NEW RECURSIONS FOR GENUS-ZERO GROMOV-WITTEN INVARIANTS

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ABSTRACT. New relations among the genus-zero Gromov-Witten invariants of a complex projective manifold X are exhibited. When the cohomology of X is generated by divisor classes and classes "with vanishing one-point invariants," the relations determine many-point invariants in terms of one-point invariants.

0. INTRODUCTION

The localization theorem for equivariant cohomology has recently been used with great success to compute the genus-zero Gromov-Witten invariants relevant to the mirror conjecture [12, 19, 5]. Zero-point invariants count expected numbers of rational curves on a projective manifold X, while the more general m-point invariants count expected numbers of rational curves meeting m given submanifolds (or cohomology classes). For the the mirror conjecture, only the zero and one-point invariants are computed, though for the construction of the quantum product (even the small version), one needs more general invariants.

In this paper we will apply the localization theorem to study genus-zero Gromov-Witten invariants involving any number of marked points. A straightforward generalization of Givental's (one variable) J-function yields homology-valued J-functions in any number of variables $t_1,\dots,t_m$ which encode all the (generalized) m-point genus-zero invariants. Our main theorem is a collection of relations among these J-functions expressing a part of the J-function for a fixed curve class and number of variables in terms of the J-functions involving fewer variables and/or "smaller" curve classes. When the cohomology of X is generated by divisor classes, or, more generally, when every class orthogonal to the subring generated by divisor classes annihilates (via cap product) all one-variable J-functions, then these new relations completely determine all m-point genus-zero Gromov-Witten invariants (of classes generated by divisor classes) in terms of one-point invariants. That is, in this setting, the one-variable J-function determines all the others. A complete intersection in $\mathbf{P}^n$ has this orthogonality property, and in that case we exhibit a formula expressing "mixed" two-point invariants in terms of one-point invariants. We apply this new formula to compute previously unknown quantum products of cohomology classes on Fano complete intersections, where the one-variable J-function is known. Since our recursions do not require any positivity of X, they would apply just as well to general-type complete intersections. Unfortunately, in those cases, the one-variable J-function is not known.

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The idea is the following. Given a stable map with several marked points, a copy of $\mathbf{P}^1$ is attached to each of the marked points (at 0 on the $\mathbf{P}^1$). This allows one to embed the moduli space of m-pointed stable maps into the moduli space of stable maps with no marked points and m parametrizations. We will call the latter space the *graph space*. There is a natural torus action on the graph space, one of whose fixed loci is the given moduli space of stable maps. There are equivariant forgetful morphisms among the graph spaces, and by comparing residues of some carefully chosen equivariant cohomology classes along the fixed loci, we obtain the recursive formulas for the J-functions. A startling (to us) feature of this approach is that it is much simpler than the computation of one-point invariants, since our argument requires no analysis of the boundary of graph spaces.

1. Kontsevich-Manin spaces

We recall the basic properties of the genus-zero stable map spaces and some results on Gromov-Witten invariants, and give an instance of our formula (to be proved in greater generality later).

Definition 1. A morphism $f\colon (C; p_1, \dots, p_m) \to X$ from a connected m-pointed complex rational curve C to a complex projective manifold X is *prestable* if C has only nodes as singularities and $p_1, \dots, p_m \in C$ are nonsingular. If in addition every irreducible component of C collapsed by f has three or more distinguished points—a *distinguished point* is a node or marked point—we say that f is *stable.*

Remark. This notion of stability is analogous to Deligne-Mumford stability for pointed curves. Indeed, a stable map to a point is a stable pointed curve.

The moduli of stable maps has been extensively studied, ever since stable maps were introduced by Kontsevich and Manin [16]. (See also [4] and [11] as good references for the following properties.)

Given $\beta \in \mathrm{H}_2(X, \mathbf{Z})$, there is a proper Deligne-Mumford stack $\overline{M}_{0,m}(X, \beta)$, representing flat families of genus-zero stable maps with m marked points and image homology class β. Each of the moduli stacks $\overline{M}_{0,m}(\mathbf{P}^n, d)$ is smooth (as a stack) of the expected dimension $n + (n+1)d + (m-3)$. For general X, there is a virtual class $\left[\overline{M}_{0,m}(X, \beta)\right]^{\mathrm{vir}}$ in the Chow group of $\overline{M}_{0,m}(X, \beta)$ of the expected dimension $n - \deg_{K_X}(\beta) + (m-3)$.

There are forgetful maps and evaluation maps:

$$\begin{array}{ccc} \overline{M}_{0,m+1}(X,\beta) & \xrightarrow{e_i} & X \\ {\scriptstyle \pi_i}\big\downarrow & & \\ \overline{M}_{0,m}(X,\beta) & & \end{array}$$

where π_i "forgets" the marked point p_i (and collapses components, if necessary), and e_i evaluates the stable map at p_i. When $i = m+1$, this diagram can be taken as part of the "universal stable map" over $\overline{M}_{0,m}(X, \beta)$. The rest of the universal stable map consists of sections

$$\rho_i \colon \overline{M}_{0,m}(X, \beta) \to \overline{M}_{0,m+1}(X, \beta)$$

of π_{m+1} corresponding to the marked points.

In case $X \subset \mathbf{P}^n$ is the transverse zero locus of a section of a vector bundle E on $\mathbf{P}^n$ which is generated by global sections, the refined top Chern class $c_{\text{top}}(\pi_{m+1*}e^*_{m+1}E)$ on $\overline{M}_{0,m}(\mathbf{P}^n, d)$ produces the virtual class on $\overline{M}_{0,m}(X, d)$.

Morphisms $\phi\colon X \to Y$ give rise to morphisms of stable map spaces

$$\overline{M}_{0,m}(X, \beta) \to \overline{M}_{0,m}(Y, \phi_*\beta).$$

Finally, the "boundary" of $\overline{M}_{0,m}(X, \beta)$ is covered by the images of the gluing maps:

$$\delta_{S,\alpha}\colon \overline{M}_{0,k+1}(X, \alpha) \times_X \overline{M}_{0,m-k+1}(X, \beta - \alpha) \to \overline{M}_{0,m}(X, \beta)$$

where $S \subseteq \{1, \ldots, m\}$ is a subset of cardinality k, which, together with the curve class α, describes how the stable map breaks into (at least) two components.

The *Gromov-Witten invariants* are usually interpreted as intersection numbers on the Kontsevich-Manin spaces of stable maps. Given cohomology classes $\gamma_1, \ldots, \gamma_m$ on X, one defines the "ordinary" invariants

$$\langle \gamma_1, \ldots, \gamma_m \rangle_\beta^X := \deg \left(\pi_1^*\gamma_1 \cup \cdots \cup \pi_m^*\gamma_m \cap \mathrm{ev}_* \left[\overline{M}_{0,m}(X, \beta)\right]^{\mathrm{vir}} \right),$$

where

$$\mathrm{ev} := (e_1, \ldots, e_m)\colon \overline{M}_{0,m}(X, \beta) \to X^m$$

is the total evaluation map and $\pi_i\colon X^m \to X$ are the projections.

The general invariants are defined using the *cotangent classes*

$$\psi_i := c_1(\rho_i^* \omega_{\pi_{m+1}}),$$

where $\omega_{\pi_{m+1}}$ is the relative dualizing sheaf. The general invariants are:

$$\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_m \psi^{a_m} \rangle_\beta^X := \\ \deg \left(\pi_1^*\gamma_1 \cup \cdots \cup \pi_m^*\gamma_m \cap \mathrm{ev}_*(\psi_1^{a_1} \cup \cdots \cup \psi_m^{a_m} \cap \left[\overline{M}_{0,m}(X, \beta)\right]^{\mathrm{vir}}) \right),$$

where $a_1, \ldots, a_m$ are non-negative integers.

The following is a very useful way to package 2-point "mixed" invariants:

$$\left\langle \gamma_1, \frac{\gamma_2}{t - \psi} \right\rangle_\beta^X := t^{-1} \langle \gamma_1, \gamma_2 \rangle_\beta^X + t^{-2} \langle \gamma_1, \gamma_2\psi \rangle_\beta^X + t^{-3} \langle \gamma_1, \gamma_2\psi^2 \rangle_\beta^X + \cdots,$$

where t is a variable. Similarly, for general 1-point invariants:

$$\left\langle \frac{\gamma}{t(t - \psi)} \right\rangle_\beta^X := t^{-2} \langle \gamma \rangle_\beta^X + t^{-3} \langle \gamma\psi \rangle_\beta^X + t^{-4} \langle \gamma\psi^2 \rangle_\beta^X + \cdots.$$

The simplest of our formulas is expressed in terms of these packages (extended t-linearly):

Formula 1.1. *Suppose $X \subset \mathbf{P}^n$ is a complete intersection of dimension $r \geq 3$ and degree l. Then for all $0 \leq a, b \leq m$ and $d \geq 0$,*

$$\left\langle H^a, \frac{H^b}{t - \psi} \right\rangle_d^X + \left\langle \frac{H^a(H - dt)^b}{-t(-t - \psi)} \right\rangle_d^X + \\ \sum_{e=1}^{d-1} \sum_{c=0}^{r} \frac{1}{l} \left\langle H^a, \frac{H^c}{t - \psi} \right\rangle_{d-e}^X \left\langle \frac{H^{r-c}(H - et)^b}{-t(-t - \psi)} \right\rangle_e^X \in \mathbf{Q}[t]$$

.

