Another way to enumerate rational curves with torus actions

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1. Introduction. In 1991 Candelas, de la Ossa, Green and Parkes used the "string-theoretic" principle of mirror symmetry to predict the numbers of rational curves of any degree on a quintic three-fold [7]. This prediction took mathematicians by surprise, as the best results at the time only counted rational curves of degree three or less. Since then, exciting new developments have led to mathematical proofs of these predictions and many other related conjectures coming from string theory. While this does not address the deeper problem of constructing a mathematical foundation for string theory, it does represent a major advance in field of enumerative algebraic geometry.

An enumerative question is usually interpreted in terms of intersection theory on a moduli space. Moduli spaces of stable maps were introduced by Kontsevich and Manin [19] on which to calculate Gromov-Witten invariants, including the expected numbers of rational curves on a quintic three-fold. Like the spaces of stable pointed curves (which are stable maps to a point), the boundary of a stable map space has a self-similar property that has been exploited in many of the important recent results in enumerative geometry, including the associativity of the quantum cohomology rings [19, 25] and the reconstruction theorems [8, 19] for genus zero invariants.

Any torus action on the target space is inherited by all spaces of stable maps. Kontsevich applied the Bott residue theorem for such torus actions in his *Enumeration of rational curves with torus actions* [18] to the problem of computing Gromov-Witten invariants of rational curves, and although his approach would in principle compute the numbers of rational curves of all degrees on the quintic, the combinatorics becomes unmanageable after degree four. Givental seems to have been the first to fully appreciate the significance of the more subtle circle action induced from the **domain** of a "parametrized" stable map. An application of techniques from equivariant cohomology to both actions led to Givental's proofs [6, 12, 13, 14, 24] of the mirror conjecture for Fano and Calabi-Yau complete intersections in toric varieties, as well as the subsequent proofs and generalizations of Kim [17] and Lian-Liu-Yau [21, 22]. In particular, these verify the predicted numbers of rational curves on the quintic three-fold.

In this paper we will produce a new and direct computation of the one-point Gromov-Witten invariants for rational curves on Fano and Calabi-Yau complete interesections in complex projective space. This proof is self-contained and much simpler than previous proofs. The idea is to use the self-similar properties of the boundary of stable map space within the context of equivariant cohomology to find a self-similar decomposition of the relevant equivariant "virtual" classes. This new idea allows us to dispense entirely with torus actions inherited from the target manifold and instead focus on the more important circle action inherited from the parametrization of the genus zero curve. This proof has the additional advantage in that it gives a computation that does not rely on "already knowing the answer," as is the case in all previous proofs of the mirror conjecture. Besides being a psychological advantage, our proof thereby generalizes without modification to give a relative version of the mirror conjecture for projective bundles.

The mirror conjecture is best expressed in terms of one-point invariants. The space of stable genus-zero one-pointed maps of degree d to $\mathbf{P}^n$ is denoted by $\overline{M}_{0,1}(\mathbf{P}^n, d)$ and comes equipped with evaluation and projection maps:

$$e : \overline{M}_{0,1}(\mathbf{P}^n, d) \to \mathbf{P}^n \text{ and } \pi : \overline{M}_{0,1}(\mathbf{P}^n, d) \to \overline{M}_{0,0}(\mathbf{P}^n, d)$$

where $\overline{M}_{0,0}(\mathbf{P}^n, d)$ is the space of stable zero-pointed maps. These spaces are always projective orbifolds and when $d = 1$ the maps are the two projections:

$$e : \mathrm{Fl}(1, 2, n+1) \to \mathbf{P}^n \text{ and } \pi : \mathrm{Fl}(1, 2, n+1) \to \mathrm{G}(2, n+1)$$

from the partial flag manifold.

If $S \subset \mathbf{P}^n$ is a transverse zero section of a vector bundle E which is generated by global sections (e.g. a complete intersection), then S defines a "virtual" class:

$$[S]_d := \pi^* c_r(\pi_* e^* E)$$

on $\overline{M}_{0,1}(\mathbf{P}^n, d)$, where r is the rank of $\pi_* e^* E$. This measures the expected constraint imposed by requiring the stable map to land in S. Even simpler, given a variety $V \subset \mathbf{P}^n$ with associated cohomology class $[V]$, the pull-back:

$$e^*[V]$$

is the constraint imposed by requiring the marked point to land in V. As opposed to the previous two "enumerative" classes, Witten's cotangent class:

$$\psi := c_1(\omega_\pi)$$

(the first chern class of the relative dualizing sheaf for the projection π) is of less enumerative significance, but very useful for computations.

The mirror conjecture for $\mathbf{P}^n$ (as formulated by Givental and generalized to homogeneous spaces by Kim [17]) says that if S is a Fano or Calabi-Yau complete intersection in $\mathbf{P}^n$ of type $(l_1, ..., l_m)$, then the Laurent polynomials:

$$e_*\left(\frac{[S]_d}{t(t-\psi)}\right) := t^{-2}e_*([S]_d) + t^{-3}e_*([S]_d \cup \psi) + t^{-4}e_*([S]_d \cup \psi^2) + ...$$

with coefficients in $\mathrm{H}^*(\mathbf{P}^n, \mathbf{Q})$ are closely related to the rational functions (also expressible as Laurent polynomials):

$$\phi_d(t,h) := \frac{\prod_{i=1}^m \prod_{k=0}^{dl_i}(l_i h + kt)}{\prod_{k=1}^d (h+kt)^{n+1}}$$

where $h \in \mathrm{H}^*(\mathbf{P}^n, \mathbf{Z})$ is the hyperplane class.

