Some Applications of Localization to Enumerative Problems

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Dedicated to Bill Fulton on the occasion of his 60th birthday

1. Introduction. A problem in enumerative geometry frequently boils down to the computation of an integral on a moduli space. We have intersection theory (with Fulton's wonderful *Intersection Theory* [7] as a prime reference) to thank for allowing us to make rigorous sense of such integrals, but for their computations we often need to look elsewhere. If a torus lurks in the background, acting on the moduli space, then the Atiyah-Bott localization theorem allows one to express equivariant cohomology classes on the moduli space in terms of their "residues" living on the connected components of the locus of fixed points (i.e. the fixed submanifolds). This can be very useful for computations, particularly when the fixed submanifolds are points.

We will use localization in a different way. Here, the moduli space itself will be a fixed submanifold for a torus action on a larger ambient space. Localization is applied in this context to relate *residues* on the moduli space to residues on simpler spaces by means of suitable equivariant maps. This point of view can lead immediately to remarkably simple derivations of some complicated-looking formulas, for example when applied to:

(a) Schubert calculus on the flag manifold, and

(b) Gromov-Witten invariants of rational curves.

In (a) the partial flag manifold $\mathrm{Fl}(1, 2, ..., m, n)$ is realized as a fixed submanifold of a blown-up projective space $\mathbf{P}(\mathrm{Hom}(W, V))$ where W and V are vector spaces of ranks m and n respectively, and all torus actions come from the "standard" torus action of $(\mathbf{C}^*)^m$ on W^*. The full locus of fixed points is a disjoint union of $m!$ fixed submanifolds in this setting, each isomorphic to the partial flag manifold, but with different (equivariant) Euler classes. For this warm-up application, we will simply list the results of Jian Kong [11], where residues on the flag manifold are computed, resulting in particular in some new methods for computing Schubert calculus on the Grassmannian $G(m, n)$. It would be quite interesting to compare this with other methods (e.g. Gröbner bases) for making such computations.

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In (b), the Kontsevich-Manin moduli space of stable maps $\overline{M}_{0,m}(X, \beta)$ of rational curves with image homology class β is realized as a fixed submanifold of the "graph space" $\overline{G}_{0,m}(X, \beta) := \overline{M}_{0,0}(X \times (\mathbf{P}^1)^m, (\beta, 1^m))$ again with a standard torus action. The main applications take place in this setting.

When $m = 1$ we investigate the J-functions introduced by Givental in his generalization of the enumerative side of mirror symmetry to arbitrary projective manifolds (see [10]). The J-function is a polynomial associated to a complex projective variety X and ample system of nef divisors which encodes all the one-point Gromov-Witten invariants. The coefficients of the J-function are push-forwards of residues, and our point of view on residues leads to a simple proof of the multiplicativity of the J-functions. Our point of view also leads to a non-obvious property of the J-function under push-forward. The J-function of projective space is computed in this context as an immediate consequence of the existence of a nice "linear" approximation to the graph space. Following Givental's proof of the enumerative mirror conjecture for complete intersections in toric varieties, Kim was led to the formulation of a "quantum Lefschetz principle" relating the J-function for X with J-functions for very ample divisor classes in X [10]. This has recently been proved by Y.P. Lee [13] in the general case building on the proof in [3] of the case $X = \mathbf{P}^n$, which we briefly discuss here.

When $m > 1$ there are many other fixed submanifolds in the graph space besides $\overline{M}_{0,m}(X, \beta)$, but they all are built out of Kontsevich-Manin spaces involving smaller m's and/or smaller β's. This has been exploited in joint work with Holger Kley [4] to produce recursive formulas for m-point Gromov-Witten invariants, and in particular to prove that when the cohomology is generated by divisor classes, the m-point Gromov-Witten invariants can be "reconstructed" from one-point Gromov-Witten invariants. We will give the formula and an outline of the proof of reconstruction in the two-point case as a final application of localization. Another proof of reconstruction has been achieved with very different techniques and different formulas by Lee and Pandharipande [14]. As a direct consequence of reconstruction, the small quantum cohomology of Fano complete intersections in $\mathbf{P}^n$, or indeed any toric variety, can be explicitly computed, since the one-point invariants are computed from the quantum Lefschetz principle. As another consequence, the quantum cohomology of products are determined by reconstruction, since the J-functions multiply.

