Id} \ast x = x \text{Id}^{-1} = x$ and for $g, h \in G$:

$$(gh) \ast x = x(h^{-1}g^{-1}), \quad g \ast (h \ast x) = g \ast (xh^{-1}) = xh^{-1}g^{-1}.$$ //

Question 2 (Artin 7.1.2). Let $H$ be a subgroup of a group $G$. Describe the orbit for the operation of $H$ on $G$ by left multiplication.

Solution. The orbits are the right cosets of $H$ in $G$; Namely, they are of the form $Hg$ for some $g \in G$. Indeed, say $k \in Hg$. Then $k = hg$ for some $h \in H$, so $k$ is in the same orbit as $g$ under $H$-action. Conversely, suppose $k$ is in the same orbit as $g$. Then $k = hg$ for some $h \in H$, so $k \in Hg$. //

Question 3 (Artin 7.2.3). A group $G$ of order 12 contains a conjugacy class of order 4. Prove that the center of $G$ is trivial.

Proof. Let $C$ be a conjugacy class of order 4 in $G$, and pick $x \in C$. Then by counting formula the centralizer $Z(x)$ of $x$ has order 3. Hence, Since the center $Z$ is a subgroup of $Z(x)$ and $|Z(x)| = 3$, it follows that $Z$ is trivial or $Z = Z(x)$. However the latter cannot happen as $x$ is not central; its conjugacy class is nontrivial. Therefore, $Z = 1$. \hfill \Box

Question 4 (Artin 7.3.2). Let $Z$ be the center of a group $G$. Prove that if $G/Z$ is a cyclic group, then $G$ is abelian, and therefore $G = Z$.

Proof. Say $G/Z = \langle xZ \rangle = \langle x \rangle Z$ for some $x \in G$. Then for every $a, b \in G$, we can write $a = x^iz$ and $b = x^jw$ for some $z, w \in Z$ and $i, j \in \mathbb{Z}$. Now

$$ab = x^izx^jw = zx^i + jw = wx^i + jz = x^jzw = x^jwx^i = ba,$$

so $a$ and $b$ commute. It follows that $G$ is abelian, so $Z = G$. \hfill \Box

Question 5 (Artin 7.4.2). Is $A_5$ the only proper normal subgroup of $S_5$?

Solution. Suppose $H < S_5$ is a proper normal subgroup. Then $H \cap A_5$ is also normal subgroup of $S_5$, as for every $x \in H \cap A_5$ and $g \in G$, we have $gxg^{-1} \in H$ and $gxg^{-1} \in A_5$ as both $H$ and $A_5$ are normal subgroups of $G$. Hence in particular, we have that $H \cap A_5 \leq A_5$. However as $A_5$ is simple, it is either $H \cap A_5 = A_5$ or $H \cap A_5 = 1$. If $H \cap A_5 = A_5$, then we conclude $H = A_5$.

Now, we suppose $H \cap A_5 = 1$. This means that every permutation in $H$ has sign $-1$ other than the identity element. Because the square of every permutation has sign $1$, it follows that $p^2 = \text{Id}$ for every $p \in H$. Thus, the cyclic decomposition of $p \in H$ should only consist of 2-cycles, but cannot contain $(2, 2)$-type such as $(1 2)(3 4)$ as it is nontrivial positive permutation of sign $1$. However as $H$ is normal in $S_5$, $H$ has to contain all the permutations of the same cyclic decomposition type. As $H$ is closed under multiplication it inevitably has to contain a $(2, 2)$-type permutation, so contradiction.

All in all, the only proper normal subgroup of $S_5$ is $A_5$. //
Question 6 (Bonus: Artin 7.2.13). Let $N$ be a normal subgroup of a group $G$. Suppose that $|N| = 5$ and that $|G|$ is an odd integer. Prove that $N$ is contained in the center of $G$.

Proof. Using the hint given by Dr. Bestvina, one can consider the homomorphism

$$\varphi : G \to \text{Aut}(N) \cong \mathbb{Z}_4,$$

induced by the conjugation action of $G$ on $N$. From the first isomorphism theorem $G/\ker(\varphi) \cong \varphi(G) \leq \mathbb{Z}_4$, and that $|G|$ is odd, it follows that $\ker(\varphi) = G$. This means that the conjugation action of $G$ on $N$ is trivial, so $N$ commutes with every element in $G$. Therefore, we conclude $N \leq Z(G)$.

Alternatively, we can argue as follows. Since $|G|$ is odd and $Z(G) \leq G$, by the class equation every conjugacy class has to have odd order. On the other hand, since $N$ is normal in $G$, $N$ is a union of such conjugacy classes. From $|N| = 5$ and that $N$ has at least one central element (which is just the identity), either $5 = 1 + 1 + 3$ or $5 = 1 + 1 + 1 + 1 + 1$. Suppose the former, $5 = 1 + 1 + 3$. This means $|Z(G) \cap N| = 2$. However, we know that $Z(G) \cap N$ is a subgroup of $G$, so $2 \nmid |G|$, contradiction. Therefore, the latter class equation for $N$ holds, which means that every element of $N$ is central, so $N \leq Z(G)$. $\square$