This formula implies a special case of our reconstruction theorem 5.2:

Corollary 1.2. *The mixed two-point invariants of complete intersections in* $\mathbf{P}^n$ *involving only powers of* H *are determined by the one-point invariants.*

Proof. The first term in the formula is clearly determined by the others. By induction on d, the mixed invariants of degree d are therefore determined by the one-point invariants of degree d or less. $\square$

In the appendix, we use Formula 1.1 to compute small quantum products of cohomology classes on Fano complete intersections. For now, we point out the identities that follow from the formula when the classes (in the second slot) are of codimensions 0, 1 and 2.

codim 0**:** $\langle H^a, 1\rangle_d^X = 0$.
codim 1**:** $\langle H^a, \psi\rangle_1^X = -\langle H^a\rangle_d^X$ and $\langle H^a, H\rangle_d^X = d\langle H^a\rangle_d^X$.
codim 2**:**

$$\left\langle H^a, \psi^2\right\rangle_d^X = \langle H^a\psi\rangle_d^X - \frac{1}{l}\sum_{e=1}^{d-1}\sum_{c=0}^{r}\langle H^a, H^c\rangle_{d-e}^X\left\langle H^{r-c}\right\rangle_e^X$$

$$\langle H^a, H\psi\rangle_d^X = -\left\langle H^{a+1}\right\rangle_d^X - d\langle H^a\psi\rangle_d^X + \frac{1}{l}\sum_{e=1}^{d-1}\sum_{c=0}^{r} e\langle H^a, H^c\rangle_{d-e}^X\left\langle H^{r-c}\right\rangle_e^X$$

$$\left\langle H^a, H^2\right\rangle_d^X = 2d\left\langle H^{a+1}\right\rangle_d^X + d^2\langle H^a\psi\rangle_d^X - \frac{1}{l}\sum_{e=1}^{d-1}\sum_{c=0}^{r} e^2\langle H^a, H^c\rangle_{d-e}^X\left\langle H^{r-c}\right\rangle_e^X.$$

Notice that the codimension two identities are not self-contained, since they inductively involve classes of higher codimension.

Remark. The codimension 0 and the two codimension 1 identities are special cases of the string, dilaton and divisor equations, respectively. The identities for codimension 2 classes, however, are not special cases of any general equations that we are aware of (though we've been informed that Lee and Pandharipande [18] have another method for producing such identities).

To state our main theorem, we will use J-functions of several variables, generalizing Givental's one-variable definition.

Definition 2.

$$\begin{aligned} J_\beta^X(t_1,\dots,t_m) &:= \mathrm{ev}_*\left(\frac{\left[\overline{M}_{0,m}(X,\beta)\right]^{\mathrm{vir}}}{t_1(t_1-\psi_1)\cdots t_m(t_m-\psi_m)}\right) \\ &:= \mathrm{ev}_*\left(\prod_{i=1}^m t_i^{-2}\left(1+\frac{\psi_i}{t_i}+\frac{\psi_i^2}{t_i^2}+\cdots\right)\cap\left[\overline{M}_{0,m}(X,\beta)\right]^{\mathrm{vir}}\right) \\ &\in \mathrm{H}_*(X^m,\mathbf{Q})[t_1^{-1},\dots,t_m^{-1}] \end{aligned}$$

with initial conditions:

$$J_0^X(t_1) := [X]$$

and

$$J_0^X(t_1,t_2) := \frac{\Delta}{t_1t_2(t_1+t_2)},$$

where $\Delta\in\mathrm{H}_*(X\times X,\mathbf{Q})$ is the diagonal class.

Remark. When $m=1$, our J-function is the Poincaré dual of Givental's.

The J-functions encode all genus-zero Gromov-Witten invariants. The following result concerning the one-variable J-function was first proved in [12]; see [19] and [5] for alternate approaches.

Theorem 1.3 (Givental). (a) *If $X \subset \mathbf{P}^n$ is a complete intersection of type $(l_1, \ldots, l_m)$ which is Fano of index two or more (i.e. $l_1 + \cdots + l_m < n$), let H be the hyperplane class. Then*

$$J_d^X(t) = I_d^X(t) := \frac{\prod_{i=1}^m \prod_{k=1}^{dl_i}(l_i H + kt)}{\prod_{k=1}^d (H+kt)^{n+1}} \cap [X].$$

(b) *If $l_1 + \cdots + l_m = n$ or $n+1$, then the following generating functions coincide after an explicit "mirror transformation" (see [12], [19] or [5]):*

$$J^X(q) := \sum_{d=0}^{\infty} J_d^X(t) q^d \quad \text{and} \quad I^X(q) := \sum_{d=0}^{\infty} I_d^X(t) q^d.$$

We introduce a tool for manipulating J-functions of several variables:

Definition 3. Given classes $\Gamma_1 \in \mathrm{H}_*(X^k \times X, \mathbf{Q})$, $\Gamma_2 \in \mathrm{H}_*(X \times X^{m-k}, \mathbf{Q})$ and $\gamma \in \mathrm{H}^*(X, \mathbf{Q})$, we use the Künneth formula and Poincaré duality to regard the tensor product as a $\mathbf{Q}$-linear map:

$$\Gamma_1 \otimes \Gamma_2 \colon \mathrm{H}^*(X^2, \mathbf{Q}) \to \mathrm{H}_*(X^m, \mathbf{Q})$$

where the factors of X^2 are the distinguished factors of $X^k \times X$ and $X \times X^{m-k}$. We then define the *twisted product* $\Gamma_1 \otimes_\gamma \Gamma_2 \in \mathrm{H}_*(X^m, \mathbf{Q})$ by setting:

$$\Gamma_1 \otimes_\gamma \Gamma_2 := (\Gamma_1 \otimes \Gamma_2)(\delta \cup \pi^* \gamma)$$

where δ is the diagonal class and $\pi \colon X^2 \to X$ is either of the two projections.

Examples. Let γ_1, γ_2 be Poincaré dual to Γ_1, Γ_2.

(a) If $k = m = 0$, then $\Gamma_1 \otimes_\gamma \Gamma_2 \in \mathbf{Q}$ is the triple intersection:

$$\int_X \gamma \cup \gamma_1 \cup \gamma_2.$$

More generally, if $k = m$, then $\Gamma_1 \otimes_\gamma \Gamma_2 = \pi_*^X(\pi_X^*(\gamma \cup \gamma_2) \cap \Gamma_1)$, where $\pi_X \colon X^m \times X \to X$ and $\pi^X \colon X^m \times X \to X^m$ are the two projections.

(b) If $k = m-1$, then $\Gamma_1 \otimes_\gamma \Delta = \pi_X^* \gamma \cap \Gamma_1$ for $\pi_X \colon X^{m-1} \times X \to X$.

Theorem 1.4 (The main theorem—rank one case). *If H is an ample divisor class generating $\mathrm{H}^2(X, \mathbf{Q})$ on a complex projective manifold X, then for each choice of $m > 0$, $d = \deg_H(\beta) \geq 0$ and $0 \leq b \leq \dim(X)$,*

$$t(t_1+t)\Bigg(\sum_{1 \in S \subseteq [m]} \sum_{e=0}^{d} J_{d-e}^X(\vec{t}_S, t) \otimes_{(H-et)^b} J_e^X(-t, \vec{t}_{S^c}) + \sum_{j=2}^{m} J_d^X(\vec{t}_{\hat{j}}, t_j) \otimes_{H^b} J_0^X(-t_j, t) \Bigg) \in \mathrm{H}_*(X^m, \mathbf{Q})[t_1, t_1^{-1}, \ldots, t_m, t_m^{-1}, t].$$

Notation. We set $[m] := \{1, \ldots, m\}$. For subsets $S = \{s_1, \ldots, s_k\} \subseteq [m]$, we define $\vec{t}_S := (t_{s_1}, \ldots, t_{s_k})$. Since the J-functions are symmetric in their variables, the expressions $J_d^X(\vec{t}_S, t)$ are well-defined. We also set $\hat{j} := [m] - \{j\}$.

We will prove the main theorem later, as well as a more general version where the rank one condition on $\mathrm{H}^2(X,\mathbf{Q})$ is removed. To finish this section, we show how Formula 1.1 follows from the main theorem.

Proof of Formula 1.1. We apply the main theorem in the case $m = 1$. Only the double sum appears, in this case as the single sum

$$\sum_{e=0}^{d} J^X_{d-e}(t_1,t) \otimes_{(H-et)^b} J^X_e(-t). \tag{1}$$

The first and last of the terms are:

$$J^X_d(t_1,t) \otimes_{H^b} J^X_0(-t) = J^X_d(t_1,t) \otimes_{H^b} [X] = \pi_{1*}(J^X_d(t_1,t) \cup \pi_2^* H^b)$$

and

$$J^X_0(t_1,t) \otimes_{(H-dt)^b} J^X_d(-t) = \frac{\Delta \otimes_{(H-dt)^b} J^X_d(-t)}{t_1 t(t_1+t)} = \frac{(H-dt)^b \cup J^X_d(-t)}{t_1 t(t+t_1)}.$$

Multiply (1) through by $t_1 t(t_1+t)$, and the main theorem tells us we obtain an element of $\mathbf{Q}[t_1, t_1^{-1}, t]$ when we integrate against H^a. For example, by the projection formula, the first term gives

$$t_1 t(t_1+t) \deg\left(H^a \cap \pi_{1*}(\pi_2^* H^b \cap J^X_d(t_1,t))\right) = (t_1+t)\left\langle \frac{H^a}{t_1-\psi}, \frac{H^b}{t-\psi}\right\rangle^X_d$$

and similarly for the other the terms. When we consider only the terms that are constant in t_1, we obtain the following formula:

$$\left\langle H^a, \frac{H^b}{t-\psi}\right\rangle^X_d + \left\langle \frac{H^a(H-dt)^b}{-t(-t-\psi)}\right\rangle^X_d + \sum_{e=1}^{d-1}\sum_{i,j=1}^{N} \left\langle H^a, \frac{\gamma_i}{t-\psi}\right\rangle^X_{d-e} g^{ij} \left\langle \frac{\gamma_j (H-et)^b}{-t(-t-\psi)}\right\rangle^X_e \in \mathbf{Q}[t],$$

where $\gamma_1,\ldots,\gamma_N \in \mathrm{H}^*(X,\mathbf{Q})$ are a basis, with respect to which g^{ij} is the inverse of the intersection matrix. This much holds for any ample H generating $\mathrm{H}^2(X,\mathbf{Q})$.

The fact that X is a complete intersection tells us that all the one-point invariants of the form $\langle \gamma\psi^c\rangle^X_\beta$ vanish when γ is a primitive cohomology class. This can either be seen using Givental's formulas, or by a monodromy argument. Since a basis for the cohomology may be chosen consisting of powers of H and (orthogonal) primitive classes, this tells us that we may replace the basis $\{\gamma_i\}$ by the smaller set $\{H^c\}$ of powers of H, resulting in Formula 1.1. $\square$

2. Graph spaces

Graph spaces are particular Kontsevich-Manin spaces which come equipped with a natural torus action. In this section, will describe some of the fixed loci under this torus action in order to eventually apply the Atiyah-Bott localization theorem to prove the main theorem.

