On the quantum cohomology of a symmetric product of an algebraic curve

Aaron Bertram

Department of Mathematics, University of Utah,
Salt Lake City, Utah 84112 USA

Michael Thaddeus

Department of Mathematics, Columbia University,
New York, N.Y. 10027 USA

Quantum cohomology is a novel multiplication on the cohomology of a smooth complex projective variety, or even a compact symplectic manifold. It can be regarded as a deformation of the ordinary cup product, defined in terms of the Gromov-Witten invariants of the manifold. Since its introduction in 1991, there has been enormous interest in computing quantum cohomology for various target spaces. Attention has focused on homogeneous spaces, complete intersections, surfaces, and of course on Calabi-Yau threefolds, where it is a key part of mirror symmetry. However, to the authors' knowledge, no one has yet studied the quantum cohomology of a symmetric product of a smooth curve. This is strange, because the problem is attractive from several points of view.

First, though quantum cohomology is clearly a fundamental invariant of a variety or a symplectic manifold, it has been explicitly computed in relatively few cases: just some of those mentioned above.

A more specific motivation comes from a link with Seiberg-Witten theory. For a complex curve C, the Seiberg-Witten-Floer cohomology of the real 3-manifold $C \times S^1$ is isomorphic to the cohomology of the dth symmetric product C_d [7, 24, 25], and is expected to carry a natural product that corresponds to the quantum product [6, 22, 26]. The index d depends on the spin-c structure on $C \times S^1$ chosen in Seiberg-Witten theory. In the ordinary Seiberg-Witten theory, this isomorphism only holds when $d < g - 1$, where g is the genus of C. However, higher symmetric products enter the picture if the Seiberg-Witten functional is perturbed in the manner of Taubes [30] and Morgan-Szabó-Taubes [24].

The quantum cohomology of a symmetric product is therefore the base ring of the "quantum category" in Seiberg-Witten theory, as introduced by Donaldson [23, §10.4].

On the other hand, there is also a link with Brill-Noether theory and the stratification of C_d by the special linear series. This makes it feasible to compute the quantum cohomology using algebraic geometry. To carry out the necessary enumerative computations on the strata, crucial use is made of a generalization of Porteous's formula, due to Harris and Tu [16].

There are several kinds of quantum cohomology; this paper is concerned only with the so-called "quantum cohomology algebra" [12] or "little quantum cohomology" [10] of C_d. This is a ring generated by the rational cohomology together with one deformation parameter for

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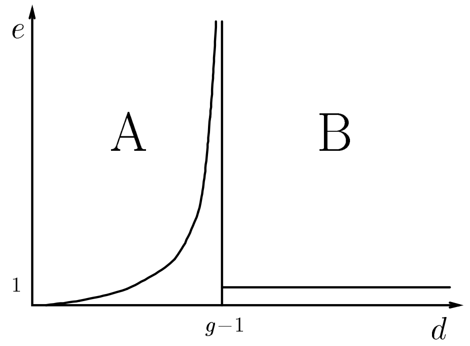
each generator of $H_2(C_d;\mathbb{Z})$. Its structure constants are given by 3-point Gromov-Witten invariants. Not quite all of these invariants are calculated herein, but in a sense most of them are. More precisely, the following results are proved.

- The number of possible deformation parameters is the second Betti number of C_d, which is fairly large. Nevertheless, the quantum product is shown to depend nontrivially only on a single parameter q (Proposition 2.6).
- Explicit formulas are given for the coefficients of q (Corollary 4.3) and q^2 (Corollary 5.4) in the quantum product.
- All the terms in the quantum product are computed for C_{g-1} (Corollary 6.2 ii).
- The coefficient of q^e in the quantum product for C_d is shown to vanish if $d < g-1$ and $e > \frac{d-3}{g-1-d}$ (Corollary 2.10), or if $d > g-1$ and $e > 1$ (Corollary 6.2 i).

Schematically, the coefficients vanish in the regions marked A and B in the diagram below. A follows from a straightforward dimension count; B is more subtle. Only in the regions under the hyperbola, or under the line $e = 1$, are there nonzero coefficients.



Putting these results together completely determines the quantum product on C_d in all cases except $d \in [\frac{3}{4}g, g-1)$.

Instead of seeking to characterize the quantum product completely, one can alternatively ask just for generators and relations for the quantum ring. This is a weaker question, because the quantum relations do not determine the additive isomorphism between the quantum and classical rings. It is answered in all cases except $d \in [\frac{4}{5}g - \frac{3}{5}, g-1)$, even using only the first order terms (Proposition 7.3 and the discussion following). In some cases, one encounters the curious fact that the quantum and classical rings are isomorphic, but only by an automorphism which is not the identity.

The organization of the paper is straightforward. Section 1 recalls some basic facts on quantum cohomology. Section 2 recalls some basic facts on symmetric products of a curve, shows why only one deformation parameter is involved, and explains the vanishing in region A. Section 3 introduces the Brill-Noether methods which will be used to compute the 3-point Gromov-Witten invariants. In particular, the result of Harris and Tu mentioned above is reviewed. Using this, the degree 1 invariants are computed in section 4, and the degree 2 invariants in section 5. Section 6 is devoted to the vanishing of higher-degree invariants in region B, and to the computation for $d = g - 1$. Section 7 explains how to find generators and relations for the quantum ring. Finally, sections 8 and 9 are essentially appendices: section 8 outlines a remarkable connection with Givental's work on the rational curves on a quintic threefold, while section 9 is concerned with the first two homotopy groups of C_d, explaining Proposition 2.6 from the point of view of symplectic topology.

A few conventions: cohomology of a space is with rational coefficients unless otherwise mentioned, and cohomology of a sheaf is over the curve C unless otherwise mentioned. If $n!$ appears in the denominator of some expression and $n < 0$, this means that the whole expression vanishes: that is, $1/n! = 0$ for $n < 0$.

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1 Preliminaries on quantum cohomology

Let X be a smooth complex projective variety with fixed polarization. Witten [32] introduced two rings associated to X, the *big* and *little* quantum cohomology. The big quantum cohomology was used by Kontsevich [18] to count rational curves in the plane. However, there are very few spaces for which it has been characterized in full. Most work, including that of Givental and Kim [12, 13], and the whole of the present paper, is concerned with the little quantum cohomology.

(1.1) Definition. A *stable map* is a morphism ϕ from a complete nodal curve Σ to X such that $\{\psi \in \operatorname{Aut}\Sigma \mid \phi\psi = \phi\}$ is finite. An *n-pointed stable map* is similar, except that n distinct smooth points $x_1, \ldots, x_n \in \Sigma$ are also chosen, and the automorphisms ψ are required to fix each x_i.

The arithmetic genus $g = p_a(\Sigma)$ and the homology class $e = \phi_*[\Sigma] \in H_2(X; \mathbb{Z})$ are discrete invariants of a stable map. When these are fixed, a fundamental theorem asserts the existence of projective moduli spaces $\overline{M}_g(X, e)$ of stable maps and $\overline{M}_{g,n}(X, e)$ of n-pointed stable maps, as well as a forgetful morphism $f_e : \overline{M}_{g,n}(X, e) \to \overline{M}_g(X, e)$ and an evaluation morphism $\mathrm{ev}_e : \overline{M}_{g,n}(X, e) \to X^n$. See Fulton-Pandharipande [10] for details. The subscript e will be suppressed when there is no danger of confusion. Also, this paper will be concerned only with the case $g = 0$, $n = 1$.

It is most accurate to regard these moduli spaces as stacks rather than schemes. They are stratified by smooth substacks on which the dimension of the deformation space is constant. Since each stable map has finitely many automorphisms, these strata are Deligne-Mumford stacks.

This paper, however, adopts a more naïve point of view, regarding the moduli spaces as schemes stratified by subschemes with a natural orbifold structure. As pointed out by the referee, this is a bit tricky, since smooth Deligne-Mumford stacks need not be orbifolds in the optimal algebraic sense of being locally a quotient of an affine scheme by a finite group. So strictly speaking, the word *orbifold* should henceforth be taken to mean a smooth Deligne-Mumford stack, and an *orbifold vector bundle* should mean a vector bundle over such a stack. Fortunately, in the only place where an explicit calculation with orbifolds is carried out, namely the proof of **(5.2)**, the space in question is an orbifold in the optimal sense mentioned above.

The moduli space $\overline{M}_0(X,e)$ has expected dimension $\dim X - 3 + c_1(TX)\cdot e$. It is not generally of this dimension, or even equidimensional. But it is endowed with a natural *virtual fundamental class*, an equivalence class of algebraic cycles $[\overline{M}_0(X,e)]^{\mathrm{vir}} \in A_*(\overline{M}_0(X,e))$ having the expected dimension. The general construction of this virtual class involves the deformation theory of stable maps and is rather complicated. It has been carried out by several authors [4, 19, 29]. We will use only the following three basic facts.

(1.2) Proposition.

(i) *If $T^1_X(\phi)$, $T^2_X(\phi)$ are the first-order deformation and obstruction spaces of the map ϕ, and $T^0(\Sigma)$, $T^1(\Sigma)$ are the first-order endomorphism and deformation spaces of the curve Σ, then there is a natural exact sequence*

$$\begin{array}{ccccc} & 0 & \longrightarrow & T^0(\Sigma) & \longrightarrow & H^0(\Sigma,\phi^*TX) \\ \longrightarrow & T^1_X(\phi) & \longrightarrow & T^1(\Sigma) & \longrightarrow & H^1(\Sigma,\phi^*TX) \\ \longrightarrow & T^2_X(\phi) & \longrightarrow & 0. & & \end{array}$$

(ii) *On any reduced locus where the first-order obstruction spaces of the map have constant dimension, the virtual class is the Euler class of the orbifold vector bundle formed by these spaces. In particular, if the first-order obstructions vanish, then the virtual class is simply the orbifold fundamental class.*

(iii) *The forgetful morphism f is flat, and*

$$[\overline{M}_{0,n}(X,e)]^{\mathrm{vir}} = f^*[\overline{M}_0(X,e)]^{\mathrm{vir}}.$$

Proof. See the work of Ran [27], Li-Tian [19], Behrend [3], Behrend-Fantechi [4] and Behrend-Manin [5]. □

(1.3) Example. If $X = \mathbb{P}^r$, then from the long exact sequence of

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{r+1} \longrightarrow T\mathbb{P}^r \longrightarrow 0,$$

it follows that $H^1(\Sigma,\phi^*T\mathbb{P}^r) = 0$ for a curve Σ of arithmetic genus 0. Hence the obstructions vanish and $\overline{M}_0(\mathbb{P}^r,e)$ is an orbifold of the expected dimension $r-3+e(r+1)$.

(1.4) Definition. For $a_1,\ldots,a_n \in H^*(X;\mathbb{Q})$, the *$n$-point degree e Gromov-Witten invariant* is defined as

$$\langle a_1,\ldots,a_n\rangle_e = \mathrm{ev}_e^*\Big(\prod_i \pi_i^* a_i\Big)[\overline{M}_{0,n}(X,e)]^{\mathrm{vir}},$$

where $\pi_i : X^n \to X$ is projection on the ith factor.