More precisely, the relationship depends on the positivity of S as follows:

Fano of Index Two or More: If $l_1 + ... + l_m < n$, then:

$$e_*\left(\frac{[S]_d}{t(t-\psi)}\right) = \phi_d$$

Fano of Index One: If $l_1 + ... + l_m = n$, then:

$$e_*\left(\frac{[S]_d}{t(t-\psi)}\right) = \sum_{r=0}^{d} \frac{(-\prod l_i!)^r \phi_{d-r}}{r! t^r}$$

Calabi-Yau: If $l_1 + ... + l_m = n+1$, then there exist power series:

$$f(q) = \sum_{d=1}^{\infty} a_d q^d \quad \text{and} \quad g(q) = \sum_{d=1}^{\infty} b_d q^d$$

with constant coefficients such that the two power series:

$$\Phi(q) = \sum_{d=0}^{\infty} \phi_d q^d \quad \text{and} \quad \Sigma(q) = [S] + \sum_{d=1}^{\infty} e_*\left(\frac{[S]_d}{t(t-\psi)}\right) q^d$$

(with Laurent polynomial coefficients) satisfy:

$$\Sigma(q) = e^{\frac{h}{t}f(q)+g(q)}\Phi(qe^{f(q)})$$

If we expand the right side, we get:

$$e^{\frac{h}{t}f(q)+g(q)}\Phi(qe^{f(q)}) = \sum_{d=0}^{\infty}\phi_d(q^d e^{(d+\frac{h}{t})f(q)+g(q)})$$

$$= \sum_d \sum_{0\le d_1<d_2<\ldots<d_{r+1}=d} \frac{\phi_{d_1}\prod_{i=1}^r(a_{d_{i+1}-d_i}(d_1+\frac{h}{t})+b_{d_{i+1}-d_i})}{r!}q^d$$

so that the mirror conjecture is equivalent to the existence of constants a_d and b_d for $d = 1, ..., \infty$ such that:

$$e_*\left(\frac{[S]_d}{t(t-\psi)}\right) = \sum_{0\le d_1<d_2<\ldots<d_{r+1}=d} \frac{\phi_{d_1}\prod_{i=1}^r(a_{d_{i+1}-d_i}(d_1+\frac{h}{t})+b_{d_{i+1}-d_i})}{r!}$$

All the previous proofs proceed by finding sufficient conditions shared by the power series $\Sigma(q)$ and the change of variables of the power series $\Phi(q)$ to uniquely characterize them, hence to conclude that they are the same. It is in this sense that one needs to "know the answer" beforehand. Our proof is completely different. We will explicitly produce the coefficients of the power series one coefficient at a time. Out of the proof it becomes clear how the self-similar properties of the coefficients of the $\Sigma(q)$ in the Calabi-Yau and Fano of index one cases are a reflection of the self-similar properties of the boundary of the moduli spaces of stable maps.

Acknowledgements: This paper grew out of a seminar in the summer of 1998 co-organized with Holger Kley during which Chris Hacon and Herb Clemens made many useful suggestions. The use of MacPherson's graph construction in Lemma 4.4 was suggested by Bill Fulton. Alexander Givental was very helpful when I reported on the Fano version of the result at MSRI in December 1998, and the key step in the Calabi-Yau proof arose out of a conversation with Ravi Vakil. Rahul Pandharipande, Pavel Etinghof and especially Bumsig Kim helped relate my original Calabi-Yau formula to the mirror conjecture and Bumsig Kim and Bong Lian made some useful remarks on an earlier version of the paper. Holger Kley very helpful with a critical revision of the paper and I'd like to thank Michael Thaddeus for explaining the basics of equivariant cohomology one sunny afternoon in Park City.

Note: A very recent preprint of Gathmann [11] explains how one might compute rational Gromov-Witten invariants of hypersurfaces without any torus actions. This seems promising, particularly in the general-type case.

2. Genus Zero Gromov-Witten Invariants on Projective Space.

Definition: If $(C; p_1, ..., p_k)$ is a pointed genus-zero curve, then $f : C \to \mathbf{P}^n$ is stable if:

(a) C has only nodes for singularities and the p_i are nonsingular points.

(b) Every component of C which is collapsed by f has at least three distinguished points (a point is distinguished if it is either a node or marked).

Existence Theorem (see [19],[1]): The moduli problem for isomorphism classes of flat families of genus-zero k-pointed stable maps of degree d to $\mathbf{P}^n$ is represented by a projective orbifold. This moduli space is denoted by $\overline{M}_{0,k}(\mathbf{P}^n, d)$ with universal curve $\mathcal{C} \cong \overline{M}_{0,k+1}(\mathbf{P}^n, d)$ and maps:

$$\overline{M}_{0,k}(\mathbf{P}^n, d) \stackrel{\pi}{\leftarrow} \mathcal{C} \stackrel{e}{\to} \mathbf{P}^n$$

For each $1 \leq i \leq k$, if σ_i is the section of π determined p_i, then $e_i := e \circ \sigma_i$ is the evaluation of a stable map at the marked point p_i.

The relevant Chern classes on $\overline{M}_{0,k}(\mathbf{P}^n, d)$ come from three sources.

• The pull-back classes $e_i^*(h^b)$

• Witten's cotangent classes $\psi_i := c_1(N_{\sigma_i}^*)$ for $i = 1, ..., k$. When $k = 1$, then $\psi := \psi_1$ is the relative canonical class $c_1(\omega_\pi)$.

• The top chern class $c_r(\pi_* e^*(E))$ whenever E is a vector bundle on $\mathbf{P}^n$ which is generated by its global sections. If $S \subset \mathbf{P}^n$ is the zero locus of a general section of E, then this chern class will be denoted by $[S]_d$ as before.

The classes $[S]_d$ are all pulled back from $\overline{M}_{0,0}(\mathbf{P}^n, d)$, leading to relations:

Example: If $S \subset \mathbf{P}^4$ is a quintic hypersurface, then the "physical" number of rational curves of degree d on S is given by:

$$\int_{\overline{M}_{0,0}(\mathbf{P}^4,d)} [S]_d = \frac{1}{d} \int_{\overline{M}_{0,1}(\mathbf{P}^4,d)} e^*(h) \cup [S]_d = \frac{1}{d} \int_{\mathbf{P}^4} e_*[S]_d \cup h$$

so that the cohomology classes $e_*[S]_d$ determine all the "physical" numbers of rational curves on the quintic (we will mostly ignore the subtleties of multiple covers in this paper).

The general k-point genus zero Gromov-Witten invariant of a complete intersection $S \subset \mathbf{P}^n$ of type $(l_1, ..., l_m)$ is an intersection number of the form:

$$\int_{\overline{M}_{0,k}(\mathbf{P}^n, d)} p(e_i^*(h), \psi_i) \cup [S]_d$$

where $p(x_i, y_i)$ is a polynomial (or power series) in $2k$ variables.

Notice that in contrast to zero-point invariants, there are interesting one-point invariants on **any** complete intersection (because of the ψ's), including projective space itself.

Definition: Let $(C \supset \mathbf{P}^1; p_1, ..., p_k)$ be a genus-zero curve with k marked points and a distinguished parametrized component $\mathbf{P}^1 \subset C$. Then a map $f : C \to \mathbf{P}^n$ is stable if:

(a) C has only nodes for singularities and the p_i are nonsingular points.