Definition 4. The *m-parametrized graph space* is

$$\overline{G}_{0,m}(X,\beta) := \overline{M}_{0,0}(X \times (\mathbf{P}^1)^m, (\beta, 1^m)).$$

It is often useful to think of the graph space in the following way. Let a *parametrization* of a rational curve C be an isomorphism from $\mathbf{P}^1$ to one of the irreducible components of C. A morphism $f\colon (C; \mathbf{P}^1_1, \ldots, \mathbf{P}^1_m) \to X$ from a connected rational curve with m parametrizations is*prestable* if C has only nodes as singular points. If in addition every component of C is either parametrized (possibly in several ways) or has at least three nodes (or both), we say the f is *stable*.

Then $\overline{G}_{0,m}(X, \beta)$ is the moduli stack of stable maps with m parametrizations and no marked points. This stack admits the action of the torus $(\mathbf{C}^*)^m$ via its action on $(\mathbf{P}^1)^m$. To carefully give this torus action, we need some more precise notation.

Fix vector spaces $W_i \cong \mathbf{C}^2$ for $i = 1, \ldots, m$. On W_i, choose coordinates $x_i, y_i \in W_i^*$, and fix the action of $\mathbf{C}^* = \mathbf{C}^*_i$ via

$$\mu_i \cdot (x_i, y_i) = (x_i, \mu_i y_i).$$

Let $0_i := (0 : 1)$ and $\infty_i := (1 : 0) \in \mathbf{P}^1_i$ be the fixed points of the action of $\mathbf{C}^*_i$ on $\mathbf{P}^1_i = \mathbf{P}(W_i)$. Let $\mathbf{T} := \prod \mathbf{C}^*_i$ acting diagonally on $\prod \mathbf{P}^1_i$ and hence on each of the graph spaces $\overline{G}_{0,m}(X, \beta)$.

Important Special Case. The two-parametrized graph space of a point

$$\overline{G}_{0,2}(\mathrm{pt}, 0) = \overline{M}_{0,0}(\mathbf{P}^1 \times \mathbf{P}^1, (1,1)) \cong \mathbf{P}^3.$$

A stable map to $\mathbf{P}^1 \times \mathbf{P}^1$ either embeds C as a smooth curve of type $(1,1)$ or as a pair of intersecting rulings. Thus the stable map space is the linear series.

Observation. A stable parametrized map $[f] \in \overline{G}_{0,m}(X, \beta)$ is a fixed point for the action of $\mathbf{T}$ described above exactly when:

- f is constant on each parametrized component
- Each parametrized component is uniquely parametrized
- Each node on a parametrized component is at 0 or ∞.

For our purposes, we will only need to consider the following "types" of fixed loci for the action of $\mathbf{T} = \prod \mathbf{C}^*_i$ on the graph space $\overline{G}_{0,m+1}(X, \beta)$:

Type 1. A single copy of $\overline{M}_{0,m+1}(X, \beta)$ "embedded at zeroes". (See Figure 1.)

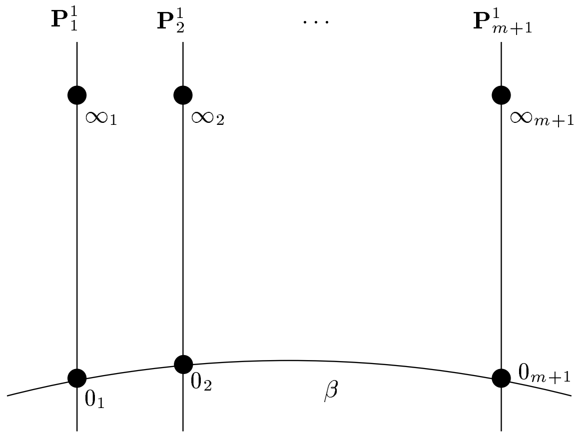

FIGURE 1. Type 1 fixed locus

Let $Y = X \times \prod_{i=1}^{m+1} \mathbf{P}^1_i$, and consider the gluing morphism:

$$\overline{M}_{0,m+1}(Y,\beta) \times_Y \prod_{i=1}^{m+1} \overline{M}_{0,1}(Y,1_i) \to \overline{G}_{0,m+1}(X,\beta),$$

where $\beta = (\beta, 0^{m+1}) \in \mathrm{H}_2(Y,\mathbf{Z})$ and likewise for 1_i. Each $\overline{M}_{0,1}(Y,1_i) \cong Y$ and $\overline{M}_{0,m+1}(Y,\beta) \cong \overline{M}_{0,m+1}(X,\beta) \times \prod \mathbf{P}^1_i$, and we obtain a regular embedding

$$i_{[m],\beta} \colon F_{[m],\beta} := \overline{M}_{0,m+1}(X,\beta) \hookrightarrow \overline{G}_{0,m+1}(X,\beta)$$

by embedding $\overline{M}_{0,m+1}(X,\beta) \times \prod 0_i \hookrightarrow \overline{M}_{0,m+1}(X,\beta) \times \prod \mathbf{P}^1_i$ and using the gluing morphism above to further embed in the graph space.

Type 2. $\overline{M}_{0,k+1}(X,\alpha) \times_X \overline{M}_{0,m-k+1}(X,\beta-\alpha)$ "with $\mathbf{P}^1_{m+1}$ in the middle." (Copies indexed by subsets $1 \in S \subseteq [m]$ with $|S| = k$ and $\alpha \in \mathrm{H}_2(X,\mathbf{Z})$). (See Figure 2.)

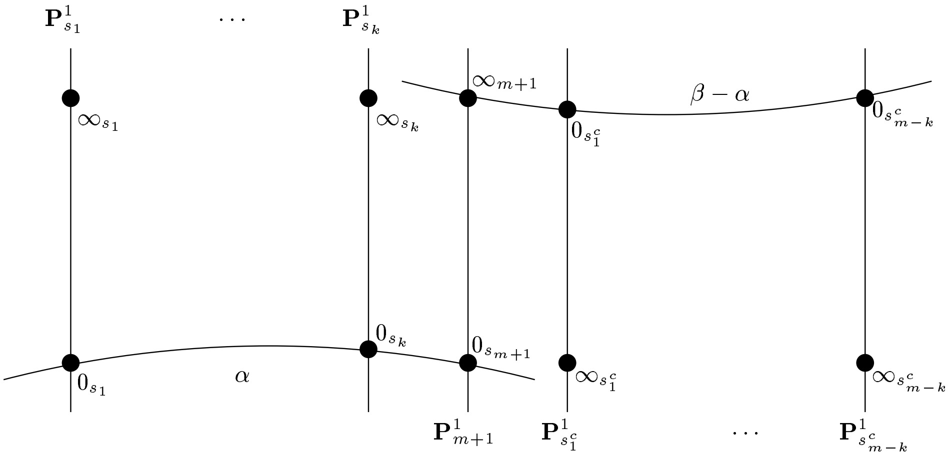

FIGURE 2. Type 2 fixed locus

In this case, we consider the composition of gluing maps taking:

$$\prod_{s_i \in S} \overline{M}_{0,1}(Y,1_i) \times_Y \overline{M}_{0,k+1}(Y,\alpha) \times_Y \overline{M}_{0,2}(Y,1_{m+1}) \times_Y$$
$$\times_Y \overline{M}_{0,m-k+1}(Y,\beta-\alpha) \times_Y \prod_{s^c_i \in S^c} \overline{M}_{0,1}(Y,1_i) \to \overline{G}_{0,m}(X,\beta),$$

assuming $(S,\alpha) \neq (\{1\},0)$, $([m],\beta)$ or any $(\hat{j},\beta)$. (These appear as other types!) The product is isomorphic to

$$\prod_{s_i \in S} \mathbf{P}^1_{s_i} \times \mathbf{P}^1_{m+1} \times \overline{M}_{0,k+1}(X,\alpha) \times_X \overline{M}_{0,m-k+1}(X,\beta-\alpha) \times \mathbf{P}^1_{m+1} \times \prod_{s^c_i} \mathbf{P}^1_{s^c_i}$$

and we identify the embedding:

$$i_{S,\alpha} \colon F_{S,\alpha} = \overline{M}_{0,k+1}(X,\alpha) \times_X \overline{M}_{0,m-k+1}(X,\beta-\alpha) \to \overline{G}_{0,m+1}(X,\beta)$$

with $(0_S, 0_{m+1}) \times \overline{M}_{0,k+1}(X,\alpha) \times_X \overline{M}_{0,m-k+1}(X,\beta-\alpha) \times (\infty_{m+1}, 0_{S^c})$.

Type 3. $\overline{M}_{0,m}(X,\beta)$ "with $\mathbf{P}^1_{m+1}$ in various places" (three subtypes).

(a) $\mathbf{P}^1_1$ as a tail off of $\mathbf{P}^1_{m+1}$ (a single copy). Let

$$i_{\{1\},0}\colon F_{\{1\},0} := \overline{M}_{0,m}(X,\beta) \to \overline{G}_{0,m+1}(X,\beta)$$

be the embedding associated to Figure 3.

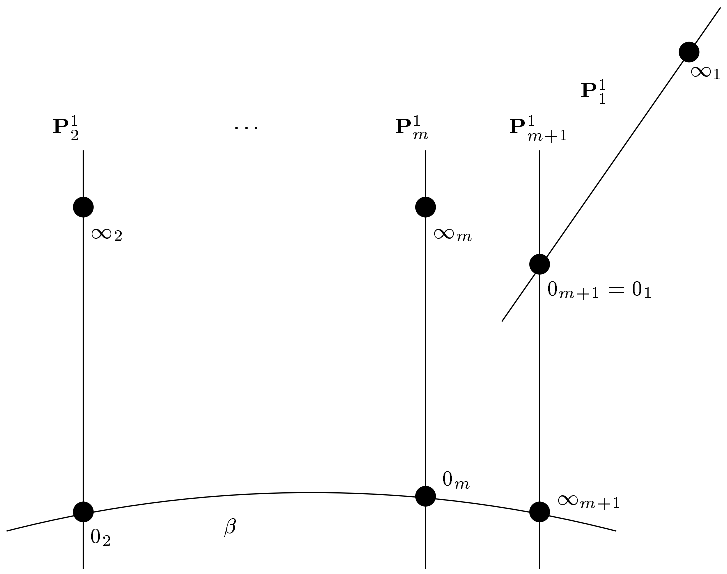

FIGURE 3. Type 3a fixed locus

(b) $\mathbf{P}^1_j$ as a tail off of $\mathbf{P}^1_{m+1}$ (one for each $1 < j \le m$). Let

$$i_{\hat{j},\beta}\colon F_{\hat{j},\beta} := \overline{M}_{0,m}(X,\beta) \to \overline{G}_{0,m+1}(X,\beta)$$

be the embedding associated to Figure 4.