In practice, we will always evaluate Gromov-Witten invariants using the following equivalent formulation. Suppose that $\tilde{M}_\alpha$ are a collection of orbifold resolutions of the closures of the smooth strata M_α of $\overline{M}_0(X,e)$ such that $\tilde{M}_\alpha \times_{\overline{M}_0(X,e)} \overline{M}_{0,1}(X,e)$ are also orbifold resolutions of the strata of $\overline{M}_{0,1}(X,e)$. Let $\tilde{f}_\alpha$ and $\widetilde{\mathrm{ev}}_\alpha$ be the liftings of f_e and ev_e to these fibered products. Choose a cycle representing the virtual class of $\overline{M}_0(X,e)$, and let $[\tilde{M}_\alpha]^{\mathrm{vir}}$ be the cycle whose projection to the closure $\overline{M}_\alpha$ consists of all components of the virtual cycle supported in $\overline{M}_\alpha$ but not in the closure of any smaller stratum. The rational equivalence classes of the individual cycles $[\tilde{M}_\alpha]^{\mathrm{vir}}$ might depend on the choice of the representative, but this will not impair our arguments.

(1.5) Proposition. *With the above notation,*

$$\langle a_1, \dots, a_n \rangle_e = \sum_\alpha \Big(\prod_i (\tilde{f}_\alpha)_* \, \widetilde{\mathrm{ev}}_\alpha^* \, a_i \Big) [\tilde{M}_\alpha]^{\mathrm{vir}}.$$

In particular, if $\overline{M}_0(X,e)$ is an orbifold of the expected dimension,

$$\langle a_1, \dots, a_n \rangle_e = \Big(\prod_i f_* \ \mathrm{ev}^* \, a_i \Big) [\overline{M}_0(X,e)].$$

Proof. First of all, the forgetful morphism f_e factors through a birational morphism to the nth fibered power $\overline{M}_{0,1}(X,e)^n_{\overline{M}_0(X,e)}$, which is also flat over $\overline{M}_0(X,e)$. An evaluation map to X^n is still defined on this space, so it suffices to perform the computation here.

It is convenient to denote the disjoint union of the $\tilde{M}_\alpha$ by $\tilde{M}_0$, its fibered product over $\overline{M}_0(X,e)$ with $\overline{M}_{0,1}(X,e)$ by $\tilde{M}_{0,1}$, and the forgetful and evaluation maps on $\tilde{M}_{0,1}$ by $\tilde{f}$ and $\widetilde{\mathrm{ev}}$. Then the proposition follows immediately from **(1.2)**(iii) and the fact that the diagram

$$\begin{array}{ccccc} (\tilde{M}_{0,1})^n_{\tilde{M}_0} & \longrightarrow & \tilde{M}^n_{0,1} & \xrightarrow{\widetilde{\mathrm{ev}}} & X^n \\ {\scriptstyle \tilde{f}}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{f}} & & \\ \tilde{M}_0 & \longrightarrow & \tilde{M}^n_0 & & \end{array}$$

is a fiber square, where the top and bottom of the square are diagonal embeddings. □

A fundamental result states that Gromov-Witten invariants are deformation invariants [4, 8, 19]. Also, they satisfy the following properties.

(1.6) Lemma.

(i) *Gromov-Witten invariants are symmetric on classes of even degree, antisymmetric on classes of odd degree.*

(ii) *If $e \neq 0$ and $a_1 \in H^0$ or H^1, then $\langle a_1, \dots, a_n \rangle_e = 0$.*

(iii) *If $e \neq 0$ and $a_1 \in H^2$, then $\langle a_1, \dots, a_n \rangle_e = (a_1 \cdot e) \langle a_2, \dots, a_n \rangle_e$.*

Proof. See for example Behrend [3] and Behrend-Manin [5]. □

The quantum product has coefficients in the following ring.

(1.7) Definition. The *Novikov ring* Λ is the subring of $\mathbb{Q}[[H_2(X;\mathbb{Z})]]$ consisting of formal power series $\sum_{e\in H_2}\lambda_e q^e$, where q is a formal variable, $\lambda_e\in\mathbb{Q}$, and

$$\{e\in H_2 \mid \lambda_e\neq 0, \omega\cdot e\leq\kappa\}$$

is finite for all $\kappa\in\mathbb{Q}$, ω being the class of a fixed polarization on X.

As an example, if $H_2\cong\mathbb{Z}$, then $\Lambda\cong\mathbb{Q}[[q]][q^{-1}]$.
One extends the Gromov-Witten invariants linearly to $H^*(X;\Lambda)$.

(1.8) Definition. Let γ_i be a basis for $H^*(X;\mathbb{Q})$ and γ^i the dual basis with respect to the Poincaré pairing. For $a,b\in H^*(X;\Lambda)$, the *little quantum product* is

$$a*b=\sum_e\sum_i q^e\gamma_i\langle a,b,\gamma^i\rangle_e.$$

Alternatively, one can define the product without choosing a basis or mentioning the Gromov-Witten invariants explicitly, as

$$a*b=\sum_e q^e\,\mathrm{ev}_*\,f^*((f_*\,\mathrm{ev}^*\,a)\cdot(f_*\,\mathrm{ev}^*\,b)),$$

where f is restricted to the virtual cycle. This reveals that the cup product has in some sense been transferred from X to the moduli space of stable maps.

Since $\overline{M}_{0,1}(X,0)=\overline{M}_0(X,0)=X$, the quantum product equals the cup product modulo q. Furthermore, it follows from **(1.6)**(ii) that the quantum product with any element of H^1 equals the cup product.

A fundamental theorem asserts that the little quantum product is associative [4, 8, 19, 28].

(1.9) Definition. The *little quantum cohomology* $QH^*(X)$ is defined to be the ring additively isomorphic to $H^*(X;\Lambda)$, but with the little quantum product as multiplication.

The expected complex dimension of $\overline{M}_0(X,e)$ is $\dim X-3+c_1(TX)\cdot e$, so $QH^*(X)$ is graded if q^e is given degree $2c_1(TX)\cdot e$.

2 Symmetric products of a curve

We begin by recalling some basic facts on the cohomology of symmetric products of a curve. Good references are Arbarello et al. [1] and Macdonald [20].

Let C be a smooth projective curve of genus g, and let C_d be the dth symmetric product, which is a smooth projective variety of dimension d. It can be regarded as the moduli space of effective divisors D on C of degree $|D|=d$. Accordingly, there exists a universal divisor $\Delta\subset C_d\times C$ having the obvious property.

The Poincaré dual of Δ is a class in $H^2(C_d\times C;\mathbb{Z})$, which determines a map $\mu: H_*(C;\mathbb{Z})\to H^*(C_d;\mathbb{Z})$. Let e_i be a basis of $H_1(C;\mathbb{Z})$ for which the intersection form is the standard symplectic form. Then define $\xi_i = \mu(e_i) \in H^1(C_d)$ and $\eta = \mu(1) \in H^2(C_d)$. It is also convenient to define $\sigma_i = \xi_i \xi_{i+g}$ for $i \leq g$ and $\theta = \sum_{i=1}^{g} \sigma_i$.

There is a morphism $\iota : C_{d-1} \to C_d$ given by $D \mapsto D + p$; it is an embedding, and its image is a divisor in C_d. Indeed, this divisor is exactly $\Delta|_{C_d \times p}$, so it is Poincaré dual to η.

There is also a natural morphism $AJ : C_d \to \mathrm{Jac}_d(C)$, the *Abel-Jacobi map*, taking a divisor D to the line bundle $\mathcal{O}(D)$. Since every nonzero section of a line bundle determines a divisor, and vice versa up to scalars, every fiber $AJ^{-1}(L)$ is a projective space, namely $\mathbb{P}H^0(L)$.

Indeed, for $d \geq 2g-2$, C_d is the projectivization of a vector bundle over Jac_d, as follows. Fix $p \in C$, and let $\mathcal{L}$ be a Poincaré line bundle over $\mathrm{Jac}_d \times C$, normalized so that $\mathcal{L}|_{\mathrm{Jac}_d \times p} = \mathcal{O}$. Let $\pi : \mathrm{Jac}_d \times C \to \mathrm{Jac}_d$ be the projection, and let $U = \pi_* \mathcal{L}$. Then U is a vector bundle of rank $d - g + 1$ by Riemann-Roch, and there is a natural isomorphism $C_d = \mathbb{P}U$.

If $\mathcal{O}(1)$ is the twisting sheaf of this projective bundle, the embedding ι satisfies $\iota^*\mathcal{O}(1) = \mathcal{O}(1)$ for any $d > 2g-2$. So for $d < 2g-2$, it is reasonable to define $\mathcal{O}(1) = \iota^*\mathcal{O}(1)$ by descending induction. For any $d \geq 0$ and any $D \in C_d$, the fiber of $\mathcal{O}(-1)$ at D is then naturally isomorphic to the space of sections of $\mathcal{L}|_{AJ(D)\times C}$ vanishing at D, where $\mathcal{L}$ is again a Poincaré bundle normalized as above. To construct the isomorphism, one simply multiplies by the appropriate power of the natural section of $\mathcal{O}(p)$ vanishing at p.

(2.1) Proposition.

(i) *The twisting sheaf $\mathcal{O}(1)$ of $\mathbb{P}U$ is isomorphic to $\mathcal{O}(C_{d-1})$, so $c_1(\mathcal{O}(1)) = \eta$.*

(ii) *There is a natural isomorphism $\mathcal{L}(1) = \mathcal{O}(\Delta)$, where $\mathcal{L}$ is short for $AJ^*\mathcal{L}$.*

(iii) *If e_i is viewed as an element of $H^1(\mathrm{Jac}_d) \cong H_1(C)$, then $\xi_i = AJ^* e_i$.*

(iv) *If Θ is a theta-divisor on Jac_d, then $AJ^* c_1(\mathcal{O}(\Theta)) = \theta$.*

Proof. On $\mathbb{P}U = C_d$, evaluation at p gives a natural homomorphism of sheaves $\mathcal{O}(-1) \to \mathcal{L}|_{\mathrm{Jac}_d \times p} = \mathcal{O}$. This vanishes precisely on C_{d-1}, so $\mathcal{O}(1) = \mathcal{O}(C_{d-1})$.

To prove (ii), first assume $d > 2g-2$. Now for any $L \in \mathrm{Jac}_d$, the restriction of $\mathcal{O}(\Delta)$ to $\mathbb{P}H^0(L) \times C$ is $L(1)$, because $L(1)$ has a section with the universal property. Hence $\mathcal{L}(1) \otimes \mathcal{O}(-\Delta)$ is trivial on the fibers of the projection $C_d \times C \to \mathrm{Jac}_d$, which is locally trivial and hence flat for $d > 2g-2$. Therefore $\mathcal{L}(1) \otimes \mathcal{O}(-\Delta)$ is the pull-back of some line bundle on Jac_d; see Hartshorne [17, III Ex. 12.4]. But the restriction of $\mathcal{O}(\Delta)$ to $C_d \times p$ is $\mathcal{O}(C_{d-1}) = \mathcal{O}(1)$ by (i), and the restriction of $\mathcal{L}$ to $C_d \times p$ is $\mathcal{O}$ by construction, so this line bundle is trivial.