(b) Every component of C which is collapsed by f is either distinguished or else has at least three distinguished points.

The existence theorem applies in the context of stable maps with a parametrized component. Following Givental, we'll call these moduli spaces the "graph spaces" and denote them by $\overline{N}_{0,k}(\mathbf{P}^n, d)$. In addition to the maps of the theorem, the universal curve over the graph space admits an evaluation map $\epsilon : \mathcal{C} \to \mathbf{P}^1$ leading to corresponding evaluation maps at the points. The zero-pointed graph space is rational via an explicit birational morphism:

$$u : \overline{N}_{0,0}(\mathbf{P}^n, d) \to \mathbf{P}_d^n := \mathbf{P}^{(n+1)(d+1)-1}$$

defined as follows. The general point of $\overline{N}_{0,0}(\mathbf{P}^n, d)$ is represented by a map $f : \mathbf{P}^1 \to \mathbf{P}^n$ of degree d. Such a map is given by $n+1$ polynomials of degree d and thus represents a point in the projective space $\mathbf{P}_d^n$. Boundary points of the graph space are represented by maps $f : C \to \mathbf{P}^n$ where C has several components, one of which is $\mathbf{P}^1$. But if $q_1, ..., q_k \in \mathbf{P}^1$ are the nodes of C on $\mathbf{P}^1$, and if the curve C_i growing out of q_i is mapped to $\mathbf{P}^n$ with degree d_i, then f maps $\mathbf{P}^1$ to $\mathbf{P}^n$ with degree $d - \sum d_i$. Letting this map be given by polynomials p_i, we get a well-defined point of $\mathbf{P}_d^n$ by choosing linear forms $q_i(x, y)$ dual to the points q_i and sending the stable map f to:

$$(\prod_{i=1}^{k} q_i(x,y)^{d_i} p_0(x,y) : .... : \prod_{i=1}^{k} q_i(x,y)^{d_i} p_n(x,y))$$

(It is easy to exhibit u as a morphism. See Jun Li's argument in [21]) There is an inclusion:

$$i : \overline{M}_{0,1}(\mathbf{P}^n, d) \hookrightarrow \overline{N}_{0,0}(\mathbf{P}^n, d)$$

defined as follows. Given a stable map $f : C \to \mathbf{P}^n$ with one marked point $p \in C$, construct a new curve $C' = C \cup \mathbf{P}^1$ by joining $\mathbf{P}^1$ and C at $p \in C$ and $0 \in \mathbf{P}^1$. Extend f to a map $f' : C' \to \mathbf{P}^n$ by collapsing the $\mathbf{P}^1$ component to the image point $f(p)$. Then there is a:

Basic Diagram:

$$\begin{array}{ccc} \overline{N}_{0,0}(\mathbf{P}^n, d) & \stackrel{u}{\to} & \mathbf{P}^n_d \\ i \uparrow & & j \uparrow \\ \overline{M}_{0,1}(\mathbf{P}^n, d) & \stackrel{e}{\to} & \mathbf{P}^n \end{array}$$

where $j(a_0 : ... : a_n) = (a_0 x^d : ... : a_n x^d)$.

3. Equivariant Basics. Let X be a compact complex manifold (or orbifold) equipped with a $\mathbf{C}^*$ action. (We will let $T = \mathbf{C}^*$.) Then:

$$ET := \mathbf{C}^{\infty+1} - \{0\} \to \mathbf{CP}^\infty =: BT$$

is the "universal" principal $\mathbf{C}^*$ bundle, which we use to construct:

$$\pi_X : X_T := X \times_{\mathbf{C}^*} ET \to ET/\mathbf{C}^* = BT$$

Following Borel, we define the equivariant cohomology by setting:

$$H^*_T(X, \mathbf{Q}) := H^*(X_T, \mathbf{Q})$$

which is a module via π_X^* over $H^*(BT, \mathbf{Q}) \cong \mathbf{Q}[t]$. At one extreme, if the T action were trivial, this would be the tensor product $H^*(X, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}[t]$ while at the other extreme if the action were free, it would be the cohomology ring of the quotient $H^*(X/T, \mathbf{Q})$, which is torsion as a module over $\mathbf{Q}[t]$.

A vector bundle E over X is linearized if it is equipped with an action of T which is lifted from the action on X so that:

$$E_T \to X_T$$

is a vector bundle which pulls back to $E \to X$ on each of the fibers of X_T over BT. As such, its chern classes represent elements of the equivariant cohomology ring, and one defines:

$$c_i^T(E) := c_i(E_T) \in H_T^*(X, \mathbf{Q})$$

If $f : X \to X'$ is a T-equivariant morphism of compact complex manifolds with T actions, let $f : X_T \to X'_T$ also denote the induced map on these spaces (which commutes with the projections to BT). The pull-back on equivariant cohomology is the ordinary pull-back with respect to the induced map f:

$$f^* : H_T^*(X', \mathbf{Q}) \to H_T^*(X, \mathbf{Q})$$

and if f is proper, then the equivariant proper push-forward is defined in the same way. We may now state:

The Atiyah-Bott Localization Theorem: Let X be a compact complex manifold with a T-action, let $F_1, ..., F_n \subset X$ be the (necessarily smooth) connected components of the fixed-point locus and let $i_k : F_k \hookrightarrow X$ denote their embeddings. Each normal bundle to F_k is canonically linearized, its top equivariant chern class $\epsilon^T(N_{F_k/X})$ is invertible in $H^*(F_k, \mathbf{Q})[t, t^{-1}]$, and every torsion-free element $c_T \in H_T^*(X, \mathbf{Q})$ uniquely decomposes in $H_T^*(X, \mathbf{Q}) \otimes \mathbf{Q}(t)$ as a sum of contributions from fixed-point loci:

$$c_T = \sum_{k=1}^{n} (i_k)_* \frac{i_k^*(c_T)}{\epsilon^T(N_{F_k/X})}$$

Remark: The localization theorem as stated here and proved in [4] only applies to compact complex manifolds X. Generalizations to the case where X is an orbifold may be found in the papers of Graber-Pandharipande [15] and Kresch [20]. See [2] for a general reference on equivariant cohomology.