2. Localization. When a torus $T = (\mathbf{C}^*)^m$ acts on a compact complex manifold M, the fixed submanifolds $F \subset M$ are closed and embedded (of varying dimensions). There is an equivariant cohomology space $\mathrm{H}^*_T(M, \mathbf{Q})$ which is naturally a module over the cohomology of the classifying space $\mathrm{H}^*(BT, \mathbf{Q}) \cong \mathbf{Q}[t_1, ..., t_m]$. If E is a linearized vector bundle over M, then there are equivariant Chern classes $c^T_d(E)$ taking values in $\mathrm{H}^{2d}_T(M, \mathbf{Q})$, and in particular, the normal bundles $N_{F/M}$ to the fixed loci are canonically linearized (for the trivial action of T on F) and yield equivariant Euler classes:

$$\epsilon_T(F/M) \in \mathrm{H}^*(F, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}[t_1, ..., t_m] \cong \mathrm{H}^*_T(F, \mathbf{Q})$$

which are the top equivariant Chern classes of the normal bundles.

The Atiyah-Bott localization theorem [1] states that these Euler classes are invertible in $\mathrm{H}^*(F, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}(t_1, ..., t_m)$ and one can recover an equivariant Chern class $\alpha \in \mathrm{H}^*_T(M, \mathbf{Q})$ uniquely (modulo torsion) as a sum of residues:

$$\sum_F i_* \frac{i^*\alpha}{\epsilon_T(F/M)}$$

where i^* and i_* are the equivariant pull-back and push-forward associated to the equivariant inclusion $i : F \hookrightarrow M$. It follows from the uniqueness that taking residues is functorial. That is, if $\Phi : M \to M'$ is an equivariant map and $j : F' \hookrightarrow M'$ is the inclusion of a component of the fixed submanifold, then:

$$(\dagger) \qquad \sum_{F \subset \Phi^{-1}(F')} \Phi|_{F*} \frac{i^*\alpha}{\epsilon_T(F/M)} = \frac{j^* f_*\alpha}{\epsilon_T(F'/M')}$$

where the sum is over the components X of the fixed locus that are contained in $\Phi^{-1}(F')$ and α is any equivariant cohomology class on M (see [3] or [15]).

Thus if we are asked to integrate a cohomology class γ on a compact complex manifold F, and if F happens to be isomorphic to a component of the fixed locus of an action of T on M as above, then the formula above expresses residues at F in terms of residues at F' and at the other fixed loci contained in $\Phi^{-1}(F')$. If γ can be expressed in terms of residues of equivariant cohomology classes, then this formula yields a relation among integrals of cohomology classes related to γ. This will be our point of view throughout the rest of this paper.

3. Flag Manifolds and Grassmannians. The partial flag manifold:

$$\mathrm{Fl}(1,2,...,m,V) = \{V_1 \subset V_2 \subset ... \subset V_m \subset V \cong \mathbf{C}^n \mid V_r \cong \mathbf{C}^r\}$$

is a component of the fixed-point locus of an action of T on M.