The case $d \leq 2g-2$ follows from this one by descending induction, since the embedding $\iota : C_{d-1} \hookrightarrow C_d$ satisfies $\iota^*\mathcal{L} = \mathcal{L}(p)$, $\iota^*\mathcal{O}(1) = \mathcal{O}(1)$, and $\iota^*\mathcal{O}(\Delta) = \mathcal{O}(\Delta)(p)$.

Parts (iii) and (iv) then follow from (ii) together with well-known formulas for $c_1(\mathcal{L})$ and $c_1(\mathcal{O}(\Theta))$; see §§2.6 and 2.7 of Griffiths-Harris [14]. □

A presentation of the cohomology ring of C_d was given by Macdonald [20].

(2.2) Theorem (Macdonald). *The cohomology ring $H^*(C_d;\mathbb{Z})$ is generated by ξ_i and η with relations*

$$0 = \eta^r \prod_{i\in I}(\eta - \sigma_i) \prod_{j\in J} \xi_j \prod_{k\in K} \xi_{k+g},$$

where $I, J, K \subset \{1, \dots, g\}$ are disjoint and

$$r + 2|I| + |J| + |K| \geq d + 1. \tag{2.3}$$

For $d > 2g-2$, these relations are generated by the single relation

$$0 = \eta^{d-2g+1} \prod_{i=1}^{g} (\eta - \sigma_i). \tag{2.4}$$

For $d \leq 2g-2$, they are generated by those for which $r = 0$ or 1 and equality holds in **(2.3)**. $\square$

(Macdonald's paper contains a small error: it is asserted that $r = 0$ is enough in the last statement. However, the method of proof clearly requires $r = 1$ as well, and it is already necessary in the case $d = 2$.)

(2.5) Corollary. *The subring of $H^*(C_d)$ invariant under all monodromies of C through smooth curves is generated by η and θ.*

Proof. Certainly the monodromy invariant part of $H^*(\mathrm{Jac}_d)$ is generated by θ. Indeed, since the monodromy surjects on $\mathrm{Aut}\, H^*(C) = \mathrm{Sp}(2g, \mathbb{Z})$, and $H^*(\mathrm{Jac}_d)$ is the exterior algebra on $H^1(C)^*$, this means simply that the symplectic form and its powers are the only alternating forms invariant under the symplectic group. Macdonald's result shows that $H^*(C_d)$ is generated by η as an algebra over $H^*(\mathrm{Jac}_d)$. For $d > 2g-2$, η is monodromy invariant by **(2.1)**(i); for $d < 2g-2$, it is still monodromy invariant since the embedding $i : C_{d-1} \hookrightarrow C_d$ satisfies $i^*\eta = \eta$. $\square$

We now turn to the quantum product on C_d. First notice that since $\sigma_i \in H^1(C_d)$, the quantum product with ξ_i equals the cup product by **(1.6)**(ii). Hence the quantum product is completely determined by the values of $\eta^u * \eta^v$ for $u, v \geq 0$.

Also notice that for $d > 1$, $h_2(C_d) = \binom{2g}{2} + 1$. This is an alarmingly large number of deformation parameters, but in fact only one parameter is nontrivially involved in the quantum product, for the following reason.

(2.6) Proposition. *If $\ell \in H_2(C_d;\mathbb{Z})$ is the homology class of any line in any fiber $AJ^{-1}(L)$, then the quantum product on $H^*(C_d;\Lambda)$ preserves the subring $H^*(C_d, \mathbb{Q}[[q^\ell]])$.*

Proof. Since an abelian variety has no rational curves whatsoever, every genus 0 stable map to C_d has image contained in a fiber of AJ. But for any line ℓ in any fiber of AJ, clearly $\eta \cdot \ell = 1$, while $\nu \cdot \ell = 0$ for any class ν pulled back from Jac_d. It follows that all such lines are homologous, so every genus 0 stable map has image homologous to a non-negative multiple of ℓ. $\square$

(2.7) Remarks.

(i) Though it is not needed in the sequel, a topological version of this statement remains true: the Hurewicz homomorphism $\pi_2(C_d) \to H_2(C_d;\mathbb{Z})$ has rank 1 for $d > 1$, so the multiples of ℓ are the only spherical classes. This is proved in an appendix, §9.

(ii) This subring determines the whole quantum product by Λ-linearity, so from now on attention will focus on it alone. In particular, the choice of polarization on C_d used to define Λ is immaterial. By abuse of notation q^ℓ is henceforth denoted simply by q, $\langle\cdots\rangle_\ell$ by $\langle\cdots\rangle_1$, and so on.

(iii) Macdonald [20] also shows that $c_1(TC_d) = (d-g+1)\eta - \theta$; consequently, the degree of $q \in QH^*(C_d)$ is $2c_1(TC_d)\cdot\ell = 2(d-g+1)$. In particular, it is negative for $d < g-1$ and 0 for $d = g-1$. The latter case resembles that of a Calabi-Yau manifold: although the canonical bundle is not trivial, its restriction to every rational curve is trivial.

(2.8) Example. Take $g=2$, $d=2$. Then by Riemann-Roch, every line bundle in Jac_2 has one section, except the canonical bundle, which has two. The Abel-Jacobi map therefore collapses exactly one rational curve $E = \mathbb{P}H^0(K)$. It is therefore precisely the blow-down of E [17, V 5.4]. The Poincaré dual of E is easily seen to be $\theta - \eta$. Since E is the only rational curve on C_2, $\overline{M}_0(C_2,[E])$ is a point and $\overline{M}_{0,1}(C_2,[E]) = E$. Therefore $\eta * \eta = \eta^2 + q(\theta-\eta)$, which completely characterizes the quantum product.

In some cases the vanishing of the 3-point invariants, and hence of certain terms in the quantum product, follows immediately from a dimension count.

(2.9) Proposition. *For $d < g-1$ and $e > \frac{d-3}{g-1-d}$, $\langle a,b,c\rangle_e = 0$ for all $a,b,c \in H^*(C_d)$.*

Proof. The expected dimension of $\overline{M}_0(C_d,e)$ is $d-3+e(d-g+1)$, which is negative in this case. The result follows from **(1.2)**(iii). □

Indeed, not only the 3-point invariants, but all higher-point invariants vanish in this range by the same argument.

(2.10) Corollary. *For $d < g-1$ and $e > \frac{d-3}{g-1-d}$,* $\mathrm{Coeff}_{q^e}\, a*b = 0$. *In particular, for $d < g/2+1$, the quantum product is simply the ordinary cup product.* □

It is also relatively easy to show that the 3-point invariants vanish for $d > 2g-2$ and $e > 1$. Indeed, this follows from the vanishing of the higher degree equivariant 3-point invariants of projective space. However, we will not pursue this now, as it is subsumed in **(6.1)**(i) below.

3 The Brill-Noether approach

To compute Gromov-Witten invariants for C_d for $d \leq 2g-2$, we must understand in detail how the fibers of the Abel-Jacobi map fit together. More precisely, we must understand the enumerative geometry of the strata where the dimension of the fiber is constant. This is the subject matter of Brill-Noether theory. Some well-known definitions and results in the theory are recalled in **(3.2)**–**(3.5)** below.

We shall also make crucial use of a formula of Harris and Tu [16] for the Chern numbers of kernel and cokernel bundles on determinantal varieties. Harris and Tu prove slightly more than the main result they state; moreover, their statement contains a sign error (the very first $+$ sign in the paper should be a $-$). In the form we shall need it, the result is the following.

(3.1) Theorem (Harris-Tu). *Let M be a complex manifold, E and F locally free sheaves of ranks m and n, and $f \in H^0(M, Hom(E,F))$. Let S be the universal subbundle over $\mathrm{Gr}_k E$, and let $\tilde{f} \in H^0(\mathrm{Gr}_k E; Hom(S,F))$ be the induced map. Suppose $\tilde{f}$ intersects the zero-section transversely in a variety M_k. If $x_1, \ldots, x_k$ are the Chern roots of S^* on M_k, then any characteristic number $\nu c_1^{j_1}(S) c_2^{j_2}(S) \cdots c_k^{j_k}(S)[M_k]$, where $\nu \in H^*(M)$, can be calculated using the formal identity*

$$\nu x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}[M_k] = \nu \begin{vmatrix} c_{n-m+k+i_1} & c_{n-m+k+1+i_1} & \cdots & c_{n-m+2k-1+i_1} \\ c_{n-m+k-1+i_2} & c_{n-m+k+i_2} & & \\ \vdots & & \ddots & \\ c_{n-m+1+i_k} & & & c_{n-m+k+i_k} \end{vmatrix} [M],$$

where $c_\alpha = c_\alpha(F-E)$. □

(3.2) Definitions. The moduli space G^r_d of (possibly incomplete) linear systems on C of dimension r and degree d can be constructed just as in the above theorem, taking $M = \mathrm{Jac}_d$. Fix any reduced divisor P on C of large degree, and let $\mathcal{L}$ be the Poincaré line bundle on $\mathrm{Jac}_d \times C$. Then the natural map $\mathcal{L}(P) \to \mathcal{O}_P \otimes \mathcal{L}(P)$ pushes forward to a map $E \to F$ of locally free sheaves on Jac_d such that, for any r, G^r_d is precisely the locus M_k of the theorem for $k = r+1$. The tautological subbundle over $\mathrm{Gr}_{r+1} E$ restricts to a bundle U^r_d over G^r_d whose projectivization $\mathbb{P}U^r_d$ admits a canonical map $\tau : \mathbb{P}U^r_d \to C_d$. Its image is referred to as C^r_d, and the subscheme $AJ(C^r_d)$ of Jac_d is referred to as W^r_d. Most of these definitions are discussed at greater length by Arbarello et al. [1].

(3.3) Lemma. *The total Chern classes of the bundles E and F defined above are $c(E) = \exp(-\theta)$ and $c(F) = 1$.*

Proof. The Poincaré bundle is normalized so that for some $p \in C$, $\mathcal{L}|_{\mathrm{Jac}_d \times p} = \mathcal{O}$. Since F is just a sum of bundles deformation equivalent to this, $c(F) = 1$. As for $c(E)$, this can be calculated using Grothendieck-Riemann-Roch: see Arbarello et al. [1, VIII §2]. □

Since Gromov-Witten invariants are deformation invariants, the curve C may be taken to be general, and we will assume this henceforth. This allows us to use one of the central results of Brill-Noether theory.

(3.4) Theorem (Gieseker). *Let C be a general curve. For any effective divisor D on C, the natural map*

$$H^0(\mathcal{O}(D)) \otimes H^0(K(-D)) \to H^0(K)$$

is injective.