Corollary (Correspondence of Residues): Suppose $f : X \to Y$ is an equivariant map of compact complex manifolds (or orbifolds) with T actions and suppose $i : F \hookrightarrow X$ and $j : G \hookrightarrow Y$ are components of the fixed-point loci with the property that F is the unique fixed component to map to G. Then equivariant cohomology classes c_T on X satisfy:

$$(f|_F)_* \left(\frac{i^* c_T}{\epsilon^T(N_{F/X})} \right) = \frac{j^* f_* c_T}{\epsilon^T(N_{G/Y})}$$

Proof of the Corollary: (See also Lemma 2.1 in [22].) The free part of c_T is a push-forward of classes from the fixed-point loci of X, by the theorem. Since F is the only such locus to map to G, only its contribution survives under $j^* f_* c_T$ and under $i^* c_T$, so we may as well assume that $c_T = i_* b_T$ for $b_T \in H^*(F, \mathbf{Q})[t, t^{-1}]$. But then:

$$j^* f_* c_T = j^* f_* i_* b_T = j^* j_* (f|_F)_* b_T = \epsilon^T(N_{G/Y})(f|_F)_* b_T$$

whereas

$$b_T = \frac{i^* i_* b_T}{\epsilon^T(N_{F/X})} = \frac{i^* c_T}{\epsilon^T(N_{F/X})}$$

completing the proof.

Returning to the basic diagram of the previous section, note that the "standard" linearized action of $\mathbf{C}^*$ on $\mathbf{P}^1$:

$$\mu \cdot (a, b) \mapsto (a, \mu b)$$

induces actions on $\overline{N}_{0,0}(\mathbf{P}^n, d)$ and on $\mathbf{P}^n_d$ and that $u : \overline{N}_{0,0}(\mathbf{P}^n, d) \to \mathbf{P}^n_d$ is $\mathbf{C}^*$-equivariant. The vector bundles $\mathcal{O}_{\mathbf{P}^n_d}(1)$ and $\pi_* e^* E$ are equipped with natural linearizations, so we may define equivariant chern classes:

$$h^T := c_1^T(\mathcal{O}_{\mathbf{P}^n_d}(1)) \text{ and } [S]^T_d := c_r^T(\pi_* e^* E)$$

on the spaces $\mathbf{P}^n_d$ and $\overline{N}_{0,0}(\mathbf{P}^n, d)$ respectively. Then:

Proposition 3.1:

(1a) $\overline{M}_{0,1}(\mathbf{P}^n, d) \subset \overline{N}_{0,0}(\mathbf{P}^n, d)$ is a component of the fixed-point locus.

(1b) $[S]_d$ and $e^*(h)$ extend to the equivariant classes $[S]^T_d$ and $u^*(h^T)$.

(2) $\overline{M}_{0,1}(\mathbf{P}^n, d)$ is the only fixed component to map to $\mathbf{P}^n$.

(3) The equivariant Euler classes are:

$$\epsilon^T(N_{\overline{M}_{0,1}/\overline{N}_{0,0}}) = t(t - \psi) \text{ and } \epsilon^T(N_{\mathbf{P}^n/\mathbf{P}^n_d}) = \prod_{k=1}^{d} (h + kt)^{n+1}$$

Proof: (1a,b) and (2) are immediate. (For a careful treatment, see [6].) As for (3), there are inclusions:

$$\overline{M}_{0,1}(\mathbf{P}^n, d) \subset \overline{M}_{0,1}(\mathbf{P}^n, d) \times \mathbf{P}^1 \subset \overline{N}_{0,0}(\mathbf{P}^n, d)$$

The (equivariant!) first chern class of the normal bundle to the first inclusion is clearly t, and the second is $c_1(T_{\mathbf{P}^1}) - \psi$, which restricts to $t - \psi$ on the fixed-point locus $\overline{M}_{0,1}(\mathbf{P}^n, d)$. This gives one Euler class computation. The second Euler class computation follows from the Euler sequences for the tangent bundles to $\mathbf{P}^n$ and $\mathbf{P}^n_d$ (see §7 for a generalization).

The correspondence of residues immediately(!) therefore gives us:

$$e_*\left(\frac{[S]_d}{t(t-\psi)}\right) = \frac{j^* u_* [S]_d^T}{\prod_{k=1}^d (h+kt)^{n+1}}$$

Notice that we get **every** one-point Gromov-Witten invariant associated to S in this way by expanding $t(t-\psi) = t^{-2} + t^{-3}\psi + t^{-4}\psi^2 + ...$ so that:

$$\int_{\overline{M}_{0,1}(\mathbf{P}^n, d)} \psi^a \cup e^*(h^b) \cup [S]_d = \text{ coeff of } t^{-2-a} \text{ in } \int_{\mathbf{P}^n} \frac{h^b \cup j^* u_* [S]_d^T}{\prod_{k=1}^d (h+kt)^{n+1}}$$

Thus our challenge is the computation of:

$$j^* u_* [S]_d^T \in H^*(\mathbf{P}^n, \mathbf{Q}) \otimes \mathbf{Q}[t]$$

In one case, this is easy. Since $j^* u_* 1 = 1$, we have:

One-Point Invariants of Projective Space:

$$e_*\left(\frac{1}{t(t-\psi)}\right) = \frac{1}{\prod_{k=1}^d (h+kt)^{n+1}}$$

4. The Fano Cases: It is difficult to see how to compute $j^* u_* [S]_d^T$ when E is a non-split bundle. In this section we will develop techniques for making the computation when E is a direct sum of line bundles, and use these techniques to prove the Fano cases of the mirror conjecture.

Let $D \subset \overline{N}_{0,0}(\mathbf{P}^n, d)$ be the exceptional divisor for $u : \overline{N}_{0,0}(\mathbf{P}^n, d) \to \mathbf{P}^n_d$. This is a divisor with normal crossings which we will eventually describe in detail, but first observe that there is a birational map

$$\overline{N}_{0,1}(\mathbf{P}^n, d) \to \mathbf{P}^1 \times \overline{N}_{0,0}(\mathbf{P}^n, d)$$

and that the exceptional divisor for this map lies over D. Then:

Proposition 4.1: There is an equivariant map:

$$\Phi : \pi_* e^*(\mathcal{O}_{\mathbf{P}^n}(l)) \to \ \mathrm{Sym}^{dl} W^* \otimes u^* \mathcal{O}_{\mathbf{P}^n_d}(l)$$

of vector bundles on $\overline{N}_{0,0}(\mathbf{P}^n, d)$ which is an isomorphism when restricted to the complement of the boundary divisor D.