In this case, M is the blow-up of $\mathbf{P}(\mathrm{Hom}(W,V))$ along:

$$Z_1 \cong \mathbf{P}(W^*) \times \mathbf{P}(V) \subset Z_2 \subset ... \subset Z_{m-1} \subset \mathbf{P}(\mathrm{Hom}(W,V))$$

where $W \cong \mathbf{C}^m$ and Z_r is the locus of maps of rank $\leq r$. That is, M is obtained by blowing up along Z_1, followed by the proper transform of Z_2, followed by the proper transform of Z_3, etc. If we choose a basis $e_1,...,e_m$ of W and let T act on the dual space $W^*$with weights $(t_1,...,t_m)$, then this induces an action of T on M, and the following are checked in [11]:

- The intersection of the $m-1$ exceptional divisors on M is:

$$E_1 \cap ... \cap E_{m-1} \cong \mathrm{Fl}(1,2,...,W^*) \times \mathrm{Fl}(1,2,...,m,V)$$

- The fixed-point loci for the action of T on M are all contained in this intersection and correspond via the isomorphism above to:

$$\Lambda_I \times \mathrm{Fl}(1,2,...,m,V)$$

where Λ_I are the (isolated) fixed points of the action of T on $\mathrm{Fl}(1,2,...,W^*)$, indexed by the permutations of m letters so that the permutation $(i_1,...,i_m)$ corresponds to the flag:

$$\Lambda_{(i_1,...,i_m)} = \{\langle x_{i_1}\rangle \subset \langle x_{i_1}, x_{i_2}\rangle \subset \cdots\}$$

- Let ζ_i be the relative hyperplane class for the projection:

$$\mathrm{Fl}(1,2,...,i+1,V) \to \mathrm{Fl}(1,2,...,i,V)$$

pulled back to $\mathrm{Fl}(1,...,m,V)$. Then the equivariant Euler class to the fixed locus $F_I = \Lambda_I \times \mathrm{Fl}(1,...,m,V)$ is:

$$\epsilon_T(F_I/M) = \prod_{1\leq j<k\leq m}(t_{i_k} - t_{i_j}) \prod_{s=1}^{m-1}(t_{i_{s+1}} - t_{i_s} - \zeta_s)$$

We are therefore in a position to apply the formula (†) to the diagram:

$$\begin{array}{ccc} M & \stackrel{\Phi}{\to} & M' = \mathbf{P}(\mathrm{Hom}(W,V)) \\ \uparrow & & \\ F_I & & \end{array}$$

On the right side, each fixed locus belongs to $Z_1 \subset \mathbf{P}(\mathrm{Hom}(W,V))$ as $F_i' = p_i \times \mathbf{P}(V)$ where $p_i \in \mathbf{P}(W^*)$ is the fixed point $\langle x_i \rangle$. In that case, one computes:

$$\epsilon_T(F_i'/M') = \prod_{s \neq i} (h + t_s - t_i)^n$$

where h is the hyperplane class on $\mathbf{P}(V)$.

An F_I belongs to $\Phi^{-1}(F_i')$ exactly when I is of the form $(i, i_2, ..., i_m)$. In that case, the induced map:

$$\Phi|_{F_I} : \Lambda_I \times \mathrm{Fl}(1, ..., m, V) \to p_i \times \mathbf{P}(V)$$

is the natural projection, which we will denote by π. Thus (†) with $\alpha = 1$ gives us the following interesting formula for Schubert calculus:

Schubert Formula 1:

$$\sum_{\{I | i_1 = i\}} \pi_* \left(\frac{1}{\prod_{1 \leq j < k \leq m} (t_{i_k} - t_{i_j}) \prod_{s=1}^{m-1} (t_{i_{s+1}} - t_{i_s} - \zeta_s)} \right) = \frac{1}{\prod_{s \neq i} (h + t_s - t_i)^n}$$

This formula encodes all the information about intersection numbers on the flag manifold of the form:

$$\int_{\mathrm{Fl}(1,2,...,m,V)} h^a \cup \zeta_1^{a_1} \cup ... \cup \zeta_{m-1}^{a_{m-1}}$$

Of course, the same intersection numbers could be obtained by applying the Grothendieck relation to each of powers of the ζ_i. But there is a second formula which is much more interesting, involving cohomology classes pulled back from the Grassmannian under:

$$\rho : \mathrm{Fl}(1, 2, ..., m, V) \to G(m, V)$$

Recall that such a cohomology class is a symmetric polynomial:

$$\tau(q_1, ..., q_m)$$

in the Chern roots $-q_i$ of the universal subbundle $S \subset V \otimes \mathcal{O}_{G(m,n)}$.