(c) $\mathbf{P}^1_{m+1}$ as a tail off of $\mathbf{P}^1_j$ (indexed by $1 < j \le m$). Let

$$i_j\colon F_j := \overline{M}_{0,m}(X,\beta) \to \overline{G}_{0,m+1}(X,\beta)$$

be the embedding associated to Figure 5.

Lemma 2.1. *There is a* $\mathbf{T}$*-equivariant birational morphism:*

$$\Phi\colon \overline{G}_{0,m+1}(X,\beta) \to \overline{G}_{0,m}(X,\beta) \times \mathbf{P}^3$$

which (when projected onto the first factor) forgets the last parametrization and (when projected onto the second factor) forgets the map to X *and all parametrizations except for the first and last.*

Proof. The existence of Φ follows from the functoriality of Kontsevich-Manin spaces. The two projections are just the two maps:

$$\overline{M}_{0,0}(X \times \prod_{i=1}^{m+1} \mathbf{P}^1_i, (\beta, 1^{m+1})) \to \overline{M}_{0,0}(X \times \prod_{i=1}^{m} \mathbf{P}^1_i, (\beta, 1^m))$$

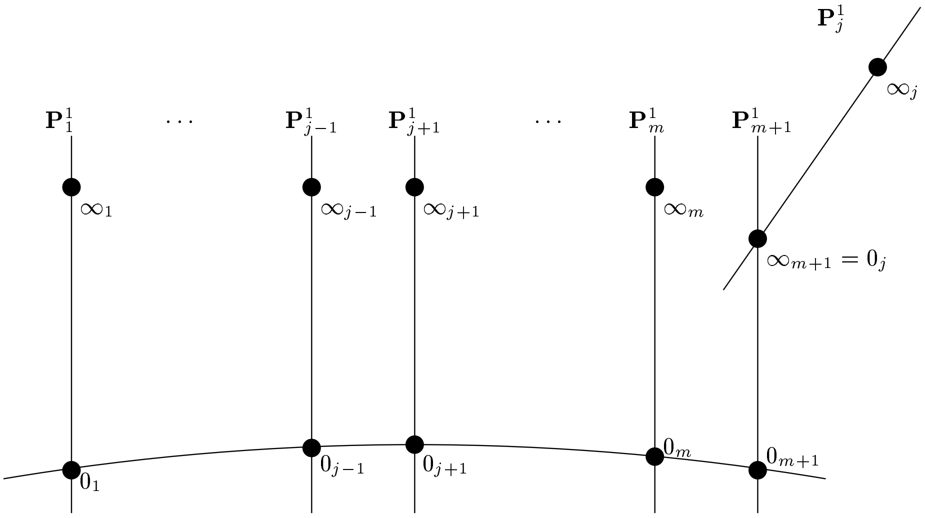

FIGURE 4. Type 3b fixed locus

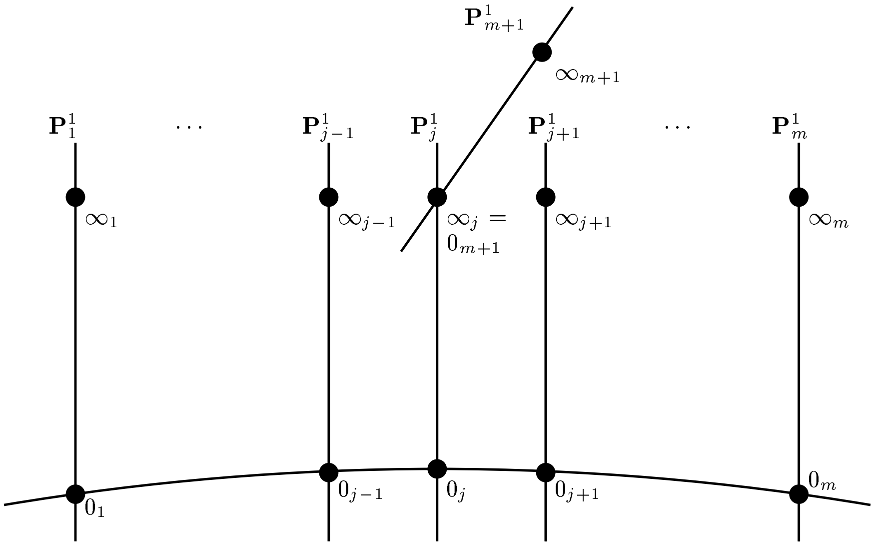

FIGURE 5. Type 3c fixed locus

and

$$\overline{M}_{0,0}(X \times \prod_{i=1}^{m+1} \mathbf{P}^1 e_i, (\beta, 1^{m+1})) \to \overline{M}_{0,0}(\mathbf{P}^1_1 \times \mathbf{P}^1_{m+1}, (1,1))$$

which clearly commute with the action of $\mathbf{T}$.

Over the open subset of $\mathbf{P}^3$ consisting of smooth curves, Φ is an isomorphism. The last parametrization is of the same component as the first, and is given by the correspondence $\mathbf{P}^1_1 \xrightarrow{\sim} \mathbf{P}^1_{m+1}$ induced by the curve in $\mathbf{P}^1_1 \times \mathbf{P}^1_{m+1}$. $\square$

Lemma 2.2. *Let $(0,0) \in \overline{G}_{0,2}(\mathrm{pt},0) = \mathbf{P}^3$ be the fixed point corresponding to the "coordinate axes." Then the embeddings of Types 1–3 listed above are a complete list of the fixed loci that are contained in*

$$\Phi^{-1}\left(F_{[m-1],\beta} \times (0,0)\right) \subset \overline{G}_{0,m+1}(X,\beta).$$

Moreover, the induced maps $\Phi|_F \colon F \to F_{[m-1],\beta} \cong \overline{M}_{0,m}(X,\beta)$ are:

(Type 1) $$\pi_{m+1} \colon \overline{M}_{0,m+1}(X,\beta) \to \overline{M}_{0,m}(X,\beta)$$

(Type 2) $$\delta_{S,\alpha} \colon \overline{M}_{0,k+1}(X,\alpha) \times_X \overline{M}_{0,m-k+1}(X,\beta-\alpha) \to \overline{M}_{0,m}(X,\beta)$$

(Type 3) $$\overline{M}_{0,m}(X,\beta) \xrightarrow{\mathrm{id}} \overline{M}_{0,m}(X,\beta).$$

Proof. Given a stable map $f \colon C \to X$, represent C by a tree with vertices and edges corresponding to the nodes and components of C, respectively. For an $f \in \overline{G}_{0,m+1}(X,\beta)$ to map to $F_{[m-1],\beta} \subset \overline{G}_{0,m}(X,\beta)$ under the forgetful map, each $\mathbf{P}^1_i$ must parametrize a different curve (edge of the tree) mapping with degree 0 to X, and each 0_i must be a node (vertex) for $i = 1,\ldots,m$. Also, the shortest path between two such vertices of the tree cannot contain any such edges, and if one of those edges is removed, the tree either stays connected or it has two components, one of which is an edge corresponding to $\mathbf{P}^1_{m+1}$ mapping with degree 0 to X. In order for f to map to $(0,0)$ under the other forgetful map to $\mathbf{P}^3$, $\mathbf{P}^1_1$ and $\mathbf{P}^1_{m+1}$ must represent different edges, 0_1 and 0_{m+1} must represent vertices (possibly the same one) and the shortest path from 0_1 to 0_{m+1} may not contain either of the two edges.

The only fixed points under the torus action which satisfy both conditions are those of types 1–3. This proves the first part of the lemma.

Under the map to $F_{[m-1],\beta}$, the parametrization of $\mathbf{P}^1_{m+1}$ is forgotten. This may result in an unparametrized component with 1 or 2 nodes, which is then collapsed. Moreover, in the 1 node case, the resulting marked point must also be forgotten. This gives the second part of the lemma. □

3. Localization and the main theorem for $\mathbf{P}^n$.

When a complex Lie group G acts on a complex manifold X, there is an equivariant cohomology ring:

$$H^*_G(X,\mathbf{Q})$$

which is an algebra over the cohomology ring of the classifying space BG. If $G = \mathbf{T} = (\mathbf{C}^*)^m$, then $\mathrm{H}^*(BG,\mathbf{Q}) \cong \mathbf{Q}[t_1,...,t_m]$. The equivariant cohomology ring for a trivial action of $\mathbf{T}$ is the polynomial algebra $\mathrm{H}^*(X,\mathbf{Q})[t_1,\ldots,t_m]$, but in general it is more complicated. Linearized vector bundles E on X have equivariant Chern classes $c^G_d(E)$ taking values in equivariant cohomology, and equivariant cohomology pulls back and pushes forward (for proper maps).

The set-up is similar for smooth Deligne-Mumford stacks. In this setting, there is an equivariant Chow ring (see [9]) $\mathrm{CH}^*_G(X)$ which always pulls back and pushes forward under equivariant proper maps. A linearized vector bundle E in this setting has equivariant Chern classes $c^G_d(E) \in \mathrm{CH}^*_G(X)$.