Proof. See Gieseker [11]. $\square$

(3.5) Corollary. *For a general curve C, G^r_d, and hence $W^r_d \backslash W^{r+1}_d$ and $C^r_d \backslash C^{r+1}_d$, are smooth of dimension ρ, ρ, and $\rho + g$ respectively, where ρ is the Brill-Noether number $g-(r+1)(g-d+r)$.*

Proof. See Arbarello et al. [1, V 1.6]. $\square$

A further useful consequence of Gieseker's result is the following.

(3.6) Lemma.

(i) *At $L \in W^r_d \backslash W^{r+1}_d$, there is a natural isomorphism $N_{W^r_d/\operatorname{Jac}_d} = \operatorname{Hom}(H^0(L), H^1(L))$.*

(ii) *At $D \in C^r_d \backslash C^{r+1}_d$, there is a natural isomorphism*

$$N_{C^r_d/C_d} = \operatorname{Hom}\left(\frac{H^0(\mathcal{O}(D))}{\langle D\rangle}, H^1(D)\right).$$

Proof. Part (i) follows easily from Proposition 4.2(i) in Chapter IV of Arbarello et al. [1]. Part (ii) follows in the same way from Lemma 1.5 in that chapter, provided that the composition of natural maps

$$H^0(\mathcal{O}(D)) \otimes H^0(K(-D)) \to H^0(K) \to H^0(K \otimes \mathcal{O}_D)$$

has kernel $\langle D\rangle \otimes H^0(K(-D))$. But the latter map has kernel exactly $H^0(K(-D))$, so this follows from Gieseker's theorem. $\square$

(3.7) Proposition. *For general C, the reduced induced subscheme of the moduli space $\overline{M}_0(C_d, e)$ is a disjoint union of orbifolds*

$$\overline{M}_0(C_d, e)^{\mathrm{red}} = \bigcup_{r=1}^{e} \bigcup_{i=1}^{r} M_{r,i},$$

consisting of the stable maps whose image spans a linear system of dimension i contained in a complete linear system of dimension r (or $\geq r$ if $r = e$). The closure of each stratum has a resolution $\tilde{M}_{r,i}$ which is the $\overline{M}_0(\mathbb{P}^i, e)$ bundle associated to the tautological subbundle over $\operatorname{Gr}_{i+1} U^r_d$.

Proof. The resolutions $\tilde{M}_{r,i}$ can be viewed as moduli spaces of triples consisting of a linear system of dimension r, a projective subspace of dimension i, and a stable map to that subspace. By **(3.5)** they are orbifolds. The open subset where the map spans the i-dimensional subspace is a subvariety of $\overline{M}_0(C_d, e)$. These subvarieties partition $\overline{M}_0(C_d, e)$, because a rational curve of degree e in projective space spans at most an e-dimensional subspace. $\square$

Since every complete linear system has dimension $\geq d-g$, $M_{r,i} = \emptyset$ if $r < d-g$. Otherwise $M_{r,i} \neq \emptyset$, and $M_{r,i-1}$ is in its closure. Hence the irreducible components of $\overline{M}_0(C_d, e)^{\mathrm{red}}$ are the closures of $M_{r,r}$ for $\min(d-g, e) \leq r \leq e$.

(3.8) Proposition. *At any point of $M_{r,r}$, $\overline{M}_0(C_d, e)$ is reduced.*

Before the proof, a few observations that will be useful again later.

(3.9) Lemma. *For any stable map $\phi : \Sigma \to C_d$,*

(i) *the natural map $T^1_{C_d}(\phi) \to T^1(\Sigma)$ of deformation spaces is surjective; and*

(ii) *if $\mathbb{P}H^0(L)$ is the complete linear system containing $\phi(\Sigma)$, the obstruction space $T^2_{C_d}(\phi)$ is naturally isomorphic to $H^1(\Sigma, \phi^* N_{\mathbb{P}H^0(L)/C_d})$.*

Proof. Any stable map has image in some $\mathbb{P}^r = \mathbb{P}H^0(L) \subset C_d$, so there is a natural diagram of deformation spaces

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T^0(\Sigma) & \longrightarrow & H^0(\Sigma, \phi^* T\mathbb{P}^r) & \longrightarrow & T^1_{\mathbb{P}^r}(\phi) & \longrightarrow & T^1(\Sigma) \\
 & & \| & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & T^0(\Sigma) & \longrightarrow & H^0(\Sigma, \phi^* TC_d) & \longrightarrow & T^1_{C_d}(\phi) & \longrightarrow & T^1(\Sigma).
\end{array}$$

By **(1.3)**, $H^1(\Sigma, \phi^* T\mathbb{P}^r) = 0$, so by **(1.2)**(i) the last arrow in the first row is surjective, which proves (i). Then **(1.2)**(i) also implies that $T^2_{C_d}(\phi) = H^1(\Sigma, \phi^* TC_d)$, and the long exact sequence of

$$0 \longrightarrow \phi^* T\mathbb{P}^r \longrightarrow \phi^* TC_d \longrightarrow \phi^* N_{\mathbb{P}^r/C_d} \longrightarrow 0$$

completes the proof of (ii). □

Proof of **(3.8)**. It suffices to show that for $\phi \in M_{r,r}$, the deformation space $T^1_{C_d}(\phi)$ has dimension equal to that of $M_{r,r}$ itself. Now $M_{r,r}$ is an $\overline{M}_0(\mathbb{P}^r, e)$ bundle over an open set in G^r_d, so

$$\dim M_{r,r} = \dim \overline{M}_0(\mathbb{P}^r, e) + \dim G^r_d.$$

On the other hand, the number of deformations of ϕ as a map to $\mathbb{P}^r$ equals $\dim \overline{M}_0(\mathbb{P}^r, e)$ since the obstructions vanish. By **(3.9)**(i), the number of deformations as a map to C_d exceeds this by

$$\dim H^0(\Sigma, \phi^* TC_d) - \dim H^0(\Sigma, \phi^* T\mathbb{P}^r) = \dim H^0(\Sigma, \phi^* N_{\mathbb{P}^r/C_d}).$$

Hence it suffices to show $\dim H^0(\Sigma, \phi^* N_{\mathbb{P}^r/C_d}) = \dim G^r_d$. Because of the definition of $M_{r,r}$, there are two cases, depending on whether or not $r = e$.

If $r < e$, the image of ϕ is in $C^r_d \backslash C^{r+1}_d$, which is a $\mathbb{P}^r$-bundle over the open set $W^r_d \backslash W^{r+1}_d \subset G^r_d$. On any fiber $\mathbb{P}^r = \mathbb{P}H^0(L)$ of this bundle, there is an exact sequence

$$0 \longrightarrow T_L W^r_d \otimes \mathcal{O} \longrightarrow N_{\mathbb{P}^r/C_d} \longrightarrow N_{C^r_d/C_d} \longrightarrow 0.$$

Hence it suffices to show that $H^0(\Sigma, \phi^* N_{C_d^r/C_d}) = 0$. By Lemma **(3.6)**(ii), there is a short exact sequence on $\mathbb{P}^r$

$$0 \longrightarrow N_{C_d^r/C_d} \longrightarrow \mathcal{O} \otimes \operatorname{Hom}(H^0(L), H^1(L)) \longrightarrow \mathcal{O}(1) \otimes H^1(L) \longrightarrow 0.$$

By the definition of $M_{r,r}$, the image of ϕ spans $\mathbb{P}H^0(L)$. Hence the natural map $H^0(L)^* \to H^0(\Sigma, \phi^*\mathcal{O}(1))$ is injective, so

$$H^0(\Sigma, \mathcal{O} \otimes \operatorname{Hom}(H^0(L), H^1(L))) \longrightarrow H^0(\Sigma, \phi^*\mathcal{O}(1) \otimes H^1(L))$$

is also and $H^0(\Sigma, \phi^* N_{C_d^r/C_d}) = 0$ as desired.

If $r = e$, then the image of ϕ spans a linear system $\mathbb{P}^r$ which may not be complete. Let $\mathbb{P}H^0(L)$ be the complete linear system containing it, and let ℓ be its projective dimension. Then there are three short exact sequences on $\mathbb{P}^r$:

$$0 \longrightarrow N_{\mathbb{P}^r/\mathbb{P}H^0(L)} \longrightarrow N_{\mathbb{P}^r/C_d} \longrightarrow N_{\mathbb{P}H^0(L)/C_d} \longrightarrow 0,$$

$$0 \longrightarrow T_L W_d^\ell \otimes \mathcal{O} \longrightarrow N_{\mathbb{P}H^0(L)/C_d} \longrightarrow N_{C_d^\ell/C_d} \longrightarrow 0,$$

$$0 \longrightarrow N_{C_d^\ell/C_d} \longrightarrow \operatorname{Hom}(H^0(L), H^1(L)) \otimes \mathcal{O} \longrightarrow H^1(L) \otimes \mathcal{O}(1) \longrightarrow 0,$$

the last by **(3.6)**(ii). But $N_{\mathbb{P}^r/\mathbb{P}H^0(L)} \cong \mathcal{O}^{\ell-r}(1)$ since $\mathbb{P}^r \subset \mathbb{P}H^0(L)$ is a projective subspace. Since $H^1(\Sigma, \mathcal{O}) = H^1(\Sigma, \phi^*\mathcal{O}(1)) = 0$, all three of the long exact sequences contain only H^0 terms. Hence $\dim H^0(\Sigma, \phi^* N_{\mathbb{P}^r/C_d})$ can be expressed in terms of $\dim H^0(\Sigma, \phi^*\mathcal{O}(1))$, which is $r+1$ since ϕ spans $\mathbb{P}^r$, and the ranks of the bundles above. A little high-school algebra gives the desired result. □

4 The degree 1 invariants

(4.1) Proposition. *As schemes,* $\overline{M}_0(C_d, 1) = G_d^1$ *and* $\overline{M}_{0,1}(C_d, 1) = \mathbb{P}U_d^1$.

Proof. The identification of sets is clear from **(3.7)**, and the moduli space is reduced by **(3.8)**. □

(4.2) Theorem. *For* $u, v, w \geq 0$,

$$\langle \eta^u, \eta^v, \theta^{g-m}\eta^w \rangle_1 = \frac{g!}{m!} \sum_{i=0}^{u-1} \binom{m}{g-d+i+v} - \binom{m}{g-d+i},$$

where $m = 2g - 2d - 1 + u + v + w$.

For example, consider the case when the expected dimension $\rho = g - 2(g-d+1)$ of G_d^1 is zero, which occurs when $g = 2d-2$. Then $\langle \eta, \eta, \eta \rangle_1$ counts the number of points in G_d^1. According to **(4.2)**, this is $\binom{2d-2}{d-1} - \binom{2d-2}{d-2}$, which agrees with the Catalan number $\frac{1}{d}\binom{2d-2}{d-1}$ computed by Castelnuovo [1, V 1.2].