The main theorem of Kong's thesis [11] is the following:

Schubert Formula 2:

$$\sum_{\{I|i_1=i\}} \pi_* \left(\frac{\rho^*\tau(q_1, ..., q_m)}{\prod_{1\le j<k\le m}(t_{i_k} - t_{i_j}) \prod_{s=1}^{m-1}(t_{i_{s+1}} - t_{i_s} - \zeta_s)} \right)$$

$$= \frac{\tau(h + t_1 - t_i, ..., h + t_m - t_i)}{\prod_{s\neq i}(h + t_s - t_i)^n} + \text{ irrelevant terms}$$

where the irrelevant terms are monomials in the t_i which do not appear on the left side of the equation.

Example: When $m = 2$, set $i = 1$ above, $t_1 = 0$ and $t_2 = t$. Then:

$$\pi_* \frac{\rho^*\tau(q_1, q_2)}{t(t - \zeta_1)} = \frac{\tau(h, h + t)}{(h + t)^n} + \text{irrelevant terms}$$

If we consider the coefficients of t^{-2} on both sides and integrate, we get the following new way of doing Schubert calculus on $\mathrm{G}(2, V)$:

$$\int_{G(2,V)} \tau(q_1, q_2) = \int_{\mathrm{Fl}(1,2,V)} \pi^*h \cup \rho^*\tau = \text{coeff of } h^n t^{-2} \text{ in} \frac{h \cdot \tau(h, h + t)}{(h + t)^n}$$

Kong proves this formula by finding a suitable equivariant class α on M which restricts to the given τ on each of the fixed components F_I. This τ is well enough approximated by the pull-back of the corresponding equivariant class of a split bundle on M' to give the formula.

The example above for $m = 2$ can be similarly worked out for $m > 2$ with the main difference being that there are $(m-1)!$ terms on the left which sum together to the attractive formula on the right. It can be shown that this suffices to compute Schubert calculus, and it seems that an analysis of the complexity of this computation ought to be done.

Finally, there is no obstruction to carrying out this program when V is replaced by a vector bundle over a base variety X. Kong also shows how the Chern classes of V figure into this "relative" setting in [11].

4. Gromov-Witten Invariants of Rational Curves. We will describe the relevant Konstsevich-Manin spaces (and maps among them) only set-theoretically for simplicity. The interested reader may go to the literature (e.g. [8]) for rigorous constructions of the spaces and morphisms.

A map $f : C \to X$ from an m-pointed rational curve is **stable** if:

• C has only nodes as singularities, and the marked points are smooth.

• Every component of C collapsed by f has at least 3 distinguished points, i.e. marked points and/or nodes.

$$\overline{M}_{0,m}(X, \beta)$$

is the Kontsevich-Manin moduli space of isomorphism classes of stable maps with m marked points and image homology class β. If X is "convex" (e.g. a homogeneous space) then this moduli space is smooth as an orbifold, of the expected dimension. Otherwise, there is a "virtual class" on X with "all the expected properties" (see [2]). There is always an injective morphism:

$$\overline{M}_{0,m}(X, \beta) \hookrightarrow \overline{G}_{0,m}(X, \beta) = \overline{M}_{0,0}(X \times (\mathbf{P}^1)^m, (\beta, 1^m))$$

where the latter space is the "graph space" associated to the former.

Given a stable map $f : C \to X$ and points $p_1, ..., p_m \in C$, we obtain the image of $[f]$ in the graph space by attaching a copy of $\mathbf{P}^1$ to each of the points, gluing $p_i \in C$ to $0 \in \mathbf{P}^1$, and collapsing each $\mathbf{P}^1$ to construct the resulting stable map $g : C \cup \coprod \mathbf{P}^1 \to X$.