A basic result in either setting is the theorem of Atiyah-Bott (see [15]):

Theorem 3.1 (Localization). *Each fixed substack $i\colon F \hookrightarrow X$ of a torus action on a proper smooth Deligne-Mumford stack is a regularly embedded proper smooth Deligne-Mumford stack, its normal bundle is canonically linearized, its Euler class $\varepsilon_{\mathbf{T}}(F)$ (the top equivariant Chern class of the normal bundle) is invertible in*

$$\mathrm{CH}^*(F,\mathbf{Q})\otimes_{\mathbf{Q}}\mathbf{Q}(t_1,\dots,t_m),$$

and any element $c\in \mathrm{CH}^*_{\mathbf{T}}(X)$ is uniquely recovered (modulo torsion) via the following localization formula:

$$c=\sum_F i_*\frac{i^*c}{\varepsilon_{\mathbf{T}}(F)}.$$

Our main interest is in the following simple corollary:

Corollary 3.2 (Correspondence of residues). *Suppose $f\colon X\to X'$ is a $\mathbf{T}$-equivariant map of smooth proper Deligne-Mumford stacks with $\mathbf{T}$-actions. If $i'\colon F'\hookrightarrow X'$ is a fixed substack and $c\in\mathrm{CH}^*_{\mathbf{T}}(X)$, let $f_F\colon F\to F'$ be the restriction of f to each of the fixed substacks $F\subset f^{-1}(X')$. Then*

$$\sum_{F\subset f^{-1}(F')} f_{F*}\frac{i^*c}{\varepsilon_{\mathbf{T}}(F)}=\frac{i'^*f_*c}{\varepsilon_{\mathbf{T}}(F')}$$

Proof. The two sides of the formula represent the contribution of F' to localization formulas for f_*c which, by uniqueness, must coincide. □

To prove the main theorem for general X we will need virtual classes. For now we will prove it in the case $X=\mathbf{P}^n$, where the basic idea and most of the computations are the same, and are not obscured by the presence of virtual classes.

Proof of the main theorem for $\mathbf{P}^n$. Let $c\in\mathrm{CH}^*_{\mathbf{T}}(\overline{G}_{0,m+1}(\mathbf{P}^n,d))$. Then applying correspondence of residues to the map Φ of smooth Deligne-Mumford stacks (here we use $X=\mathbf{P}^n$) of Lemma 2.1, and using the enumeration of fixed loci in Lemma 2.2, we get:

$$\begin{aligned}(2)\quad \pi_{m+1*}\left(\frac{i^*_{[m],d}c}{\varepsilon_{\mathbf{T}}(F_{[m],d})}\right)+\sum_{1\in S}\sum_{e=0}^{d}\delta_{S,e*}\left(\frac{i^*_{S,d-e}c}{\varepsilon_{\mathbf{T}}(F_{S,d-e})}\right)+\frac{i^*_{\{1\},0}c}{\varepsilon_{\mathbf{T}}(F_{\{1\},0})}+\\ \sum_{j=2}^{m}\left(\frac{i^*_{\hat{j},d}c}{\varepsilon_{\mathbf{T}}(F_{\hat{j},d})}+\frac{i^*_jc}{\varepsilon_{\mathbf{T}}(F_j)}\right)=\frac{i^*_{[m-1],d}\Phi_*c}{\varepsilon_{\mathbf{T}}(F_{[m-1],d})}\cdot\frac{1}{t_1t_{m+1}(t_1+t_{m+1})}.\end{aligned}$$

(The computation $\varepsilon_{\mathbf{T}}(0,0)=t_1t_{m+1}(t_1+t_{m+1})$ is easily made.)

Now, the equivariant Euler classes $\varepsilon_{\mathbf{T}}(F)$ appearing in the denominators depend entirely on the nodes of the domain of a general representative $f\in F$. Essentially, there are two types of nodes: those of type I, where at the point $p_i\in\{0_i,\infty_i\}$, the ith parametrized component meets a component mapping in positive degree α to X, and those of type II, where at the point $p_i\in\{0_i,\infty_i\}$, the ith parametrized component meets the point $p_j\in\{0_j,\infty_j\}$ of the jth parametrized component. See Figure 6.

Any type I node is a codimension 2 condition—one for the node and one for specifying p_i—while a type II node is a codimension 3 condition—one for the node

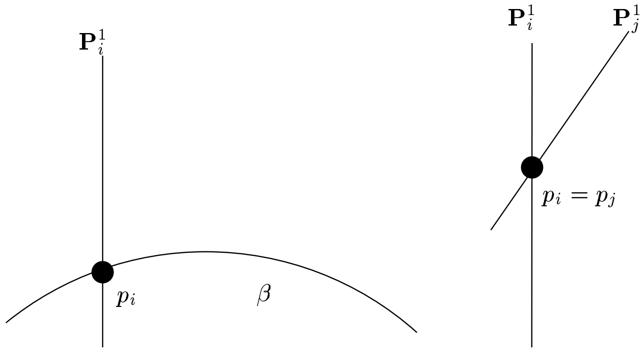

FIGURE 6. Type I and type II nodes

and two more for specifying p_i and p_j. Set

$$\nu_i := \begin{cases} 1 & \text{if } p_i = 0_i \\ -1 & \text{if } p_i = \infty_i \end{cases}.$$

Then the type I node contributes the factor

$$\nu_i t_i(\nu_i t_i - \psi_i)$$

to $\varepsilon_{\mathbf{T}}(F)$, while the type II node contributes

$$\nu_i t_i \nu_j t_j(\nu_i t_i + \nu_j t_j).$$

(For this type of computation, see [7].)

Thus, if we let

$$t := t_{m+1},$$

the following computations are valid on any $\overline{G}_{0,m+1}(X,\beta)$:

$$\varepsilon_{\mathbf{T}}(F_{[m],\beta}) = t(t - \psi_{m+1}) \prod_{i\in[m]} t_i(t_i - \psi_i). \tag{3.1}$$

$$\varepsilon_{\mathbf{T}}(F_{S,\alpha}) = t(t - \psi_{k+1})(-t)(-t - \psi'_1) \prod_{S} t_{s_i}(t_{s_i} - \psi_i) \prod_{S^c} t_{s^c_i}(t_{s^c_i} - \psi'_{i+1}) \tag{3.2}$$

where ψ_i, ψ'_i are the cotangent classes on $\overline{M}_{0,k+1}(X,\alpha)$ and $\overline{M}_{0,m-k+1}(X,\beta-\alpha)$.

$$\varepsilon_{\mathbf{T}}(F_{\{1\},0}) = t_1 t(t_1 + t)(-t)(-t - \psi_1) \prod_{i=2}^{m} t_i(t_i - \psi_i). \tag{3.3a}$$

$$\varepsilon_{\mathbf{T}}(F_{\hat{j},\beta}) = (-t)t_j(-t + t_j) \prod_{S=\hat{j}} t_{s_i}(t_{s_i} - \psi_i)t(t - \psi_m). \tag{3.3b}$$

$$\varepsilon_{\mathbf{T}}(F_j) = (-t_j)t(-t_j + t) \prod_{S=\hat{j}} t_{s_i}(t_{s_i} - \psi_i)t_j(t_j - \psi_m). \tag{3.3c}$$

Finally, let $\mathbf{P}^n = \mathbf{P}(V)$ and consider $H_{\mathbf{T}}$, the equivariant hyperplane class on the linear space $\mathbf{P}(\mathrm{Hom}(\mathrm{Sym}^d(W_{m+1}), V))$. There is an equivariant morphism

$$v\colon \overline{G}_{0,m+1}(\mathbf{P}^n, d) \to \overline{G}_{0,1}(\mathbf{P}^n, d) \to \mathbf{P}(\mathrm{Hom}(\mathrm{Sym}^d(W_{m+1}), V))$$

which is a composition of the forgetful map remembering only $\mathbf{P}^n$ and the last parametrization, and the "map to the linear sigma model." (The geometry of this second (birational) map was used in [5] to give a proof of the mirror theorem.) This $H_{\mathbf{T}}$ pulls back to the fixed loci as follows:

$$i^*_{[m],d} v^* H_{\mathbf{T}} = e^*_{m+1} H \tag{4.1}$$
$$i^*_{S,d-e} v^* H_{\mathbf{T}} = e^*_{k+1} H - et \tag{4.2}$$
$$i^*_{\hat{j},d} v^* H_{\mathbf{T}} = e^*_j H \tag{4.3a}$$
$$i^*_{\{1\},0} v^* H_{\mathbf{T}} = e^*_1 H - dt \tag{4.3b}$$
$$i^*_j v^* H_{\mathbf{T}} = e^*_j H. \tag{4.3c}$$

To see this, note that under the morphism v, the fixed loci map to various copies of $\mathbf{P}^n$ sitting as the fixed loci in $\mathbf{P}(\mathrm{Hom}(\mathrm{Sym}^d(W_{m+1}), V))$. More specifically, set

$$(\mathbf{P}^n)_e := \{x^{d-e}_{m+1} y^e_{m+1}\} \times \mathbf{P}(V) \subset \mathbf{P}\big(\mathrm{Sym}^d(W^*_{m+1})\big) \times \mathbf{P}(V) \xrightarrow{\text{Segre}} \mathbf{P}(\mathrm{Hom}(\mathrm{Sym}^d(W_{m+1}), V))$$

which are all fixed under the $\mathbf{T}$ action. One easily computes that $H_{\mathbf{T}}$ restricts to $\mathrm{H}^*_{\mathbf{T}}((\mathbf{P}^n)_e, \mathbf{Q})$ as $H - et$. Finally, fixed loci of types 1, 3a, and 3c map under v to $(\mathbf{P}^n)_0$, the fixed loci of type 3b map to $(\mathbf{P}^n)_d$, and the loci of type 2 map to $(\mathbf{P}^n)_e$ for appropriate $1 \leq e \leq d-1$.