Proof. Since the moduli space is smooth of the expected dimension, the Gromov-Witten invariants can be calculated using **(1.5)**. The evaluation map is precisely $\tau : \mathbb{P}U^1_d \to C_d$, as defined in **(3.2)**. By **(2.1)**(i) $\tau^*\eta = c_1(\mathcal{O}(1))$, and hence $f_*\tau^*\eta^u$ is the Segre class $s_{u-1}(U^1_d)$; see for example Fulton [9, §3.1]. If x_1 and x_2 are the Chern roots of $(U^1_d)^*$, then the Segre classes are the complete symmetric polynomials:

$$s_k = \sum_{i=0}^{k} x_1^i x_2^{k-i}.$$

So the 3-point invariant is

$$\langle \eta^u, \eta^v, \theta^{g-m}\eta^w \rangle_1 = \theta^{g-m} \sum_{i=0}^{u-1} \sum_{j=0}^{v-1} \sum_{k=0}^{w-1} x_1^{i+j+k} x_2^{u+v+w-3-i-j-k} [G^1_d].$$

The Harris-Tu formula implies that

$$\theta^{g-m} x_1^p x_2^{u+v+w-3-p} [G^1_d] = \theta^{g-m}(\gamma_p - \gamma_{p+1})[\mathrm{Jac}_d],$$

where

$$\begin{aligned} \gamma_p &= c_{g-d+1+p}(F-E) \cdot c_{g-d+u+v+w-2-p}(F-E) \\ &= \theta^m/(g-d+1+p)!(g-d+u+v+w-2-p)!, \end{aligned}$$

the last equality by **(3.3)**. The sum over k therefore telescopes to yield

$$\langle \eta^u, \eta^v, \theta^{g-m}\eta^w \rangle_1 = \theta^{g-m} \sum_{i=0}^{u-1} \sum_{j=0}^{v-1} (\gamma_{i+j} - \gamma_{i+j+w})[\mathrm{Jac}_d].$$

But there is another symmetry, namely $\gamma_p = \gamma_{u+v+w-3-p}$; applying this to the second term and canceling gives

$$\begin{aligned} \sum_{i=0}^{u-1} \sum_{j=0}^{v-1} \gamma_{i+j} - \gamma_{i+j+w} &= \sum_{i=0}^{u-1} \sum_{j=0}^{v-1} \gamma_{i+j} - \gamma_{(u-1-i)+(v-1-j)-1} \\ &= \sum_{i=0}^{u-1} \sum_{j=0}^{v-1} \gamma_{i+j} - \gamma_{i+j-1} \\ &= \sum_{i=0}^{u-1} \gamma_{i+v-1} - \gamma_{i-1}. \end{aligned}$$

Substituting this and using $\theta^g[\mathrm{Jac}_d] = g!$ then gives the answer as stated. □

Of course, by the same method one could easily derive a formula for 3-point invariants involving arbitrary elements of $H^*(C_d)$. But these are enough to characterize the linear term of the quantum product.

(4.3) Corollary. *For $u, v \geq 0$,*

$$\eta^u * \eta^v = \eta^{u+v} + q\sum_{i=0}^{u-1}\left(\frac{\theta^{g-d+i+v}}{(g-d+i+v)!}\eta^{u-1-i} - \frac{\theta^{g-d+i}}{(g-d+i)!}\eta^{v+u-1-i}\right) + O(q^2).$$

Proof. By the definition of quantum product, the coefficient of q in $\eta^u * \eta^v$ is μ if and only if for all $\nu \in H^*(C_d)$, $\langle \eta^u, \eta^v, \nu\rangle_1 = \mu\nu[C_d]$. However, Gromov-Witten invariants, and hence the quantum product, are deformation invariant. Since η and its powers are monodromy invariant by **(2.5)** whenever C lies in a family of smooth curves, each coefficient of $\eta^u * \eta^v$ must be as well. Again by **(2.5)**, this means that it is in the subring generated by η and θ. This subring is the algebraic part of $H^*(C_d)$ for C general: see Arbarello et al. [1, VIII §5]. In particular, it satisfies Poincaré duality; hence it suffices to take ν in the subring as well. This circumvents the cumbersome task of dealing with arbitrary monomials in the ξ_i.

It is easy to check against **(4.2)** that the coefficient stated above satisfies the required condition. One simply applies the convenient formula

$$\theta^i\eta^{d-i}[C_d] = \begin{cases} \frac{g!}{(g-i)!} & \text{if } i \leq g \\ 0 & \text{if } i > g \end{cases}$$

which follows from **(2.1)**(i) and **(2.4)** using the fact that $C_{d-1} \subset C_d$ is Poincaré dual to η. The required identity follows term-by-term from a comparison of the coefficient above with the Gromov-Witten invariant in **(4.2)**. □

5 The degree 2 invariants

The degree 2 invariants can also be calculated using ideas from Brill-Noether theory. Here things are considerably more complicated; in particular, the virtual class comes into play. However, some delightful cancellations make the computations tractable.

In the decomposition of **(3.7)**, the moduli space $\overline{M}_0(C_d, 2)$ has three strata, M_{11}, M_{22}, and M_{12}. If $d \geq g+2$, M_{11} is empty, but this will not affect the results.

Of these strata, M_{22} has the expected dimension, while M_{11} exceeds it by $g+1-d$. The third, M_{12}, is in the closure of M_{22} and has less than the expected dimension. The virtual class is therefore a sum of cycles pushed forward from $\tilde{M}_{11}$ and $\tilde{M}_{22}$. By **(3.8)**, the former is simply the ordinary fundamental class of $\overline{M}_{22}$. The Gromov-Witten invariant therefore can be computed as in **(1.5)**:

$$\langle a_1, a_2, a_3\rangle_2 = \Big(\prod_k \tilde{f}_*\widetilde{\mathrm{ev}}^* a_k\Big)([\tilde{M}_{11}]^{\mathrm{vir}} + [\tilde{M}_{22}]).$$

(5.1) Proposition. *For all $a_1, a_2, a_3 \in H^*(C_d)$,*

$$\Big(\prod_k \tilde{f}_*\widetilde{\mathrm{ev}}^* a_k\Big)[\tilde{M}_{22}] = 0.$$

Proof. On $\tilde{M}_{22} \times_{\overline{M}_0(C_d,2)} \overline{M}_{0,1}(C_d, 2)$, the evaluation maps factor through the $\mathbb{P}^2$ bundle $\mathbb{P}U_d^2$, where any cohomology class can be expressed as a polynomial over $H^*(G_d^2)$ of degree ≤ 2. So on $\tilde{M}_{22}$, $\tilde{f}_* \tilde{\mathrm{ev}}^* a_i$ is in the submodule $H^{\geq \deg a_i - 4}(G_d^2) \cdot H^*(\tilde{M}_{22})$. It certainly vanishes on the fundamental class for dimensional reasons unless $\sum \deg a_i = 2 \dim \tilde{M}_{22} + 6 = 6d - 4g + 4$. But then the product is in the submodule $H^{\geq 6d-4g-8}(G_d^2) \cdot H^*(\tilde{M}_{22})$, which is zero since $\dim G_d^2$ is the Brill-Noether number $\rho = 3d - 2g - 6$. □

The above is really a special case of the more general statement that the equivariant 3-point invariants of projective space vanish in degree > 1.

We now attack the virtual class $[\tilde{M}_{11}]^{\mathrm{vir}}$, or rather its push-forward to G_d^1, where the invariant will be calculated. For simplicity U_d^1 will be denoted U in the rest of this section.

(5.2) Theorem. *Let $p : \tilde{M}_{11} \to G_d^1$ be the projection. Then $p_*[\tilde{M}_{11}]^{\mathrm{vir}}$ is Poincaré dual to the degree $2(g-1-d)$ part of*

$$\frac{\frac{1}{8}\exp\theta}{1 + c_1(U)/2}.$$

Proof. First, it suffices to work only on $p : M_{11} \to W_d^1 \backslash W_d^2$, because the codimension of the missing locus in G_d^1 is $g + 2 - d$, which is greater than $g - 1 - d$.

Take the short exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(\Delta) \longrightarrow \mathcal{O}_\Delta(\Delta) \longrightarrow 0$$

on $(C_d^1 \backslash C_d^2) \times C$, and push forward to a long exact sequence on the first factor:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (R^0\pi)_*\mathcal{O} & \longrightarrow & (R^0\pi)_*\mathcal{O}(\Delta) & \longrightarrow & (R^0\pi)_*\mathcal{O}_\Delta(\Delta) \\ & \longrightarrow & (R^1\pi)_*\mathcal{O} & \longrightarrow & (R^1\pi)_*\mathcal{O}(\Delta) & \longrightarrow & 0. \end{array}$$

The third nonzero term is TC_d, and the image of the second is the tangent space to the pencils. The quotient is therefore the vector bundle N whose fibers along each pencil are the normal spaces to that pencil in C_d. So there is a short exact sequence on $C_d^1 \backslash C_d^2$

$$0 \longrightarrow N \longrightarrow (R^1\pi)_*\mathcal{O} \longrightarrow (R^1\pi)_*\mathcal{O}(\Delta) \longrightarrow 0,$$

and hence a long exact sequence

$$0 \longrightarrow H^0(\Sigma, \phi^*N) \longrightarrow H^1(C, \mathcal{O}) \longrightarrow H^0(\Sigma, \phi^*(R^1\pi)_*\mathcal{O}(\Delta)) \longrightarrow H^1(\Sigma, \phi^*N) \longrightarrow 0.$$

By **(3.9)**(ii), the obstruction space $T^2_{C_d}(\phi)$ is nothing but the last term of this sequence. Moreover, by **(1.2)**(ii), the virtual class on M_{11} is exactly the Euler class of the orbifold vector bundle whose fiber at $\phi : \Sigma \to C_d$ is this obstruction space.

How can the terms in this long exact sequence be described better? Well, note that N also fits into the exact sequence on $C_d^1 \backslash C_d^2$

$$0 \longrightarrow TW_d^1 \longrightarrow N \longrightarrow N_{C_d^1/C_d} \longrightarrow 0.$$

By **(3.6)**(ii), $N_{C_d^1/C_d}$ restricted to the pencil $\mathbb{P}H^0(L)$ is $\mathcal{O}(-1) \otimes H^1(L)$, so $H^0(\Sigma, \phi^* N_{C_d^1/C_d})$ vanishes and hence $H^0(\Sigma, \phi^*N)$ is simply $T_L W_d^1$. On the other hand, if $\mathcal{O}(1)$ on C_d denotes the relative hyperplane bundle of the Abel-Jacobi map, and $\mathcal{L}$ is a Poincaré line bundle pulled back from $\mathrm{Jac}_d \times C$, then on $C_d \times C$, $\mathcal{O}(\Delta) = \mathcal{L}(1)$ by **(2.1)**(ii). Hence on $C^1_d \backslash C^2_d$, $(R^1\pi)_*\mathcal{O}(\Delta)$ is the tensor product $V(1)$, where $V = (R^1\pi)_*\mathcal{L}$ is a vector bundle pulled back from $W^1_d \backslash W^2_d$. The term $H^0(\phi^*(R^1\pi)_*\mathcal{O}(\Delta))$ which appeared above is therefore the tensor product $V \otimes H^0(\Sigma, \phi^*\mathcal{O}(1))$.