It is convenient to number the $\mathbf{P}^1$'s, so $\mathbf{P}^1_i = \mathbf{P}(W_i)$ is the particular $\mathbf{P}^1$ which we attach to p_i. The actions of $\mathbf{C}^*$ on the dual spaces W_i^* with weights $(0, t_i)$ give a natural action of the torus T on the product of the $\mathbf{P}^1$'s and hence on the graph space above. Moreover, the m-pointed Kontsevich-Manin space is one of the components of the fixed-locus for the torus action.

One computes, using for example [8]:

$$\epsilon_T(\overline{M}_{0,m}(X, \beta)/\overline{G}_{0,m}(X, \beta)) = \prod_{i=1}^{m} t_i(t_i - \psi_i)$$

where the ψ_i are the "gravitational descendants" $\psi_i = c_1(\sigma_i^*(\omega))$. Here ω is the relative dualizing sheaf of the universal curve $\mathcal{C}$ over $\overline{M}_{0,m}(X, \beta)$ and σ_i is the section of $\mathcal{C}$ corresponding to the ith marked point.

The Case $m = 1$: Here we let $t = t_1$ and $\psi = \psi_1$.

If H is an ample divisor on X, then following Givental, we define:

$$J_{X,H}(q) = 1 + \sum_{\beta \neq 0} e_{\beta *}\left(\frac{1}{t(t-\psi)}\right) q^{\deg_{\mathrm{H}}(\beta)}$$

where $e_\beta : \overline{M}_{0,1}(X, \beta) \to X$ is the evaluation map $e_\beta([f]) = f(p)$. Since only a finite number of classes β have a given degree against H, this sum makes sense. More generally, we will suppose H is a system $H = (H_1, ..., H_r)$ of (linearly independent) nef divisors, and that some linear combination of the H_i is ample. In that case, we define:

$$J_{X,H}(q) = 1 + \sum_{\beta \neq 0} e_{\beta *}\left(\frac{1}{t(t-\psi)}\right) q_1^{\deg_{H_1}(\beta)} ... q_r^{\deg_{H_r}(\beta)}$$

The following "functorial" properties of the J-function are easily proved once we recognize that the coefficients are push-forwards of residues.

Product Formula: Suppose X and X' are simply connected projective manifolds (so the curve classes on $X \times X'$ are all of the form (β, β')) and H and H' are ample systems of divisors, as above. Then:

$$J_{X \times X', (\pi_1^* H, \pi_2^* H')}(q, q') = \pi_1^* J_{X,H}(q) \cdot \pi_2^* J_{X',H'}(q')$$

Proof: Kontsevich-Manin spaces are functorial, in the sense that a map $f : X \to Y$ gives rise to maps:

$$f_{0,m} : \overline{M}_{0,m}(X, \beta) \to \overline{M}_{0,m}(Y, f_*\beta)$$

and analogous compatible equivariant maps on the graph spaces. Thus the projection maps give rise to a diagram of "lifts" of the identity map:

$$\begin{array}{ccc}
\overline{G}_{0,1}(X \times X', (\beta, \beta')) & \stackrel{\Phi}{\to} & \overline{G}_{0,1}(X, \beta) \times \overline{G}_{0,1}(X', \beta') \\
\uparrow & & \uparrow \\
\overline{M}_{0,1}(X \times X', (\beta, \beta')) & \stackrel{\phi}{\to} & \overline{M}_{0,1}(X, \beta) \times \overline{M}_{0,1}(X', \beta') \\
\downarrow & & \downarrow \\
X \times X' & \to & X \times X'
\end{array}$$

Φ is birational when X and X' are convex (and "virtally birational" always) even though ϕ is not birational (the two sides have different dimensions!). Thus $\Phi_*1 = 1$ and we may apply (†) to the class 1 to obtain:

$$\phi_*\left(\frac{1}{t(t-\psi)}\right) = \pi_1^*\left(\frac{1}{t(t-\psi)}\right)\pi_2^*\left(\frac{1}{t(t-\psi)}\right)$$

Further pushing forward to $X \times X'$ yields the desired product formula.