Proof: For $W \cong \mathbf{C}^2$ and $V \cong \mathbf{C}^{n+1}$, the map v below:

$$\overline{N}_{0,1}(\mathbf{P}^n, d) \stackrel{v}{\to} \mathbf{P}(W) \times \mathbf{P}(\mathrm{Sym}^d W^* \otimes V) - - > \mathbf{P}^n$$

resolves a rational map which is linear in the second factor and has degree d in the first. The composition is the evaluation map e.

Corollary 4.2: Let $f : D \hookrightarrow \overline{N}_{0,0}(\mathbf{P}^n, d)$ be the inclusion. Then:

$$u_*[S]^T_d = \prod_{i=1}^{m} \prod_{k=0}^{dl_i} (l_i h^T + kt) + u_* f_* c$$

for some equivariant class c supported on D.

Proof: This follows from the projection formula, the computation of the top equivariant chern class of $\mathrm{Sym}^{dl} W^* \otimes \mathcal{O}_{\mathbf{P}^n_d}(l)$ and an application of the proposition to the sum of vector bundles $\pi_* e^* \mathcal{O}_{\mathbf{P}^n}(l_i)$.

The Fano of Index 2 or More Case: If $l_1 + ... + l_m < n$ then:

$$e_* \left(\frac{[S]_d}{t(t-\psi)} \right) = \frac{j^* u_* [S]^T_d}{\prod_{k=1}^{d} (h+kt)^{n+1}} = \frac{\prod_{i=1}^{m} \prod_{k=0}^{dl_i} (l_i h + kt)}{\prod_{k=1}^{d} (h+kt)^{n+1}} =: \phi_d$$

i.e. the difference supported on D contributes nothing.

To prove this, we need to understand D better, as well as the behavior of the map Φ when restricted to D. The self-similar properties of both will lead to this formula and all the others. Our description will roughly follow Fulton-Pandharipande [10]:

Components of the boundary divisor: There are natural maps:

$$f_i : \widetilde{D}_i := \overline{N}_{0,1}(\mathbf{P}^n, d-i) \times_{\mathbf{P}^n} \overline{M}_{0,1}(\mathbf{P}^n, i) \to \overline{N}_{0,0}(\mathbf{P}^n, d)$$

that desingularize the d components of D. The map u pulls back as follows:

$$\begin{array}{ccc} \overline{N}_{0,0}(\mathbf{P}^n, d) & \stackrel{u}{\to} & \mathbf{P}^n_d \\ f_i \uparrow & & j_i \uparrow \\ \overline{N}_{0,1}(\mathbf{P}^n, d-i) \times_{\mathbf{P}^n} \overline{M}_{0,1}(\mathbf{P}^n, i) & \stackrel{\pi_i}{\to} & \mathbf{P}^1 \times \mathbf{P}^n_{d-i} \end{array}$$

Here π_i is the projection $\rho_i : \widetilde{D}_i \to \overline{N}_{0,1}(\mathbf{P}^n, d-i)$ followed by the map v defined in Proposition 4.1, and j_i is the "multi-linear" map:

$$\mathbf{P}(W) \times \mathbf{P}(\mathrm{Sym}^{d-i} W^* \otimes V) \to \mathbf{P}(\mathrm{Sym}^d W^* \otimes V)$$

(we identify $\mathbf{P}(W)$ with $\mathbf{P}(W^*)$ via the canonical $W \cong W^* \otimes \wedge^2 W$).

A stratification of the graph space: We partially stratify the boundary of $\overline{N}_{0,0}(\mathbf{P}^n, d)$ in terms of "comb" types, i.e. sequences μ of the form:

$$0 \leq d_1 < d_2 < ... < d_r < d_{r+1} = d$$

We define $\Delta_i := d_{i+1} - d_i$ and

$$\widetilde{D}_\mu := \overline{N}_{0,r}(\mathbf{P}^n, d_1) \times_{(\mathbf{P}^n)^r} \prod_{i=1}^{r} \overline{M}_{0,1}(\mathbf{P}^n, \Delta_i)$$

and note that the natural finite map $f_\mu : \widetilde{D}_\mu \to D_\mu \subset \overline{N}_{0,0}(\mathbf{P}^n, d)$ is a "desingularization" of its image D_μ. The general point of D_μ is a stable map of degree d_1 on the parametrized component and Δ_i on r other components (the "teeth" of the comb) each of which meets the parametrized component. Different comb types with the same r and d_1 and the same sets of Δ_i's will share the same image, and the map:

$$\coprod_\mu \widetilde{D}_\mu \to D_\mu$$

from the union over all comb types with image D_μ has degree $r!$, the order of the permutation group of the set of Δ_i's. The map u pulls back to $\widetilde{D}_\mu$ according to:

$$\begin{array}{ccc} \overline{N}_{0,0}(\mathbf{P}^n, d) & \stackrel{u}{\to} & \mathbf{P}^n_d \\ f_\mu \uparrow & & j_\mu \uparrow \\ \widetilde{D}_\mu & \stackrel{\pi_\mu}{\to} & (\mathbf{P}^1)^r \times \mathbf{P}^n_{d_1} \end{array}$$

Here $\rho_\mu : \widetilde{D}_\mu \to \overline{N}_{0,r}(\mathbf{P}^n, d_1)$ and π_μ and j_μ are defined as before, with the obvious generalized map $v : \overline{N}_{0,r}(\mathbf{P}^n, d) \to (\mathbf{P}^1)^r \times \mathbf{P}^n_d$.

The following two lemmas are the heart of the Fano proof.