Substitute in the summands of (2) for the equivariant Euler classes and for the choice $c = v^* H^b_{\mathbf{T}}$ and push forward under the total evaluation map ev. Then by the projection formula and the computations above we obtain the following:

$$\mathrm{ev}_* \, \pi_{m+1*} \frac{i^*_{[m],d} c}{\varepsilon_{\mathbf{T}}(F_{[m],d})} = J^{\mathbf{P}^n}_d(t_1, \ldots, t_m, t) \otimes_{H^b} J^{\mathbf{P}^n}_0(-t) \tag{5.1}$$
$$\mathrm{ev}_* \, \delta_{S,\alpha*} \frac{i^*_{S,\alpha} c}{\varepsilon_{\mathbf{T}}(F_{S,\alpha})} = J^{\mathbf{P}^n}_{d-e}(\vec{t}_S, t) \otimes_{(H-et)^b} J^{\mathbf{P}^n}_e(-t, \vec{t}_{S^c}) \tag{5.2}$$
$$\mathrm{ev}_* \frac{i^*_{\{1\},0} c}{\varepsilon_{\mathbf{T}}(F_{\{1\},0})} = J^{\mathbf{P}^n}_0(t_1, t) \otimes_{(H-dt)^b} J^{\mathbf{P}^n}_d(-t, \vec{t}_{\hat{1}}) \tag{5.3a}$$
$$\mathrm{ev}_* \frac{i^*_{\hat{j},d} c}{\varepsilon_{\mathbf{T}}(F_{\hat{j},d})} = J^{\mathbf{P}^n}_d(\vec{t}_{\hat{j}}, t) \otimes_{H^b} J^{\mathbf{P}^n}_0(-t, t_j) \tag{5.3b}$$
$$\mathrm{ev}_* \frac{i^*_j c}{\varepsilon_{\mathbf{T}}(F_j)} = J^{\mathbf{P}^n}_d(\vec{t}_{\hat{j}}, t_j) \otimes_{H^b} J^{\mathbf{P}^n}_0(-t_j, t) \tag{5.3c}$$

We get the theorem by multiplying both sides of (2) by $t_1 t(t_1 + t)$, collecting types 1, 2, 3a, and 3b under the double sum, and noting that

$$\mathrm{ev}_* \frac{i^*_{[m-1],d} \Phi_* c}{\varepsilon_{\mathbf{T}}(F_{[m-1],d})} \in \mathrm{H}^*((\mathbf{P}^n)^m, \mathbf{Q})[t_1, t_1^{-1}, \ldots, t_m, t_m^{-1}, t]$$

because $i^*_{[m-1],d} \Phi_* c$ is **polynomial** in $t_1, \ldots, t_m, t$ and the inverse to $\varepsilon_{\mathbf{T}}(F_{[m-1],d})$ belongs to $\mathrm{H}^*(\overline{M}_{0,m}(\mathbf{P}^n, d), \mathbf{Q})[t_1^{-1}, \ldots, t_m^{-1}]$. □

4. Virtual classes

In order to prove our main theorem in the general case, we will need to establish a simple property of equivariant virtual classes. We begin by briefly recalling the construction of the virtual class on Kontsevich-Manin spaces, following Behrend and Fantechi [2, 3] (but see also Li-Tian [20]) and of the equivariant virtual class on graph spaces, following Graber-Pandharipande [13].

Fix a complex projective manifold X and an embedding $X \hookrightarrow \mathbf{P}^n$. For each $\beta \in \mathrm{H}_2(X, \mathbf{Z})$, let d be the degree of the image of β in $\mathbf{P}^n$. Then there is a commuting diagram of stacks:

$$\begin{array}{ccc} \overline{M}_{0,m}(X,\beta) & \stackrel{i}{\to} & \overline{M}_{0,m}(\mathbf{P}^n, d) \\ & \searrow \quad\quad \rho \swarrow & \\ & \mathfrak{M}_{0,m} & \end{array}$$

where $\mathfrak{M}_{0,m}$ is the Artin (not Deligne-Mumford!) stack of prestable m-marked curves. The map i is a closed embedding (let $\mathcal{I}$ be the associated ideal sheaf) and ρ is smooth. It follows that the relative intrinsic normal cone of Behrend-Fantechi is the cone stack associated to the following map of sheaves on $\overline{M}_{0,m}(X,\beta)$:

$$\mathcal{I}/\mathcal{I}^2 \to i^*\Omega_{\overline{M}_{0,m}(\mathbf{P}^n,d)/\mathfrak{M}_{0,m}}$$

This relative normal cone, which we denote by $\mathfrak{C}_{\overline{M}_{0,m}(X,\beta)/\mathfrak{M}_{0,n}}$, embeds in the smooth h^1/h^0 cone stack $V_{\overline{M}_{0,m}(X,\beta)/\mathfrak{M}_{0,m}}$ associated to the object

$$(R^1\pi_* e^* TX)^\vee$$

of the derived category of coherent sheaves on $\overline{M}_{0,m}(X,\beta)$. (The dual is the Verdier dual and $\pi\colon \mathcal{C} \to \overline{M}_{0,m}(X,\beta)$ and $e\colon \mathcal{C} \to X$ come from the universal stable map). The virtual class $\left[\overline{M}_{0,m}(X,\beta)\right]^{\mathrm{vir}}$ is then obtained by pulling back the class of $\mathfrak{C}_{\overline{M}_{0,m}(X,\beta)/\mathfrak{M}_{0,m}}$ via the zero section of $V_{\overline{M}_{0,m}(X,\beta)/\mathfrak{M}_{0,m}}$.

Similarly, for graph spaces, there is a diagram of $\mathbf{T}$-invariant morphisms

$$\begin{array}{ccc} \overline{G}_{0,m}(X,\beta) & \stackrel{i}{\to} & \overline{G}_{0,m}(\mathbf{P}^n, d) \\ & \searrow \quad\quad \rho \swarrow & \\ & \mathfrak{G}_{0,m} & \end{array},$$

where $\mathfrak{G}_{0,m}$ is the stack of prestable zero-pointed maps to $(\mathbf{P}^1)^m$ of multi-degree $(1,\ldots,1)$. The (equivariant) intrinsic relative normal cone $\mathfrak{C}_{\overline{G}_{0,m}(X,\beta)/\mathfrak{G}_{0,m}}$ and h^1/h^0 cone $V_{\overline{G}_{0,m}(X,\beta)/\mathfrak{G}_{0,m}}$ are defined exactly as before, and the (equivariant) virtual class $\left[\overline{G}_{0,m}(X,\beta)\right]^{\mathrm{vir}}_{\mathbf{T}}$ may also be defined as before, using equivariant Chow groups. This definition is simpler than the definition in [13], but is equivalent. The simplification in our case comes from the existence of the $\mathbf{T}$-invariant embedding i into a relatively smooth graph space. The significance of the equivariant virtual class is in the following "virtual" version of the localization theorem:

Theorem 4.1 (Graber-Pandharipande)**.** *In the equivariant Chow group of the graph space $\overline{G}_{0,m}(X,\beta)$ the virtual class satisfies*

$$\left[\overline{G}_{0,m}(X,\beta)\right]^{\mathrm{vir}}_{\mathbf{T}} = \sum_F i_* \frac{i^*\left[\overline{G}_{0,m}(X,\beta)\right]^{\mathrm{vir}}_{\mathbf{T}}}{\varepsilon_{\mathbf{T}}(F)}$$

where $i\colon F \hookrightarrow \overline{G}_{0,m}(X,\beta)$ *are the (regular) embeddings of the fixed substacks.*

In order to use this theorem, we need the following:

Lemma 4.2.

(a) *The forgetful map*

$$\phi\colon \overline{G}_{0,m}(X,\beta) \to \overline{M}_{0,0}(X,\beta)$$

is flat and equivariant for the trivial action of $\mathbf{T}$ *on* $\overline{M}_{0,0}(X,\beta)$.

(b) *The equivariant virtual class satisfies*

$$\left[\overline{G}_{0,m}(X,\beta)\right]_{\mathbf{T}}^{\mathrm{vir}} = \phi^* \left[\overline{M}_{0,0}(X,\beta)\right]^{\mathrm{vir}},$$

where $\left[\overline{M}_{0,0}(X,\beta)\right]^{\mathrm{vir}}$ *is the ordinary virtual class, regarded as an equivariant class for the trivial action of* $\mathbf{T}$*. In particular, each* $i^* \left[\overline{G}_{0,0}(X,\beta)\right]_{\mathbf{T}}^{\mathrm{vir}} = [F]^{\mathrm{vir}}$ *in the theorem above, where* $[F]^{\mathrm{vir}}$ *is the "ordinary" virtual class on* F*, thought of as a fiber product of Kontsevich-Manin spaces.*

Proof. It suffices by induction to prove the lemma for the case $m = 1$. In that case, we will consider a (non-commuting!) diagram of stacks:

$$\begin{array}{ccccc} \overline{M}_{0,3}(X,\beta) & \xrightarrow{g} & \overline{G}_{0,1}(X,\beta) & \xrightarrow{\phi} & \overline{M}_{0,0}(X,\beta) \\ \downarrow & & \downarrow & & \\ \mathfrak{M}_{0,3} & \xrightarrow{\mathfrak{g}} & \mathfrak{G}_{0,1} & & \end{array}$$

where the horizontal maps are the "cross-ratio" maps defined as follows. The universal curve $\mathcal{C} \cong \overline{M}_{0,4}(X,\beta)$ over $\overline{M}_{0,3}(X,\beta)$ maps to $\overline{M}_{0,4} \cong \mathbf{P}^1$ via the forgetful map. Together with the evaluation map to X, this defines g. If $f\colon C \to X$ is a stable map with 3 marked points p, q, $r \in C$, then the map $C \to \mathbf{P}^1$ defined by $g(f)$ may be taken to be the unique map with the property that $f(p) = 0, f(q) = 1$ and $f(r) = \infty$. This is the cross-ratio if p, q, r belong to the same component of C, but is well-defined even if they lie on different components.

For $\mathfrak{g}$, we apply the prestabilization map $\mathcal{C} \to \mathfrak{M}_{0,4}$ (see [2]) to the universal curve over $\mathfrak{M}_{0,3}$ followed by the stabilization map $\mathfrak{M}_{0,4} \to \overline{M}_{0,4} \cong \mathbf{P}^1$. This map has the same pointwise description as g.

The diagram doesn't commute because g stabilizes unstable maps to $X \times \mathbf{P}^1$, while $\mathfrak{g}$ does not. On the other hand, there is a "good" open substack

$$U := \{f : C \to \mathbf{P}^1 \mid f \text{ is an isomorphism over } 0,\ 1,\ \infty\} \subset \mathfrak{G}_{0,1}$$

with the following properties:

- g and $\mathfrak{g}$ are both isomorphisms over U.
- The diagram above is Cartesian when restricted to U.
- Translates of U by elements $m \in \mathrm{PGL}(2,\mathbf{C})$ cover $\mathfrak{G}_{0,1}$,

If $f \in U$, then p, q, and r are the preimages of 0, 1, and ∞, so $\mathfrak{g}$ is invertible at f. If $f \in \overline{M}_{0,3}(X,\beta)$ lies over U, then p, q, and r all belong to same component $C_0 \subset C$ of the curve associated to f, and $g(f)$ imposes the unique parametrization on C_0 taking p, q, and r to 0, 1, and ∞. Clearly, then, g and $\mathfrak{g}$ are isomorphisms over U and the diagram is Cartesian over U. Since every prestable map $f\colon C \to \mathbf{P}^1$

of degree one is *generically* an isomorphism over $\mathbf{P}^1$, it follows that the translates of U cover $\mathfrak{G}_{0,1}$.