Push-Forward Formula: Suppose $f : X \to Y$ is given. Then there are equivariant classes $f_{\beta_*}1 \in \mathrm{H}^*(\overline{M}_{0,1}(Y, f_*\beta), \mathbf{Q})[t]$ so that:

$$f_*J_{X,H}(q) = f_*1 + \sum_{\beta\neq 0}(e_{f_*\beta})_*\left(\frac{f_{\beta_*}1}{t(t-\psi)}\right)q_1^{\deg_{H_1}(\beta)}...q_r^{\deg_{H_r}(\beta)}$$

Proof: Here we consider the diagram of lifts of f:

$$\begin{array}{ccc}
\overline{G}_{0,1}(X,\beta) & \xrightarrow{\Phi} & \overline{G}_{0,1}(Y, f_*\beta) \\
\uparrow & & j\uparrow \\
\overline{M}_{0,1}(X,\beta) & \xrightarrow{\phi} & M_{0,1}(Y, f_*\beta) \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}$$

and note that applying (†) to the class 1 again, we get:

$$f_*e_{\beta_*}\left(\frac{1}{t(t-\psi)}\right) = (e_{f_*\beta})_*\phi_*\left(\frac{1}{t(t-\psi)}\right) = (e_{f_*\beta})_*\left(\frac{j^*\Phi_*1}{t(t-\psi)}\right)$$

hence the push-forward formula with $f_{\beta_*}1 := j^*\Phi_*1$.

Remark: If f is an embedding, then $\phi^*\psi = \psi$, in which case the projection formula tells us that $f_{\beta_*}1 = \phi_*1$ is constant in t. It seems that in general, however, $f_{\beta_*}1$ is not constant in t. It would be very interesting to compute it, for instance, in case f is the inverse of a blow-up along a submanifold.

The J-function of Projective Space: Let H be the hyperplane class on $\mathbf{P}^n$. Then:

$$J_{\mathbf{P}^n,H}(q) = \sum_{d=0}^{\infty} e_{d*}\left(\frac{1}{\prod_{k=1}^{d}(H+kt)^{n+1}}\right)q^d$$

Proof: $\overline{G}_{0,1}(\mathbf{P}(V), d)$ has a natural birational map to a "linear" space $\mathbf{P}(\mathrm{Hom}(\mathrm{Sym}^d(W), V)$, where $W = W_1$. A general element of the graph space is represented by a degree d morphism $f : \mathbf{P}^1 \to \mathbf{P}^n$ which maps to an $n+1$-tuple of degree d polynomials $(p_0(x,y) : ... : p_n(x,y))$ with no common factors. When the curve underlying the stable map picks up extra components, then the $n+1$-tuple of polynomials picks up common factors. In particular, the image of $\overline{M}_{0,1}(\mathbf{P}^n, d)$ under this weighted blow-down is a copy of $\mathbf{P}^n$, embedded via:

$$\{x^d\} \times \mathbf{P}(V) \hookrightarrow \mathbf{P}(\mathrm{Sym}^d(W^*)) \times \mathbf{P}(V) \hookrightarrow \mathbf{P}(\mathrm{Hom}(\mathrm{Sym}^d(W), V)$$

Thus, we have the diagram:

$$\begin{array}{ccc} \overline{G}_{0,1}(\mathbf{P}^n, d) & \stackrel{\Phi}{\to} & \mathbf{P}(\mathrm{Hom}(\mathrm{Sym}^d(W), V) \\ i \uparrow & & j \uparrow \\ \overline{M}_{0,1}(\mathbf{P}^n, d) & \stackrel{e_d}{\to} & \mathbf{P}^n \end{array}$$

One computes (see [5]) $\epsilon_T(\mathbf{P}^n/\mathbf{P}(\mathrm{Hom}(\mathrm{Sym}^d(W), V))) = \prod_{k=1}^d (H+kt)^{n+1}$ so that (†) now applies with the class 1, giving us:

$$e_{d*}\left(\frac{1}{t(t-\psi)}\right) = \frac{1}{\prod_{k=1}^d (H+kt)^{n+1}}$$

proving the formula.