Putting it all together yields an exact sequence

$$0 \longrightarrow N_{W^1_g/\,\mathrm{Jac}_d} \longrightarrow V \otimes H^0(\Sigma, \phi^*\mathcal{O}(1)) \longrightarrow H^1(\Sigma, \phi^*N) \longrightarrow 0$$

for any $\phi \in M_{11}$. The first two terms clearly have dimension independent of ϕ, so they determine an orbifold vector bundle on M_{11}, and hence so does the third term.

Now consider the orbifold structure on M_{11}. It is an $\overline{M}_0(\mathbb{P}^1, 2)$-bundle over $W^1_d \backslash W^2_d$. In fact, a degree 2 stable map to a line is characterized by its base points, so as schemes $\overline{M}_0(\mathbb{P}^1, 2) \cong \mathbb{P}^2$ and $M_{11} = \mathbb{P}\,\mathrm{Sym}^2 U^*$ where $U = U^2_d$. But every stable map in M_{11} has an involution, so as an orbifold M_{11} is a global quotient by $\mathbb{Z}_2$ of a double cover, say $\hat{M}_{11}$, branched over the bundle of conics $\mathbb{P}U^* \subset \mathbb{P}\,\mathrm{Sym}^2 U^*$.

The involution splits $H^0(\Sigma, \phi^*\mathcal{O}(1))$ into ± 1-eigenspaces. The $+1$-eigenspace consists of those sections pulled back from the pencil, so the corresponding bundle is just U^*. The -1-eigenspace is generated by the square root of the section of $\mathcal{O}(2)$ on the pencil vanishing on the branch points of the double cover, so the corresponding bundle is an orbifold line bundle on M_{11}, coming from a $\mathbb{Z}_2$-equivariant line bundle on $\hat{M}_{11}$ whose tensor square is the pull-back of $\mathcal{O}(-1)$ from M_{11}. Call this orbifold bundle $\mathcal{O}(-\frac{1}{2})$.

Now it follows from **(3.6)**(i) that the normal bundle $N_{W^1_g/\,\mathrm{Jac}_d}$ is $V \otimes U^*$, and its image in $V \otimes H^0(\Sigma, \phi^*\mathcal{O}(1))$ is clearly in the $+1$-eigenspace. This splits the exact sequence mentioned above. Hence the orbifold bundle with fiber $H^1(\Sigma, \phi^*N)$ is none other than $V(-\frac{1}{2})$, and the virtual class is the Euler class of this.

It only remains to push forward the virtual class to G^1_d. If ξ_j are the Chern roots of V, and h is the hyperplane class of $\mathbb{P}\,\mathrm{Sym}^2 U^*$, then the virtual class is $\prod_j (\xi_j - h/2)$. This pushes forward to

$$\tfrac{1}{2} \sum_{n=0}^{g-d-1} (-\tfrac{1}{2})^{n+2} c_{g-1-d-n}(V)\, s_n(\mathrm{Sym}^2 U^*),$$

where s_i denotes the Segre class. The extra factor of $\frac{1}{2}$ appears because of the orbifold structure on M_{11}.

The Chern roots of $\mathrm{Sym}^2 U^*$ are $2x_1$, $2x_2$, and $x_1 + x_2$. The factors of 2 cancel and one gets the degree $2(g-d-1)$ part of:

$$\frac{\frac{1}{8}\, c(V)}{(1-x_1)(1-x_2)(1-(x_1+x_2)/2)}.$$

Now a miraculous cancellation. The $(1-x_1)(1-x_2)$ in the denominator is just $c(U)$. But U and V are exactly the kernel and cokernel of the map of bundles $E \to F$ defined in **(3.2)**. Hence by **(3.3)**

$$\begin{aligned} c(V)/c(U) &= c(F)/c(E) \\ &= 1/\exp(-\theta) \\ &= \exp\theta, \end{aligned}$$

and plugging this in yields the stated formula. □

(5.3) Theorem. *For $u,v,w\geq 0$,*

$$\begin{aligned}\langle\eta^u,\eta^v,\theta^{d+1-m}\eta^w\rangle_2 &\\ = \sum_{n=0}^{g-1-d}\sum_{p=0}^{n}\binom{n}{p}\frac{g!}{2^n(g-1-d-n)!(m+n)!}\sum_{i=0}^{u-1}&\binom{m+n}{g-d+i+v+p}-\binom{m+n}{g-d+i+p},\end{aligned}$$

where $m=2g-2d-1+u+v+w$.

Proof. The proof is parallel to that of **(4.2)**.

By **(5.1)**,

$$\langle a_1,a_2,a_3\rangle_2=\Big(\prod_k \tilde{f}_*\widetilde{\mathrm{ev}}^*a_k\Big)[\tilde{M}_{11}]^{\mathrm{vir}}.$$

Conveniently, $\tilde{M}_{11}=\mathbb{P}\,\mathrm{Sym}^2U^*$ is isomorphic to the closure $\overline{M}_{11}$. Also, $f^{-1}(\overline{M}_{11})$ is simply the fibered product $\mathbb{P}\,\mathrm{Sym}^2U^*\times_{G^1_d}\mathbb{P}U$. In contrast to M_{11}, a generic stable map in $f^{-1}(\overline{M}_{11})$ has no involution; it is rigidified by the marked point. The evaluation map $f^{-1}(\overline{M}_{11})\to C_d$ factors through $\mathbb{P}U$; hence $f_*\,\mathrm{ev}^*\eta^u$ restricted to $\overline{M}_{11}$ is a class pulled back from G^1_d, namely $2\,s_{u-1}(U)$, where s_i is the ith Segre class. The factor of 2 appears because of the orbifold structure on $\overline{M}_{11}$. In terms of the Chern roots x_1,x_2 of U^*, this is

$$2\,s_{u-1}(U)=2\sum_{i=0}^{u-1}x_1^ix_2^{u-1-i}.$$

Expanding the formula of **(5.2)** in terms of x_1 and x_2 yields the cap product

$$p_*[\tilde{M}_{11}]^{\mathrm{vir}}=\tfrac{1}{8}\sum_{n=0}^{g-1-d}\sum_{p=0}^{n}\binom{n}{p}\frac{(-1)^n\theta^{g-1-d-n}x_1^px_2^{n-p}}{2^n(g-1-d-n)!}\cap[G^1_d].$$

Consequently

$$\begin{aligned}&\langle\eta^u,\eta^v,\theta^{g-n-m}\eta^w\rangle_2\\ &\quad=\sum_{n=0}^{g-1-d}\sum_{p=0}^{n}\sum_{i=0}^{u-1}\sum_{j=0}^{v-1}\sum_{k=0}^{w-1}\binom{n}{p}\frac{\theta^{d-4-m}x_1^{p+i+j+k}x_2^{n-p+u+v+w-3-i-j-k}}{2^n(g-1-d-n)!}[G^1_d].\end{aligned}$$

As in the proof of **(4.2)**, one now applies the Harris-Tu formula **(3.1)**, telescopes the sum over k, and cancels using the additional symmetry to obtain the stated result. □

(5.4) Corollary. *For $u,v\geq 0$,*

$$\eta^u*\eta^v=\eta^{u+v}+q\sum_{i=0}^{u-1}\left(\frac{\theta^{g-d+i+v}}{(g-d+i+v)!}\eta^{u-1-i}-\frac{\theta^{g-d+i}}{(g-d+i)!}\eta^{v+u-1-i}\right)$$

$$+ q^2 \sum_{n=0}^{g-1-d} \sum_{p=0}^{n} \binom{n}{p} \frac{1}{2^n (g-1-d-n)!}$$
$$\cdot \sum_{i=0}^{u-1} \left(\frac{\theta^{2g-2d+1-n+i+v+p} \eta^{u+n-i-p-3}}{(g-d+i+v+p)!} - \frac{\theta^{2g-2d+1-n+i+p} \eta^{u+v+n-i-p-3}}{(g-d+i+p)!} \right) + O(q^3).$$

Proof. Similar to that of **(4.3)**. □

6 The higher degree invariants for $d \geq g-1$

Examining the formulas of the last two sections reveals that the degree 2 invariants vanish for $d > g-1$, and equal the degree 1 invariants for $d = g-1$. The result below shows that this is the case for all higher degree invariants.

(6.1) Theorem. *For all $e > 1$ and all $a_1, a_2, a_3 \in H^*(C_d)$,*

(i) $\langle a_1, a_2, a_3 \rangle_e = 0$ *when* $d > g-1$;

(ii) $\langle a_1, a_2, a_3 \rangle_e = \langle a_1, a_2, a_3 \rangle_1$ *when* $d = g-1$.

Proof. The virtual class is some algebraic cycle class of degree $d-3+e(d-g+1)$. Choose a cycle representing this class; it decomposes uniquely as a sum of cycles $\sum_{r,i} [M_{r,i}]^{\mathrm{vir}}$, where every component of $[M_{r,i}]^{\mathrm{vir}}$ is supported on the closure of $M_{r,i}$ but not on any smaller stratum. Then $[M_{r,i}]^{\mathrm{vir}}$ is in the image of the push-forward from the resolution $\tilde{M}_{r,i}$, so its contribution to the Gromov-Witten invariant $\langle a_1, a_2, a_3 \rangle_e$ can be computed on $\tilde{M}_{r,i}$ as

$$\Big(\prod_k \tilde{f}_* \widetilde{\mathrm{ev}}^* a_k \Big) [\tilde{M}_{r,i}]^{\mathrm{vir}}.$$

Now on $\tilde{M}_{r,i}$, all three of the evaluation maps factor through $\mathbb{P}U^r_d$. The pullback of a_i to this bundle can be written as a polynomial in the hyperplane class over $H^*(G^r_d)$ of degree $\leq r$. Hence $\tilde{f}_* \widetilde{\mathrm{ev}}^* a_i$ is in the submodule

$$H^{\geq \deg a_i - 2r}(G^r_d) \cdot H^*(\overline{M}_{r,i}).$$

Certainly $\langle a_1, a_2, a_3 \rangle_e = 0$ on dimensional grounds unless $\sum_k \deg a_k = 2d + 2e(d-g+1)$. In that case the product $\prod_k \tilde{f}_* \widetilde{\mathrm{ev}}^* a_k$ is in the submodule

$$H^{\geq 2d+2e(d-g+1)-6r}(G^r_d) \cdot H^*(\overline{M}_{r,i}).$$

But the dimension of G^r_d is the Brill-Noether number $\rho = g-(r+1)(g-d+r)$, and some high school algebra shows that

$$d + e(d-g+1) - 3r - \rho = (e-r)(d-g+1) + r(r-1).$$

This is positive for all $d > g-1$ unless $e = r = 1$, and also for $d = g-1$ unless $r = 1$.

Hence the ideal $H^{\geq 2d+2e(d-g+1)-6r}(G^r_d)$ vanishes in those cases. This immediately implies part (i) of the theorem, and it shows that when $d = g-1$, only the single stratum M_{11} contributes nontrivially to the 3-point invariants.