We finish the proof now by comparing $\overline{G}_{0,1}(X,\beta)$ with $\overline{M}_{0,3}(X,\beta)$. Suppose $f \in \overline{G}_{0,1}(X,\beta)$ lies over U. Then g is an isomorphism at f, so since $\phi \circ g$ is flat everywhere (it is a composition of the flat forgetful maps), it follows that ϕ is flat at f. But an arbitrary $f \in \overline{G}_{0,1}(X,\beta)$ lies over some translate mU, over which the composition of g with translation by m is an isomorphism, and we similarly conclude that ϕ is flat at an arbitrary f. This gives us (a).

Thus ϕ is flat, and we may use the flat pull-back to define $\phi^* \left[\overline{M}_{0,0}(X,\beta)\right]^{\mathrm{vir}}$. Behrend showed that the relative intrinsic normal cone $\mathfrak{C}_{\overline{M}_{0,0}(X,\beta)/\mathfrak{M}_{0,0}}$ pulls back under $\phi \circ g$ to $\mathfrak{C}_{\overline{M}_{0,3}(X,\beta)/\mathfrak{M}_{0,3}}$ and the same trick we employed in the previous paragraph shows that it pulls back under ϕ to $\mathfrak{C}_{\overline{G}_{0,1}(X,\beta)/\mathfrak{G}_{0,1}}$. The flatness of ϕ also tells us that $R^1\pi_* e^*TX$ pulls back to the corresponding element of the derived category of sheaves on $\overline{G}_{0,1}(X,\beta)$, and we get (b). The last sentence of (b) is a consequence of Behrend's work, since the induced maps $F \to \overline{M}_{0,0}(X,\beta)$ are always gluing maps of Kontsevich-Manin spaces. □

5. The main theorem and reconstruction.

We now return to Theorem 1.4 and its generalizations and consequences.

Proof of the main theorem (rank one case): We may assume that H is very ample. Indeed, suppose the polynomiality condition holds for the expression

$$\sum_{1\in S\subseteq[m]} \sum_{e=0}^{d} J^X_{d-e}(\vec{t}_S,t) \otimes_{(lH-et)^b} J^X_e(-t,\vec{t}_{S^c}) + \sum_{j=2}^{m} J^X_d(\vec{t}_{\hat{j}},t_j) \otimes_{(lH)^b} J^X_0(-t_j,t)$$

for some $l > 0$. Only the e's divisible by l will produce non-zero terms, because the degree of every curve (measured against lH) is a multiple of l. But replacing $lH - et$ by $H - \frac{e}{l}t$ in the twisted tensor products simply multiplies the expression by l^{-b}. If we now replace the subscript of each J by the degree of the curve against H (instead of against lH) we get the desired result for H.

The embedding $X \subset \mathbf{P}^n$ defined by H allows us to define a morphism

$$v\colon \overline{G}_{0,m+1}(X,d) \to \overline{G}_{0,1}(X,d) \hookrightarrow \overline{G}_{0,1}(\mathbf{P}^n,d) \to \mathbf{P}(\mathrm{Hom}(\mathrm{Sym}^d(W_{m+1}) \otimes V))$$

and an equivariant Chern class $v^*(H^b_{\mathbf{T}})$ as in the $\mathbf{P}^n$ case. Applying Lemma 4.2 (a) to the map $\overline{G}_{0,m+1}(X,\beta) \to \overline{G}_{0,m}(X,\beta)$, we see that Φ is a local complete intersection (l.c.i.) morphism, since it factors through the graph followed by a flat morphism:

$$\begin{array}{ccc} & \nearrow & \overline{G}_{0,m+1}(X,d) \times \mathbf{P}^3 \\ & & \downarrow \\ \overline{G}_{0,m+1}(X,d) & \xrightarrow{\Phi} & \overline{G}_{0,m}(X,d) \times \mathbf{P}^3 \end{array}.$$

Then by Lemma 4.2 (b),

$$\Phi^*\left(\left[\overline{G}_{0,m}(X,d)\right]^{\mathrm{vir}}_{\mathbf{T}} \times [\mathbf{P}^3]\right) = \left[\overline{G}_{0,m+1}(X,d)\right]^{\mathrm{vir}}_{\mathbf{T}}.$$

It follows by the projection formula that the correspondence of residues holds for $c \cap \left[\overline{G}_{0,m+1}(X,d)\right]^{\mathrm{vir}}_{\mathbf{T}}$ (and any equivariant Chern class c) with each i^*c replaced by $i^*c \cap [F]^{\mathrm{vir}}$, and $i^*_{[m-1],d}\Phi_* c$ replaced by $i^*_{[m-1],d}\Phi_* c \cap \left[F_{[m-1],d}\right]^{\mathrm{vir}}$ (again, using Lemma 4.2). The proof of the $\mathbf{P}^n$ case now carries over to prove the general rank one case. □

Next we turn to the theorem for arbitrary $\mathrm{H}^2(X, \mathbf{Q})$. It seems best to do this, not for J-functions defined intrinsically on X, but for J-functions defined in terms of a choice of (generalized) polarization on X. (See also [6].)

Definition 5.

(a) A divisor H on X is *eventually free* if some positive multiple lH defines a morphism $X \to \mathbf{P}^n$.
(b) A collection H_1,...,H_k (written H for short) of eventually free divisors is *ample* if positive $\mathbf{Z}$-linear combinations $l_1H_1 + \cdots + l_kH_k$ are ample.
(c) The J-functions associated to an H as in (b) are

$$\begin{aligned} J_d^{X,H}(t_1,\ldots,t_m) &= J_{(d_1,\ldots,d_k)}^{X,H_1,\ldots,H_k}(t_1,\ldots,t_m) \\ &:= \sum_{d(\beta)=(d_1,\ldots,d_k)} J_\beta^X(t_1,\ldots,t_m), \end{aligned}$$

where $d(\beta)$ is the multi-degree $(\deg_{H_1}(\beta), \ldots, \deg_{H_k}(\beta))$.

Theorem 5.1 (The main theorem—general case)**.** *If X is a complex projective manifold and $H = (H_1, \ldots, H_k)$ is an ample collection of eventually free divisors, then*

$$t(t_1+t)\Bigg(\sum_{1\in S\subseteq[m]}\sum_{e\preceq d} J_{d-e}^{X,H}(\vec{t}_S,t)\otimes_{\prod(H_i-e_it)^{b_i}} J_e^{X,H}(-t,\vec{t}_{S^c})+$$
$$\sum_{j=2}^m J_d^{X,H}(\vec{t}_{\hat{j}},t_j)\otimes_{\prod H_i^{b_i}} J_0^{X,H}(-t_j,t)\Bigg) \in \mathrm{H}_*(X^m,\mathbf{Q})[t_1,t_1^{-1},\ldots,t_m,t_m^{-1},t]$$

In this case, we sum over $0 \preceq e = (e_1, \ldots, e_k) \preceq d$, meaning that $0 \le e_i \le d_i$.

Proof. As in the proof of the rank one version, we may assume that H_1,...,H_k are not just eventually free, but free, by replacing them with positive multiples (which can be taken to be the same multiple). The H_i define a morphism

$$v\colon \coprod_{d(\beta)=d} \overline{G}_{0,m+1}(X,\beta) \to \coprod_{d(\beta)=d} \overline{G}_{0,1}(X,\beta) \to \prod_{i=1}^k \mathbf{P}(\mathrm{Hom}(\mathrm{Sym}^{d_i} W_{m+1}, V_i))$$

and the theorem results from applying the correspondence of residues to the class $v^*\prod_{i=1}^k H_{i\mathbf{T}}^{b_i}$, where the $H_{i\mathbf{T}}$ are the equivariant hyperplane classes pulled back from $\mathbf{P}(\mathrm{Hom}(\mathrm{Sym}^{d_i} W_{m+1}, V_i))$. □

Finally, we have the

Theorem 5.2 (Reconstruction)**.** *Let $R_H \subset \mathrm{H}^*(X,\mathbf{Q})$ be the subring generated as a $\mathbf{Q}$-algebra by 1 and an ample collection H_1,...,H_k of eventually free divisors. If the orthogonal complement to R_H annihilates each of the one-variable J-functions $J_d^{X,H}(t)$, then the Gromov-Witten invariants of the form*

$$\sum_{d(\beta)=d} \langle \gamma_1\psi^{a_1}, \ldots, \gamma_m\psi^{a_m}\rangle_\beta^X$$

for $\gamma_i \in R_H$ are completely determined by the one-point invariants, the intersection matrix on R_H, and the canonical class K_X.

Proof. The only term in the main theorem involving a J-function of $m+1$ variables and curves of (multi) degree d is

$$J_d^{X,H}(t_1,\ldots,t_m,t) \otimes_{\prod H_i^{b_i}} J_0^{X,H}(-t) = \pi_*^X\left(\left(\cup\pi_X^*\prod H_i^{b_i}\right)\cap J_d^{X,H}(t_1,\ldots,t_m,t)\right).$$

The product $t(t+t_1)J_d^{X,H}(t_1,\ldots,t_m,t)$ is a polynomial in t^{-1}, expanding as

$$t(t+t_1)J_d^{X,H}(t_1,\ldots,t_m,t) = (t+t_1)\sum_{a=1}^{N} t^{-a}\sum_{d(\beta)=d}\mathrm{ev}_*\frac{\psi_{m+1}^{a-1}\cap[\overline{M}_{0,m+1}(X,\beta)]^{\mathrm{vir}}}{\prod_{i=1}^m t_i(t_i-\psi_i)},$$

for some N depending on K_X. It follows by downward induction on the power of t^{-1} and the main theorem that every term in the expansion of $\pi_*^X(J_d^{X,H}(t_1,\ldots,t_m,t)\cup \pi_X^*\prod H_i^{b_i})$ in t^{-1} is determined inductively by J-functions involving fewer variables and/or lower degrees. Note that by stopping the induction at the t^{-1} term, we determine the constant term, about which the main theorem tells us nothing.