Lemma 4.3: The fixed-point locus $j : \mathbf{P}^n \hookrightarrow \mathbf{P}^n_d$ factors uniquely through:

$$j : \mathbf{P}^n \stackrel{j'_\mu}{\to} (\mathbf{P}^1)^r \times \mathbf{P}^n_{d_1} \stackrel{j_\mu}{\to} \mathbf{P}^n_d$$

and if b is an equivariant cohomology class on $(\mathbf{P}^1)^r \times \mathbf{P}^n_{d_1}$, then:

$$j^*(j_\mu)_* b = (j'_\mu)^* b \cup \frac{\prod_{k=1}^{d}(h+kt)^{n+1}}{t^r \prod_{k=1}^{d_1}(h+kt)^{n+1}}$$

Proof: j'_μ maps $(0, ..., 0) \times \mathbf{P}^n \hookrightarrow (\mathbf{P}^1)^r \times \mathbf{P}^n \stackrel{(1,j)}{\longrightarrow} (\mathbf{P}^1)^r \times \mathbf{P}^n_{d_1}$

If j_μ were an embedding, then the second part of the lemma would be an immediate consequence of the excess intersection formula. Since it isn't, we will instead use the localization theorem. The components of the fixed-point locus of the $\mathbf{C}^*$ action on $\mathbf{P}^n_d$ are all copies of $\mathbf{P}^n$, embedded via Segre as:

$$(e0 + (d-e)\infty) \times \mathbf{P}^n \subset \mathbf{P}^d \times \mathbf{P}^n \subset \mathbf{P}^n_d = \mathbf{P}(\mathrm{Sym}^d W^* \otimes V)$$

and similarly, the components of the fixed-point locus in $(\mathbf{P}^1)^r \times \mathbf{P}^n_{d_1}$ are all copies of $\mathbf{P}^n$, embedded via:

$$s \times (e0 + (d_1 - e)\infty) \times \mathbf{P}^n \subset (\mathbf{P}^1)^r \times \mathbf{P}^n_{d_1}$$

where $s \in \{0, \infty\}^r$. Of the fixed-point loci in this latter space, there is thus only one that maps to the image of j, namely the image of j'_μ. By the localization theorem, an equivariant class on $(\mathbf{P}^1)^r \times \mathbf{P}^n_{d_1}$ is a sum of its contributions from fixed-point loci. The contribution from the image of j'_μ is easily computed to be ${j'_\mu}_* b'$ where:

$$b' := \frac{j'^*_\mu b}{t^r \prod_{k=1}^{d_1}(h+kt)^{n+1}}$$

But this is the unique component to map to the image of j, hence:

$$j^* j_{\mu *} b = j^* j_* b' = b' \cup \prod_{k=1}^{d}(h+kt)^{n+1}$$

Lemma 4.4: The equivariant virtual class $[S]_d^T$ decomposes as:

$$[S]_d^T = \sum_\mu \frac{1}{r!} f_{\mu *}(c_\mu \cup \pi_\mu^* \prod_{i=1}^{m} \prod_{k=0}^{d_1 l_i} (l_i h^T + kt))$$

where h^T is pulled back from the projection to $\mathbf{P}^n_{d_1}$ and the c_μ are equivariant cohomology classes which will be explicitly described in the proof.

Proof: We'll prove this first when S is a hypersurface of degree l.

MacPherson's graph construction [9, 23] gives a decomposition of the difference $[S]_d^T - u^* \prod_{k=0}^{ld}(lh^T + kt)$ as follows.

Let $E_d = \pi_* e^* \mathcal{O}(l)$, $F_d = \mathrm{Sym}^{dl} W^* \otimes u^* \mathcal{O}_{\mathbf{P}^n_d}(l)$, and let $G(dl+1, E_d \oplus F_d)$ be the Grassmann bundle. Consider the locus of scaled graphs:

$$\overline{N}_{0,0}(\mathbf{P}^n, d) \times \mathbf{A}^1 \subset G(dl+1, E_d \oplus F_d); \quad (x, \lambda) \mapsto \Gamma(\lambda \Phi_x)$$

where $\Gamma(\lambda\Phi_x) = \{(e, \lambda\Phi_x(e)) | e \in E_d(x)\}$, and let $V \subset G(dl+1, E_d \oplus F_d) \times \mathbf{P}^1$ be the closure of this locus. Let $V_\infty \subset G(dl+1, E_d \oplus F_d)$ be the fiber of V over $\{\infty\} \in \mathbf{P}^1$. One of the components of V_∞ is a (reduced) copy of $\overline{N}_{0,0}(\mathbf{P}^n, d)$ itself, embedded in the Grassmann bundle via the fibers of F_d, and all other components map to proper subvarieties of $\overline{N}_{0,0}(\mathbf{P}^n, d)$. If V_Z is such a component, surjecting onto $Z \subset \overline{N}_{0,0}(\mathbf{P}^n, d)$ via the projection map $\pi_{V_Z} : V_Z \to \overline{N}_{0,0}(\mathbf{P}^n, d)$, let n_{V_Z} be its multiplicity in V_∞. Then:

$$[S]_d^T - u^* \prod_{k=0}^{ld} (lh^T + kt) = \sum_{V_Z} n_{V_Z} \pi_{V_Z *} c_{dl+1}(T_{V_Z})$$

where T_{V_Z} is the pull-back of the tautological sub-bundle on $G(dl+1, E_d \oplus F_d)$ (see [9], Example 18.1.6).

Now notice that Φ is the push-forward of the inclusion:

$$e^* \mathcal{O}_{\mathbf{P}^n}(l) \cong L(-\sum_{i=1}^{d} ilC_i) \hookrightarrow L \cong v^* \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^n_d}(dl, l)$$

where $C_i \cong \overline{N}_{0,1}(\mathbf{P}^n, d-i) \times_{\mathbf{P}^n} \overline{M}_{0,2}(\mathbf{P}^n, i)$ is the unparametrized component of the universal curve over $D_i \subset D$.

Since the divisors C_i are exactly the exceptional divisors for the map $\overline{N}_{0,1}(\mathbf{P}^n, d) \to \mathbf{P}^1 \times \overline{N}_{0,0}(\mathbf{P}^n, d)$ it is clear that the sheaf inclusion holds with some negative coefficients of the C_i. The computation of the coefficients follows from the observation that the pull-back $f^*\mathcal{O}_{\mathbf{P}^n}(l)$ under a stable map $f : C \to \mathbf{P}^n$ corresponding to a general point of D_i has degree il on the unparametrized component of C.

Alternatively, as in the pointwise description of u, if we are given a stable map $f : C \to \mathbf{P}^n$ such that the parametrized component C_0 has degree d_1 and such that r curves "bubble" off this component with degrees $\Delta_1, ..., \Delta_r$ at nodes $q_1, ..., q_r \in C_0$, then Φ is the following linear map at the point of $\overline{N}_{0,0}(\mathbf{P}^n, d)$ corresponding to f:

$$H^0(C, f^*\mathcal{O}(l)) \to H^0(C_0, f^*\mathcal{O}(l)|_{C_0}) \cong H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d_1 l)) \subset H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(dl))$$

where the last inclusion comes from multiplying by the $\Delta_i l$th powers of the equations of the linear forms dual to the q_i. It follows that Φ enjoys the following pleasant properties:

(i) Φ successively drops rank in codimension one, always along the self-intersection strata of the boundary divisor D. Moreover, if such a stratum is not a D_μ (I call these other strata "hairy combs"), then it can be embedded in a comb stratum such that the generic corank of Φ is the same along the two strata.