Quantum Lefschetz Hyperplane: We will limit ourselves to considering hypersurfaces in $\mathbf{P}^n$, as in [3]. See [13] for the general version. Let $f : X \hookrightarrow \mathbf{P}^n$ be a hypersurface of degree l, and let H denote the hyperplane class, either on $\mathbf{P}^n$ or on X. Let:

$$I_{X/\mathbf{P}^n,H}(q) = \sum_{d=0}^{\infty} \frac{\prod_{k=0}^{dl}(lH+kt)}{\prod_{k=1}^{d}(H+kt)} q^d$$

(a) If $l < n$, then $f_* J_{X,H}(q) = I_{X/\mathbf{P}^n,H}(q)$.

(b) If $l = n$, then $f_* J_{X,H}(q) = e^{\frac{-l}{t}q} I_{X/\mathbf{P}^n,H}(q)$.

(c) If $l = n+1$, then there are $a(q), b(q) \in q\mathbf{Q}[[q]]$ so that:

$$f_* J_{X,H}(q) = e^{\frac{H}{t}a(q)+b(q)} I_{X/\mathbf{P}^n,H}(qa(q))$$

To prove this, one uses the diagram for $\mathbf{P}^n$ and observes that

$$f_*J_{X,H}(q) = lH + \sum_{d>0} e_{d*}\left(\frac{j^*\Phi_*[X]_T}{t(t-\psi)}\right) q^d$$

where $[X]_T$ is the equivariant Chern class:

$$[X]_T = c^T_{dl+1}(\pi_* e_d^* \mathcal{O}_{\mathbf{P}^n}(l)) \in \mathrm{H}^*_T(\overline{G}_{0,1}(\mathbf{P}^n, d), \mathbf{Q})$$

Thus the proof of quantum Lefschetz amounts to a detailed analysis of the class $j^*\Phi_*[X]_T \in H^*(\mathbf{P}^n, \mathbf{Q})[t]$. This is obtained by decomposing $[X]_T$ along boundary strata of the graph space by means of intersection theory. In particular, the open stratum of the graph space contributes $\prod_{k=0}^{dl}(lH + kt)$ via an approximation of $\pi_* e_d^* \mathcal{O}_{\mathbf{P}^n}(l)$ by $\mathrm{Sym}^{dl}(W^*) \otimes \Phi^*\mathcal{O}(l)$ in much the same way that the graph space is approximated by $\mathbf{P}(\mathrm{Hom}(\mathrm{Sym}^d(W), V))$. In the case $l < n$, this is the only stratum which contributes to $j^*\Phi_*[X]_T$, giving us (a). In the other cases, the boundary strata do contribute, but in a self-similar manner. When tallied up, these contributions give formulas (b) and (c) in the cases $l = n$ and $l = n+1$ respectively. It is unknown whether a more general "change of coordinates" analogous to (b) and (c) occurs in the general type cases $l > n+1$.

The case $m > 1$. Reconstruction: In [4], reconstruction theorems make use of the following diagrams of K-M spaces and graph spaces:

$$\begin{array}{ccc}
\overline{G}_{0,m}(X,\beta) & \stackrel{\Phi}{\to} & \overline{G}_{0,m-1}(X,\beta) \times \overline{M}_{0,0}(\mathbf{P}^1_1 \times \mathbf{P}^1_m, (1,1)) \\
i \uparrow & & j \uparrow \\
\overline{M}_{0,m}(X,\beta) & \stackrel{\pi}{\to} & \overline{M}_{0,m-1}(X,\beta)
\end{array}$$

Φ is derived, as in the product formula, from projections. $\overline{M}_{0,m-1}(X,\beta)$ is included in the graph space in the ordinary way, and the inclusion j is given by the additional inclusion of the point corresponding to the inclusion of the intersecting lines $\{0\} \times \mathbf{P}^1_m \cup \mathbf{P}^1_1 \times \{0\}$ in $\mathbf{P}^1_1 \times \mathbf{P}^1_m$.