This argument only proves the reconstruction theorem when all cohomology is generated by the H_i since it (inductively) requires knowledge of the classes $\pi_*^X((\pi_X^*\gamma)\cap J_d^{X,H}(t_1,\ldots,t_m,t))$ where γ is an *arbitrary* cohomology class. This argument does, however, capture the main idea of the proof.

We now prove the following by induction on $(m+1,d)$:

Claim 1.

(a) For all $\gamma_1,\ldots,\gamma_m \in R_H$ and $\alpha \in R_H^{\perp}$,

$$\sum_{d(\beta)=d}\langle\gamma_1\psi^{a_1},\ldots,\gamma_m\psi^{a_m},\alpha\psi^{a}\rangle_\beta^X = 0.$$

(b) For all $\gamma_1,\ldots,\gamma_m \in R_H$, the invariants

$$\sum_{d(\beta)=d}\langle\gamma_1\psi^{a_1},\ldots,\gamma_m\psi^{a_m},\gamma_{m+1}\psi^{a_{m+1}}\rangle_\beta^X$$

are determined by the one-point invariants and the intersection matrix on R_H.

In terms of J-functions (using the symmetry), this claim is equivalent to

Claim 2.

(a) If $\gamma_1,\ldots,\gamma_m \in R_H$ and $\alpha \in R_H^{\perp}$ then

$$\deg\left((\pi_1^*\alpha\cup\pi_2^*\gamma_1\cup\cdots\cup\pi_{m+1}^*\gamma_m)\cap J_d^{X,H}(t_1,\ldots,t_m,t)\right) = 0.$$

(b) For all $\gamma_1,\ldots,\gamma_m \in R_H$,

$$\deg\left((\pi_1^*\gamma_1\cup\pi_2^*\gamma_2\cup\cdots\cup\pi_{m+1}^*\gamma_{m+1})\cap J_d^{X,H}(t_1,\ldots,t_m,t)\right)$$

is determined by one-point invariants and the intersection matrix on R_H.

To start our induction, note that the claim holds for $m = 0$ by assumption. Also, the claim holds for $d = 0$:

$$\deg((\pi_1^*\alpha \cup \pi_2^*\gamma) \cap J_0^{X,H}(t_1, t_2)) = \frac{1}{t_1 t_2(t_1 + t_2)} \int_X \alpha \cup \gamma = 0$$

by orthogonality, and

$$\deg((\pi_1^*\gamma_1 \cup \pi_2^*\gamma_2) \cap J_0^{X,H}(t_1, t_2)) = \frac{1}{t_1 t_2(t_1 + t_2)} \int_X \gamma_1 \cup \gamma_2$$

and hence is determined by the intersection matrix on R_H.

Using the argument at the beginning of this proof, the vanishing in Claim 2(a) will follow by induction (on the power of t^{-1}), once we establish vanishing for all expressions of the form

$$I_a := \deg\left((\pi_1^*\alpha \cup \pi_2^*\gamma_1 \cup \cdots \cup \pi_m^*\gamma_{m-1}) \cap \left(J_{d-e}^{X,H}(\vec{t}_S, t) \otimes_{\prod(H_i - e_i t)^{b_i}} J_e^{X,H}(-t, \vec{t}_{S^c})\right)\right)$$

and

$$I_b := \deg\left((\pi_1^*\alpha \cup \pi_2^*\gamma_1 \cup \cdots \cup \pi_m^*\gamma_{m-1}) \cap \left(J_d^{X,H}(\vec{t}_{\hat{j}}, t_j) \otimes_{\prod H_i^{b_i}} J_0^{X,H}(-t_j, t)\right)\right).$$

But these expressions may be rewritten:

$$I_b = \deg\left(\left(\pi_1^*\alpha \cup \pi_2^*\gamma_1 \cup \cdots \cup \pi_m^*(\gamma_{m-1} \cup \prod H_i^{b_i})\right) \cap J_d^{X,H}(\vec{t}_{\hat{j}}, t_j)\right).$$

To rewrite I_a, choose an orthogonal basis $\lambda_j, \alpha_l \in \mathrm{H}^*(X, \mathbf{Q})$ such that $\lambda_j \in R_H$ with intersection matrix $g_{jj'}$ and $\alpha_l \in R_H^\perp$ with intersection matrix $h_{ll'}$. Then

$$\begin{aligned} I_a = \sum_{j,j'} &\deg\left(\left(\pi_1^*\alpha \cup \cdots \cup \pi_{k+1}^*(\prod(H_i - e_i t)^{b_i} \cup \lambda_j)\right) \cap J_{d-e}^{X,H}(\vec{t}_S, t)\right) g^{jj'} \\ &\deg\left(\left(\pi_1^*(\lambda_{j'}) \cup \pi_2^*\gamma_k \cup \cdots \cup \pi_{m-k+1}^*\gamma_{m-1}\right) \cap J_e^{X,H}(-t, \vec{t}_{S^c})\right) \\ + \sum_{l,l'} &\deg\left(\left(\pi_1^*\alpha \cup \cdots \cup \pi_{k+1}^*(\prod(H_i - e_i t)^{b_i} \cup \alpha_l)\right) \cap J_{d-e}^{X,H}(\vec{t}_S, t)\right) h^{ll'} \\ &\deg\left(\left(\pi_1^*(\alpha_{l'}) \cup \pi_2^*\gamma_k \cup \cdots \cup \pi_{m-k+1}^*\gamma_{m-1}\right) \cap J_e^{X,H}(-t, \vec{t}_{S^c})\right). \end{aligned}$$

Now suppose Claim 2(a) holds for all $(n+1, e)$ such that either $n < m$ or $n = m$ and $e \prec d$. Then $I_b = 0$ (taking $n = m - 1$), and $I_a = 0$ since the first factors in the first double sum and the second factors in the second sum vanish. This proves Claim 1(a) by induction. Similarly, assuming Claim 1(a), we prove 1(b) by induction, noting that in this case, the second double sum in I_a (but not the first) vanishes. The first double sum and the I_b terms are explicitly determined by the intersection matrix $g_{jj'}$ and Gromov-Witten invariants for lower $(n+1, e)$. □

Appendix A. Small quantum product for complete intersections

We may turn Formula 1.1 into an algorithm for producing structure constants for the small quantum product on Fano complete intersections in $\mathbf{P}^n$.

Given the type, $(l_1, \ldots, l_m)$ of the Fano complete intersection $S \subset \mathbf{P}^n$, set

$$f := n + 1 - l_1 - \cdots - l_m$$

the Fano index of S, and

$$d_{\max} := \left\lfloor \frac{n-m+1}{f} \right\rfloor ,$$

the maximal degree d for which nonzero "unmixed" 2-point invariants $\langle H^a, H^b \rangle_d^X$ may occur (by a dimension count).

For $d = 1,\ldots,d_{\max}$, let $v(d)$ be the vector of one-point invariants, i.e., $v(d)$ is defined by

$$e_* \left(\frac{[\overline{M}_{0,1}(X,d)]^{\mathrm{vir}}}{t(t-\psi)} \right) = v(d)_0 t^{-f} + v(d)_1 H t^{-f-1} + \cdots + v(d)_{n-m} H^{n-m} t^{-f-n+m}.$$

(These are computed by Givental's formulas, Theorem 1.3.)

We define shift matrices of size $(n-m+1) \times (n-m+1)$:

$$S(d) := \begin{pmatrix} d & 0 & \ldots & 0 & 0 \\ 1 & d & \ldots & 0 & 0 \\ & & \vdots & & \\ 0 & 0 & \ldots & d & 0 \\ 0 & 0 & \ldots & 1 & d \end{pmatrix}.$$

Applying $S(d)$ to a vector corresponds to multiplication by $(H + dt)$.

We define the matrices of mixed invariants, also of size $(n-m+1) \times (n-m+1)$, indexed from 0 to $n-m$:

$$M(d)_{n-m-a,b} := \frac{(-1)^{c-1}}{\prod l_i} \langle H^a, H^b \psi^c \rangle_d^X ,$$

where $c := df + n - m - a - b$. This is the matrix associated to the operator

$$H^b \mapsto e_{1*} \left(\frac{e_2^* H^b \cap [\overline{M}_{0,2}(X,d)]^{\mathrm{vir}}}{-t-\psi} \right).$$

It is important to note that $M(d)_{n-m-a,b} = 0$ when $c < 0$.

In terms of these data structures, our formula becomes a recursive formula for the bth column of $M(d)$ in terms of the lower $M(e)$'s:

$$M(d)_{*,b} = -S(d)^b v(d) - \sum_{e=1}^{d-1} M(d-e) S(e)^b v(e),$$

except that we must set $M(d)_{n-m-a,b} = 0$ whenever $c < 0$. This amounts to truncating $M(d)$ at the upper right corner.

Finally, reading off all coefficients of $M(d)$ with $c = 0$ yields the complete list of "unmixed" two-point invariants, which in turn yield the structure constants of thesmall quantum product (via the associativity).

This algorithm is very easy to implement. For example, when X is a quintic hypersurface in $\mathbf{P}^6$, it gives the following products:

$$
\begin{array}{lcllllll}
H * H & = & H^2 & + & 120q & & & \\
H * H^2 & = & H^3 & + & 770qH & & & \\
H * H^3 & = & H^4 & + & 1345qH^2 & + & 211,200q^2 & \\
H * H^4 & = & H^5 & + & 770qH^3 & + & 692{,}500q^2H & \\
H * H^5 & = & & & 120qH^4 & + & 211{,}200q^2H^2 & + \quad 31{,}320{,}000q^3
\end{array}
$$

As a typical application, note that the last number implies the following interesting bit of enumerative geometry:

Corollary A.1. *The expected number of twisted cubics through two general points of of a quintic five-fold $X \subset \mathbf{P}^6$ is:*

$$2{,}088{,}000.$$

One similarly may produce the expected numbers of rational normal curves of degree d passing through 2 general points of a hypersurface of degree $2d-1$ in $\mathbf{P}^{2d}$ for any d.

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