We therefore turn to the contribution of this stratum, which consists of e-fold covers of pencils on C_{g-1}. The closure $\overline{M}_{11}$ is isomorphic to $\tilde{M}_{11}$, which is a bundle over G^1_d with fiber $\overline{M}_0(\mathbb{P}^1, e)$. The evaluation map factors through $\mathbb{P}U^1_d$:

$$\begin{array}{ccccc} f^{-1}(\overline{M}_{11}) & \longrightarrow & \mathbb{P}U^1_d & \xrightarrow{\tau} & C_d \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \pi} & & \\ \overline{M}_{11} & \xrightarrow{p} & G^1_d. & & \end{array}$$

The morphism $f^{-1}(\overline{M}_{11}) \to \overline{M}_{11} \times_{G^1_d} \mathbb{P}U^1_d$ is finite of degree e, so for $a \in H^*(C_d)$,

$$p_* f_* \operatorname{ev}^* a = e \cdot \pi_* \tau^* a.$$

The Gromov-Witten invariant can be expressed as

$$\langle a_1, a_2, a_3 \rangle_e = \Big(\prod_k p_* f_* \operatorname{ev}^* a_k \Big) \, p_*[\tilde{M}_{11}]^{\mathrm{vir}};$$

but since G^1_d has dimension $g-4$, exactly that of the virtual cycle, $p_*[\tilde{M}_{11}]^{\mathrm{vir}} \in H_{2g-8}(G^1_d)$ must be a scalar multiple of $[G^1_d]$.

On the other hand, by **(4.1)** $G^1_d = \overline{M}_0(C_d, 1)$ and $\mathbb{P}U^1_d = \overline{M}_0(C_d, 1)$. Hence

$$\Big(\prod_k \pi_* \tau^* a_k \Big) [G^1_d] = \langle a_1, a_2, a_3 \rangle_1.$$

So to prove part (ii) of the theorem, it suffices to show that $p_*[\tilde{M}_{11}]^{\mathrm{vir}} = 1/e^3 [G^1_d]$. One may work on any single fiber of p; choose one over W^1_d. The moduli space is an orbifold in a neighborhood of this fiber, so **(1.2)**(ii) applies. Hence we need to contribute the Euler class of the orbifold bundle whose fiber at a map ϕ is the obstruction space $T^2_{C_d}(\phi)$. By **(3.9)**(ii), $T^2_{C_d}(\phi) = H^1(\Sigma, \phi^* N_{\mathbb{P}^1/C_d})$. ¿From the short exact sequence

$$0 \longrightarrow N_{\mathbb{P}^1/C^1_d} \longrightarrow N_{\mathbb{P}^1/C_d} \longrightarrow N_{C^1_d/C_d} \longrightarrow 0,$$

together with the isomorphism $N_{\mathbb{P}^1/C^1_d} \cong \mathcal{O}^\rho$, it follows that

$$H^1(\Sigma, \phi^* N_{\mathbb{P}^1/C_d}) = H^1(\Sigma, \phi^* N_{C^1_d/C_d}).$$

As seen in **(3.6)**(ii), the normal space to C^r_d at a divisor D is naturally isomorphic to $\operatorname{Hom}(H^0(\mathcal{O}(D))/\langle D \rangle, H^1(\mathcal{O}(D)))$. In this case $\dim H^0 = \dim H^1 = 2$, so on $\mathbb{P}^1$ the normal bundle $N_{C^1_d/C_d}$ is isomorphic to $\operatorname{Hom}(\mathcal{O}(1), \mathcal{O} \oplus \mathcal{O}) = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

So, on $\overline{M}_0(\mathbb{P}^1, e)$, we want to know the Euler class of the orbifold bundle whose fiber at a map ϕ is $H^1(\phi^*\mathcal{O}(-1) \oplus \phi^*\mathcal{O}(-1))$. Very felicitously, this is exactly the number computed to be $1/e^3$, using some nontrivial combinatorics, in the work of Aspinwall-Morrison [2], Manin [21], and Voisin [31]. In their work, the motivation was to compute Gromov-Witten invariants for Calabi-Yau threefolds containing rational curves with normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$; the present case is in a sense a relative version of this, since C_{g-1} contains the family C^1_{g-1} of rational curves whose normal bundle restricts to $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ on every curve. In any case, this completes the proof. $\square$

(6.2) Corollary. *For all $u, v \geq 0$,*

(i) *when $d > g-1$,*

$$\eta^u * \eta^v = \eta^{u+v} + q\sum_{i=0}^{u-1}\left(\frac{\theta^{g-d+i+v}}{(g-d+i+v)!}\,\eta^{u-1-i} - \frac{\theta^{g-d+i}}{(g-d+i)!}\,\eta^{u+v-1-i}\right);$$

(ii) *when $d = g-1$,*

$$\eta^u * \eta^v = \eta^{u+v} + \frac{q}{1-q}\sum_{i=1}^{u}\left(\frac{\theta^{i+v}}{(i+v)!}\,\eta^{u-i} - \frac{\theta^i}{i!}\,\eta^{v+u-i}\right).$$

Proof. Similar to that of **(4.3)**. $\square$

7 Presentation of the quantum ring

The results of the previous sections **(2.5)**, **(4.3)**, **(5.4)**, **(6.2)** have given an explicit quantum multiplication table for C_d provided $d \notin [\frac{3}{4}g, g-1)$. It is natural to look for a presentation of the quantum cohomology ring as well. Such a presentation contains less information than the multiplication table, because it does not specify the additive isomorphism between $QH^*(C_d)$ and $H^*(C_d;\Lambda)$. We will be able to determine it even in the slightly more general case $d \notin [\frac{4}{5}g - \frac{3}{5}, g-1)$. To go this far requires only the degree 1 invariants and the results of the last section; in principle one could continue as far as $\frac{6}{7}g - \frac{5}{7}$ by using the degree 2 invariants, but this becomes cumbersome.

(7.1) Proposition. *The rings $QH^*(C_d)$ are generated over Λ by Macdonald's generators η and ξ_i, and there is a complete set of relations, uniquely determined by the property that it reduces mod q to Macdonald's relations.*

Proof. These facts are well-known for quantum cohomology generally in the case $\deg q > 0$. The first statement is proved by a simple induction on the degree of the cup product of an arbitrary collection of classical generators. Just take a similar monomial where all the cup products are replaced by quantum products. The difference between the two monomials is a multiple of q, so the coefficient has lower degree and by induction it can be expressed as a quantum product of the classical generators. This shows that the classical generators generate the quantum ring as well.

To extend the classical relations to quantum relations, first take a classical relation and replace all the cup products by quantum products. This quantum expression may not be a quantum relation, but it is in the ideal $\langle q\rangle$, because it reduces to a classical relation. So express it as q times a classical monomial, replace the cup products in this monomial by quantum products, and subtract the result from the quantum expression. This difference is now in $\langle q^2\rangle$. Proceed inductively; when $\deg q > 0$, the coefficient of a high power of q will eventually be in $H^{<0} = 0$.

To see that these generate all the quantum relations, just notice that imposing them gives a free module over Λ of the correct dimension.

Similar arguments work for $\deg q < 0$. To prove the first statement, for example, express the generator of top-dimensional cohomology as a cup product of generators. This identity still holds if the cup products are replaced by quantum products. Then apply descending induction.

This leaves only the case $\deg q = 0$, which for a symmetric product C_d means $d = g-1$. Since quantum and cup products with each ξ_i are the same, to prove the first statement it suffices to express η^u as a quantum polynomial in η and ξ_i for all $u \geq 0$. An induction on u shows this is possible, since by **(6.2)**(ii)

$$\eta * \eta^u = \eta^{u+1} + \frac{q}{1-q}\left(\frac{\theta^{u+1}}{(u+1)!} - \theta\eta^u\right).$$

The second statement is proved as for $\deg q > 0$, but by induction on the power of θ rather than q. □

How do we find the quantum relations alluded to in the proposition? Macdonald's classical relations from **(2.2)** can be expressed as

$$0 = \Big(\sum_\alpha \eta^{r+|I|-\alpha} s_\alpha(\sigma_i)\Big)\Big(\prod \xi_j \prod \xi_{k+g}\Big) \tag{7.2}$$

where s_α is the elementary symmetric polynomial of degree α. To find explicit quantum relations reducing to these mod q, the following result is useful. We adopt the notation $\prod^*$ for a quantum product, and η^{*u} for a quantum power.

(7.3) Proposition. *For $u \geq 0$, $\eta^u =$*

(i) $\quad \eta^{*u} - q\sum_{j=0}^{n}\eta^{*(n-j)}\theta^j/j!$ $\qquad$ *for $d > g$, where $n = u-d+g-1$;*

(ii) $\quad (\eta+q)^{*u} - q\sum_{j=0}^{u-1}(\eta+q)^{*(u-1-j)}\theta^j/j!$ $\qquad$ *for $d = g$;*

(iii) $\quad (\eta+r)^{*u} - r\sum_{j=0}^{u-1}(\eta+r)^{*(u-1-j)}\theta^j/(j+1)!$ $\qquad$ *for $d = g-1$, where $r = q\,\theta/(1-q)$;*

(iv) $\quad \eta^{*u} - q\sum_{j=0}^{u-1}\eta^{*j}\theta^{n-j}/(n-j)!$

$\quad + \; q\,u\,\eta^{*(u-1)}\theta^{g-d}/(g-d)! + O(q^2)$ $\qquad$ *for $g/2 < d \leq g-1$, where $n = u-d+g-1$.*

Proof. For $d > g$, by **(6.2)**(i)

$$\eta * \eta^v = \eta^{v+1} + q\,\frac{\theta^{g-d+v}}{(g-d+v)!}.$$

Part (i) follows inductively by taking the quantum product of both sides with η. Similarly, for $d = g$

$$(\eta + q) * \eta^v = \eta^{v+1} + q\,\theta^v/v!,$$

and for $d = g - 1$

$$(\eta + r) * \eta^v = \eta^{v+1} + r\,\theta^v/(v+1)!,$$

so parts (ii) and (iii) also follow inductively. Part (iv) follows from **(4.3)** by an easy induction. □

For $d \geq g - 1$ or $d < \frac{4}{5}g - \frac{3}{5}$, it is now easy to obtain a complete set of quantum relations which are explicit, if somewhat inelegant. Just plug the formulas of **(7.3)** into **(7.2)**. When $\frac{2}{3}g + \frac{1}{3} \leq d < \frac{4}{5}g - \frac{3}{5}$, terms of higher order in q appear in the quantum product. However, these do not affect the relations, since by the last statement in **(2.2)**, the latter are homogeneous of degree d or $d+1$, too large to contain a power of q^2. The same is true for the linear terms in q when $\frac{1}{2}g + 1 \leq d < \frac{2}{3}g - \frac{1}{3}$, leading to the amusing phenomenon that in that range, the quantum and classical rings are isomorphic, but only by a map which is not the identity.

Although the general quantum relation is no thing of beauty, there are a few exceptions.