The fixed loci contained in $\Phi^{-1}(\overline{M}_{0,m-1}(X,\beta))$, in addition to $\overline{M}_{0,m}(X,\beta)$ are isomorphic to one of the following:

$$\overline{M}_{0,k+1}(X,\beta_1) \times_X \overline{M}_{0,m-k}(X,\beta-\beta_1)) \text{ or } \overline{M}_{0,m-1}(X,\beta)$$

and the induced maps to $\overline{M}_{0,m-1}(X,\beta)$ are the gluing maps to boundary divisors (see [8]) and the identity map, respectively.

The equation (†) now tells us that given an equivariant cohomology class α on $\overline{G}_{0,m}(X,\beta)$, there is a relation among the residues of α along the fixed loci listed above, as well as the residue of $\Phi_*\alpha$ along the fixed locus $\overline{M}_{0,m-1}(X,\beta)$. So the question now becomes, how to find interesting equivariant classes α on the graph space? The only source we know of to produce good residue classes comes from the linear approximation to $\overline{G}_{0,1}(\mathbf{P}^n,d)$. Namely, suppose a morphism (not necessarily an embedding) $f : X \to \mathbf{P}^n$ is given. Then we can pull back equivariant cohomology classes via:

$$\overline{G}_{0,m}(X,\beta) \to \overline{G}_{0,1}(X,\beta) \to \overline{G}_{0,1}(\mathbf{P}^n,d) \to \mathbf{P}(\mathrm{Hom}(\mathrm{Sym}^d(W),V))$$

After all the equivariant Euler classes are computed, recursive formulas are obtained. Thus in this context the necessity of considering cohomology classes generated by divisor classes springs from our inability to find useful equivariant classes not coming from the linear approximation spaces to $\overline{G}_{0,1}(\mathbf{P}^n,d)$. As an example of the reconstruction theorems we obtain, we include the most useful one, which, in case the cohomology of X is generated by divisor classes, already suffices to express (small) quantum cohomology in terms of the J-function.

Reconstruction Theorem for 2-Point Invariants: Given $f : X \to \mathbf{P}^n$, let H be the hyperplane class on $\mathbf{P}^n$ and on X, and define:

$F_\beta(t) = e_{\beta *}\left(\frac{1}{t(t-\psi)}\right)$ (these are the coefficients of J) and

$G_\beta(\gamma,t) = e^1_{\beta *}\left(\frac{e^{2\,*}_\beta\gamma}{t-\psi_2}\right)$ for evaluation maps $e^1_\beta, e^2_\beta : M_{0,2}(X,\beta) \to X$ and cohomology class $\gamma \in \mathrm{H}^*(X,\mathbf{Q})$, extended by linearity in the first factor.

Then the expression:

$$G_\beta(H^a,t) + \left(\sum_{\beta_1+\beta_2=\beta} G_{\beta_1}(F_{\beta_2}(-t)(H-d_{\beta_2}t)^a,t)\right) + F_\beta(-t)(H-d_\beta t)^a$$

is polynomial in t, where d_β is the degree of $f_*\beta \in \mathrm{H}_2(\mathbf{P}^n,\mathbf{Z})$.

Since $G_\beta(H^a,t)$ is polynomial in t^{-1} (with no contant term), this formula expresses $G_\beta(H^a,t)$ in terms of coefficients of J and $G_{\beta'}(H^a,t)$ for smaller β'. Hence it inductively determines $G_\beta(H^a,t)$ in terms of J.

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