(7.4) Corollary.

(i) *For* $d > 2g - 2$,

$$\eta^{*(d-2g+1)} * \prod_{i=1}^{g}{}^{*}(\eta - \sigma_i) = q;$$

(ii) *For* $g < d \leq 2g - 2$,

$$\prod_{i=1}^{g}{}^{*}(\eta - \sigma_i) = q\,\eta^{*(2g-1-d)};$$

(iii) *For* $d = g$,

$$\prod_{i=1}^{g}{}^{*}(\eta - \sigma_i + q) = q\,(\eta + q)^{*(g-1)}.$$

Proof. The first relation follows directly from **(2.4)** and **(7.3)**(i), since the coefficient of q in the latter is -1 for $u = d - g + 1$ and 0 for $u < d - g + 1$. To prove (ii) and (iii), note first that by **(2.2)**, the relation

$$0 = \prod_{i=1}^{g}(\eta - \sigma_i)$$

holds in the classical ring for $d \leq 2g-2$. This is equivalent to

$$0 = \sum_{j=0}^{g} (-1)^j \eta^{g-j} \theta^j / j!.$$

One plugs in the formulas of **(7.3)**(i) and (ii) for η^u and notes that the double sum in the second term telescopes using the binomial theorem. □

8 Relation with Givental's work

In fact, relations (ii) and (iii) in the corollary above were known to the authors before any of the Gromov-Witten invariants, and were the starting point of the investigation. With a little ingenuity, they can be read off from formulas in the wonderful paper of Givental [12], which is concerned chiefly with proving that the Gromov-Witten invariants of the quintic threefold satisfy the Picard-Fuchs equation. We will sketch an outline of the connection.

At the heart of Givental's paper is a "quantum Lefschetz hyperplane theorem" explaining how the Gromov-Witten invariants of a variety are related to those of a hyperplane section. This allows him to compute a generating function for the Gromov-Witten invariants of complete intersections. He shows in Corollary 6.4 that any differential operator annihilating this generating function determines a relation in the quantum cohomology.

The symmetric products of a curve do not appear in any natural way as complete intersections in a projective space. But they do appear as complete intersections in a projective *bundle*. Indeed, for any $d \leq 2g-2$, choose a reduced divisor $P = \sum p_i$ of degree $2g-1-d$. Then $D \mapsto D+P$ gives an embedding $C_d \hookrightarrow C_{2g-1}$, whose image is a complete intersection of divisors in the linear system of $\mathcal{O}(1)$. In particular, for $d \leq 2g-2$, C_d embeds in C_{2g-1}, which is a $\mathbb{P}^{g-1}$-bundle over Jac_{2g-1}.

Givental's methods are perfectly adapted to studying this more general case. Indeed, his formulas for Gromov-Witten invariants are derived as special cases of formulas for equivariant Gromov-Witten invariants. These can be regarded as universal formulas for Gromov-Witten invariants of complete intersections in projective bundles, because they are defined as relative Gromov-Witten invariants for complete intersections in the universal projective bundle over the classifying space.

Givental works with the group $(\mathbb{C}^\times)^{n+1}$ acting on $\mathbb{P}^n$. The classifying space is then $(\mathbb{CP}^\infty)^{n+1}$, and the universal bundle is a direct sum of line bundles. This would appear to be a problem, because C_{2g-1} is not the projectivization of a direct sum of line bundles. However, a splitting principle argument shows that all of Givental's equivariant formulas extend word for word to the action of $\mathrm{GL}(n,\mathbb{C})$. The classifying space is then an infinite Grassmannian; and any projective bundle at all is pulled back from some map to a Grassmannian, even in the algebraic category.

Thus a formula in equivariant cohomology determines a formula in the cohomology of C_{2g-1}, or more properly, in the cohomology of the spaces of stable maps to C_{2g-1}. This is perhaps surprising, since no group is acting on C_{2g-1}. However, the group is present as the maximal torus of the structure group of the projective bundle $C_{2g-1} \to \mathrm{Jac}_{2g-1}$; one regards equivariant cohomology as giving universal formulas in the cohomology of all such bundles.

Another apparent problem is that the equivariant methods treat only Gromov-Witten invariants for classes $e \in H_2$ killed by the projection to the base, namely Jac_{2g-1}. But luckily, as we have seen, these are the only nonzero invariants of C_d.

The symmetric product C_d is an equivariant complete intersection in C_{2g-1} of type $(1,1,\ldots,1)$. Such an intersection in a single projective space would just be a linear subspace, but it can be less trivial equivariantly, that is to say, in families. Givental's results therefore apply with $r = 2g-1-d$, $l_1 = \cdots = l_r = 1$.

The case $d > g$ is covered by Givental's §9. The equivariant quantum potential satisfies the differential equation shown below his Theorem 9.5. As instructed in Corollary 6.4, we make the substitutions $\hbar\, d/dt = \eta$ (the relative hyperplane class), $e^t = q$, $\hbar = 0$ to obtain a quantum relation. We also substitute $\lambda_i = \sigma_i$ since by **(2.4)** these are the Chern roots of the bundle whose projectivization is C_{2g-1}, and $\lambda'_i = 0$ since these come from an additional equivariance we are not using. The result is precisely **(7.4)**(ii)!

Similarly, the case $d = g$ is covered by §10. The differential equation satisfied by the equivariant quantum potential S' is not explicitly stated, but it is

$$\prod_i (D - \lambda_i)\, S' = l_1 \cdots l_r e^t \prod_{j,m} (l_j D - \lambda'_j + m\hbar)\, S',$$

where $D = \hbar\, d/dt + l_1 \cdots l_r e^t$ as in Corollary 10.8. Making the same substitutions as before transforms D to $\eta + q$ and this equation to **(7.4)**(iii).

Finally, the case $d = g-1$ is analogous to the Calabi-Yau case, which is covered by §11. Again the differential operator determines a relation in degree $2g$, but now this is vacuous, since the real dimension of C_{g-1} is only $2g-2$. This is a familiar occurrence in Givental's work: the Picard-Fuchs equation, for example, has degree 4, so it gives a quantum relation in H^8 of the quintic threefold, which is of course trivial.

Nevertheless, the Picard-Fuchs equation on the quintic does contain valuable information: enough to determine the 3-point invariants at genus 0, and hence the virtual number of rational curves of all degrees. On C_{g-1} the corresponding invariants were all worked out in **(6.1)**, and are all determined by the lines using the Aspinwall-Morrison formula. It is like a Calabi-Yau with no higher degree rational curves. But it certainly ought to be possible to recover these invariants for C_{g-1}, or even C_d for $d \geq g-1$, by calculating Givental's quantum potential in this case.

A more daunting project would be to extend these methods to C_d for $d < g-1$, where the results of this paper are incomplete. Givental remarks that the corresponding case for complete intersections remains unsolved, but is in a sense "less interesting," since nonzero invariants appear only in finitely many degrees. For symmetric products, however, this is clearly the most interesting and mysterious case.

9 The homotopy groups of symmetric products of curves

In this appendix, we first compute the fundamental group of a symmetric product, then show that the Hurewicz homomorphism $\pi_2 \to H_2$ has rank 1 for $d > 1$. These results are not needed anywhere else, but they clarify the non-contribution of other homology classes, from the point of view of symplectic topology.

For $d > 2g-2$, the symmetric product is a projective bundle over the Jacobian, so it has fundamental group $H_1(C;\mathbb{Z})$. On the other hand, for $d = 1$ it is of course $\pi_1(C)$. What happens in between? The simplest possible thing, it turns out. We are grateful to Michael Roth for supplying the following theorem and its proof; but see also Grothendieck [15].

(9.1) Theorem. *For $d > 1$, $\pi_1(C_d) = H_1(C;\mathbb{Z})$. Indeed, the Abel-Jacobi map induces an isomorphism on fundamental groups.*

Proof. Choose a basepoint $p \in C$ and let $i : C \to C^d$ be given by $x \mapsto (x, p, \ldots, p)$. Also let $\psi : C^d \to C_d$ be the quotient by the symmetric group. We will show first, that $\pi_1(\psi i)$ surjects on $\pi_1(C_d)$, second, that its image is abelian, and third, that the kernel is exactly the commutator subgroup.

Choose as a basepoint in C_d the divisor $d{\cdot}p$. Since the big diagonal has real codimension 2, any loop can be perturbed so that it meets the big diagonal only at the basepoint. It then lifts unambiguously to a loop on C^d. Hence $\pi_1(\psi)$ is surjective. But any loop on C^d is homotopic to a composition of loops on the various factors. Since ψ is symmetric, each of these loops can be replaced with the corresponding loop on the first factor, that is, on the image of i, without changing $\pi_1(\psi)$ of the composite. Hence $\pi_1(\psi i)$ is surjective.

Let γ_2, γ_2 be two loops in $\pi_1(C)$. To show that their images in C_d commute, first include them in C^d via i. Then note that γ_2 can be transferred to the second factor without changing its image in C_d; then, however, it commutes with γ_1 in $\pi_1(C^d) = \pi_1(C)^d$.

Finally, to show that ψi kills only loops in the commutator, compose ψi with the Abel-Jacobi map. The resulting map $C \to \mathrm{Jac}_d$ is itself just the composition of the Abel-Jacobi map with the identification $\mathrm{Jac}_1 \cong \mathrm{Jac}_d$ induced by p. But it is well-known that the Abel-Jacobi map induces an isomorphism $H_1(C;\mathbb{Z}) = H_1(\mathrm{Jac}_1;\mathbb{Z})$; see for example Griffiths-Harris [14, 2.7]. □

(9.2) Theorem. *For $d > 1$, the Hurewicz homomorphism $\pi_2(C_d) \to H_2(C;\mathbb{Z})$ has rank 1.*

Proof. It is easy to see that the rank is at least 1: for example, take C to be hyperelliptic; there is then a one-dimensional linear system ℓ on C_d, which is a 2-sphere, non-trivial in rational homology because $\eta \cdot \ell = 1$.

To show it is at most 1, note that by the above theorem the universal cover $\tilde{C}_d$ is the fibered product $C_d \times_{\mathrm{Jac}_d} H^1(C,\mathcal{O})$, where the vector space $H^1(C,\mathcal{O})$ is the universal cover of Jac_d. Now the push-forward $H_2(C_d;\mathbb{Q}) \to H_2(\mathrm{Jac}_d;\mathbb{Q})$ has 1-dimensional kernel: this follows for example from **(2.1)**(iii) and **(2.2)**. Since the map $\tilde{C}_d \to C_d \to \mathrm{Jac}_d$ also factors through the contractible space $H^1(C,\mathcal{O})$, the induced map on H_2 is zero and so the map

$H_2(\tilde{C}_d) \to H_2(C_d)$ has rank at most 1. But by the Hurewicz isomorphism and the homotopy exact sequence, $H_2(\tilde{C}_d) = \pi_2(\tilde{C}_d) = \pi_2(C_d)$, and the map to $H_2(C_d)$ is exactly the Hurewicz homomorphism. □

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