(ii) Φ drops rank "transversally" along comb-type strata. That is, if Φ has generic coranks m and n along two comb-type strata which intersect transversally along a third comb-type stratum, then Φ has generic corank $m + n$ along the intersection.

Precisely, Φ factors as follows when it is pulled back to $\widetilde{D}_\mu$:

$$f_\mu^*\Phi : f_\mu^* E_d \to \rho_\mu^* E_{d_1} \xrightarrow{\rho_\mu^*\Phi} \rho_\mu^* F_{d_1} \subset f_\mu^* F_d$$

where E_{d_1}, F_{d_1} and Φ on $\overline{N}_{0,r}(\mathbf{P}^n, d_1)$ are pulled back from $\overline{N}_{0,0}(\mathbf{P}^n, d_1)$, and the map $\rho_\mu^* F_{d_1} \to f_\mu^* F_d$ is pulled back via π_μ^* from the projection to $(\mathbf{P}^1)^r \times \mathbf{P}^n_{d_1}$ of the sheaf inclusion given by multiplication by the $\Delta_i l$ powers of the equations of the appropriate diagonals on $\mathbf{P}^1 \times (\mathbf{P}^1)^r \times \mathbf{P}^n_{d_1}$:

$$\mathcal{O}(d_1 l, 0, ..., 0, l) \to \mathcal{O}(dl, \Delta_1 l, ..., \Delta_r l, l)$$

For boundary divisors, the kernel of $f_i^*\Phi$ is pulled back from the kernel of the evaluation map of bundles on $\overline{M}_{0,1}(\mathbf{P}^n, i)$:

$$0 \to E_i^1 \to \pi_* e^* \mathcal{O}_{\mathbf{P}^n}(l) \to e^* \mathcal{O}_{\mathbf{P}^n}(l) \to 0$$

via the projection $\tau_i : \widetilde{D}_i \to \overline{M}_{0,1}(\mathbf{P}^n, i)$. If $Z \subset \overline{M}_{0,1}(\mathbf{P}^n, i)$ is the image of the section σ_1, then E_i^1 is the first in a filtration of sub-bundles:

$$0 = E_i^{il+1} \subset E_i^{il} \subset ... \subset E_i^k = \pi_* \left(e^* \mathcal{O}_{\mathbf{P}^n}(l) \otimes \mathcal{O}(-kZ) \right) \subset ... \subset E_i^1 \subset E_i$$

which filter $\ker(f_i^*\Phi)$ according to higher-order behavior of Φ along D_i:

$$0 = \tau_i^* E_i^{il+1} \subset \cdots \subset \tau_i^* E_i^k = f_i^*(\ker(\Phi|_{kD_i})) \subset \cdots \subset \tau_i^* E_i^1 \subset f_i^* E_d$$

There is a filtration of $f_i^* F_d$ given by the spans of the images of $\Phi|_{kD_i}$:

$$\rho_i^* F_{d-i} = \pi_i^* F_{d-i}^1 \subset \cdots \subset \pi_i^* F_{d-i}^k \subset \cdots \subset f_i^* F_d$$

where F_{d-i}^k is the push-forward of $\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^n_{d-i}}((d-i)l + k - 1, k - 1, l)$.

From these descriptions of Φ along boundary divisors (and induction) we can conclude that infinitessimal versions of the pleasant properties (i) and (ii) also hold for Φ. Property (i) tells us immediately that any $Z \subset \overline{N}_{0,0}(\mathbf{P}^n, d)$ in the image of a component V_Z must be a comb-type boundary stratum, hence that this gives a decomposition of the form:

$$[S]_d^T = \sum_{D_\mu} \gamma_\mu$$

summed over the images (including $\overline{N}_{0,0}(\mathbf{P}^n, d)$) of the comb-types. But thanks to the filtrations of $f_i^* E_d$ and $f_i^* F_d$, we can be much more explicit. Namely, there are il components $V_{D_i}^k$ over each boundary divisor D_i, each of which is a birational image of a $\mathbf{P}^1$-bundle $V_{\widetilde{D}_i}^k$ on $\widetilde{D}_i$. By a local coordinate computation, $V_{D_i}^k$ has multiplicity k in V_∞ and tautological bundle fitting into an extension of the form:

$$0 \to \pi_i^* F_{d-i}^k \oplus \tau_i^* E_i^{k+1} \to T_{V_{\widetilde{D}_i}^k} \to \mathcal{O}(-1) \to 0$$

hence the codimension one contributions γ_i have the desired form:

$$\gamma_i = f_{i*}\left(c_i \cup c^T_{(d-i)l+1}(\pi_i^* F_{d-i}^1) \right) = f_{i*}\left(c_i \cup \pi_i^* \prod_{k=0}^{(d-i)l} (lh^T + kt) \right)$$

where $c_i = \sum_{k=1}^{il}(-k)\tau_i^* c^T_{il-k}(E_i^{k+1}) \cup \pi_i^* c^T_{k-1}(F^k_{d-i}/F^1_{d-i})$.

The second property of Φ allows us to construct the γ_μ inductively. Namely, if we fix a $V^k_{D_i}$ (or rather the corresponding $\mathbf{P}^1$-bundle over $\widetilde{D_i}$) and apply the MacPherson construction to the map:

$$f_i^* E_d/\tau_i^* E_i^1 \cong \rho_i^* E_{d-i} \stackrel{\rho_i^*\Phi}{\to} \rho_i^* F_{d-i} = \pi_i^* F^1_{d-i}\left(\subset T_{V^k_{\widetilde{D}_i}}\right)$$

then the components of this V_∞ over comb type strata of $\overline{N}_{0,1}(\mathbf{P}^n, d-i)$ map birationally to the components of the original V_∞ over comb type strata of $\overline{N}_{0,0}(\mathbf{P}^n, d)$. It follows by induction that for each μ (which includes an ordering of the Δ_i), we will obtain towers of $\mathbf{P}^1$-bundles mapping birationally to the components V_{D_μ} over D_μ, and we obtain the Lemma with the following explicit formula for the c_μ:

$$c_\mu = \prod_{i=1}^{r}\sum_{k_i=1}^{\Delta_i l}(-k_i)\tau_{\Delta_i}^* c^T_{\Delta_i l - k_i}(E^{k_i+1}_{\Delta_i}) \cup \pi_\mu^* c^T_{k_i-1}(F^{k_i}_{d_i}/F^1_{d_i})$$

where $F^1_{d_i} \subset F^{k_i}_{d_i}$ (we are abusing notation slightly in the interest of clarity) is pushed forward from the inclusion of sheaves on $\mathbf{P}^1 \times (\mathbf{P}^1)^r \times \mathbf{P}^n_{d_1}$:

$$\mathcal{O}(d_i l, \Delta_1 l, ..., \Delta_{i-1} l, 0, ..., 0, l) \hookrightarrow \mathcal{O}(d_i l + k_i - 1, \Delta_1 l, ..., \Delta_{i-1} l, k_i - 1, 0, ..., 0, l)$$

and τ_{Δ_i} is induced from the projection (so again we are abusing notation) $\overline{N}_{0,1}(\mathbf{P}^n, d_i) \times_{\mathbf{P}^n} \overline{M}_{0,1}(\mathbf{P}^n, \Delta_i) \to \overline{M}_{0,1}(\mathbf{P}^n, \Delta_i)$.

In the complete intersection case, the components $V^k_{\widetilde{D}_i}$ are only generically $\mathbf{P}^1$-bundles, and the explicit form of the c_i is thus more difficult to compute, though the existence of the decomposition of the Lemma and the inductive nature of the c_μ terms follow from the same argument. Alternatively, one can use the product:

$$[S]^T_d = \prod_{i=1}^{m}[S_{l_i}]^T_d$$

and the decompositions of each of the hypersurface virtual classes $[S_{l_i}]^T_d$ along with excess intersection theory to obtain the desired decomposition of $[S]^T_d$. There is one subtlety, in that contributions from hairy combs need to be pushed forward to their underlying (bald) comb strata.

Proof of the Fano of Index ≥ 2 Case: Using the decomposition of $[S]_d^T$ from Lemma 4.4, we have:

$$u_*[S]_d^T = \sum_\mu \frac{1}{r!} u_* f_{\mu *}(c_\mu \cup \pi_\mu^* b_\mu) = \sum_\mu \frac{1}{r!} j_{\mu *} \left(\pi_{\mu *} c_\mu \cup b_\mu \right)$$

where $b_\mu = \prod_{i=1}^m \prod_{k=0}^{d_1 l_i} (l_i h^T + kt)$. On the other hand, the codimension of the image of $j_\mu : (\mathbf{P}^1)^r \times \mathbf{P}_{d_1}^n \to \mathbf{P}_d^n$ is

$$(n+1)(d-d_1) - r \geq n(d-d_1)$$

because of the obvious inequality $r \leq d - d_1$. Thus the codimension of $j_{\mu *} b_\mu$ is already at least:

$$(\sum_{i=1}^m l_i) d_1 + n(d-d_1) > \operatorname{codim}(u_*[S]_d^T) = (\sum_{i=1}^m l_i) d$$

since $(\sum_{i=1}^m l_i) < n$ by assumption. This implies that $\pi_{\mu *} c_\mu = 0$ on all strata of positive codimension, hence $u_*[S]_d^T = \prod_{i=1}^m \prod_{k=0}^{d l_i} (l_i h^T + kt)$. Together with the argument from §3, this proves the desired result:

$$e_* \left(\frac{[S]_d}{t(t-\psi)} \right) = \phi_d$$

The Fano of Index One Case: Suppose $l_1 + ... + l_m = n$. Then:

$$e_* \left(\frac{[S]_d}{t(t-\psi)} \right) = \sum_{r=0}^d \frac{(-\prod l_i!)^r \phi_{d-r}}{r! t^r}$$

Proof: By Lemmas 4.3 and 4.4, the contribution of the stratum indexed by μ to the **ratio**:

$$\frac{j^* u_*[S]_d^T}{\prod_{k=1}^d (h+kt)^{n+1}}$$

is given by:

$$\frac{j^* j_{\mu *}(\pi_{\mu *} c_\mu \cup b_\mu)}{r! \prod_{k=1}^d (h+kt)^{n+1}} = \frac{j_\mu'^* \pi_{\mu *} c_\mu \cup j_\mu'^* b_\mu}{r! t^r \prod_{k=1}^{d_1} (h+kt)^{n+1}} = \frac{j_\mu'^* \pi_{\mu *} c_\mu \cup \phi_{d_1}}{r! t^r}$$

where b_μ is defined as in the higher index case.

When $(\sum l_i)d_1 + (n+1)(d-d_1) - r > (\sum l_i)d$, then as we've already seen, $\pi_{\mu_*} c_\mu = 0$ for dimension reasons. Under the Fano index one assumption, this inequality holds unless $r = d - d_1$, in which case equality holds. This occurs exactly once for each r from 0 to d, namely when μ is the comb type:

$$0 \leq d_1 < d_1 + 1 < ... < d_1 + r = d$$

For these comb strata, $\pi_{\mu_*} c_\mu$ is a codimension zero class, so that the formula here amounts to the identity: $\pi_{\mu_*} c_\mu = (-\prod_{i=1}^m l_i!)^r \cdot 1$.

In the hypersurface case, we note that the positive chern classes of the $\pi_1^*(F_{d-1}^k / F_{d-1}^1)$ cannot contribute to the push-forward, hence the only term which does contribute to $\pi_{1*} c_1$ is $-\tau_i^* c_{l-1}(E_1^2)$. The classes $c_1(E_1^k / E_1^{k+1})$ are easily computed to be $e^*(lh) + k\psi$. The $e^*(lh)$ do not contribute to the push-forward, and we are left with the term $-l!\psi^{l-1}$ which pushes forward to $-l!$. Using the inductive description of c_μ, we readily obtain the desired computation $\pi_{\mu_*} c_\mu = (-l!)^r$.

In the complete intersection case, from the product of decompositions $[S]_d^T = \prod_{i=1}^m [S_{l_i}]_d^T$ and excess intersection, we can conclude that modulo terms pulled back under π_1^*, the term c_1 agrees with $f_1^*(D_1)^{m-1} \prod_{i=1}^m -\tau_1^* c_{l_i - 1}(E_{l_i,1}^2)$ where $E_{l_i,1}$ is the bundle $\pi_* e^* \mathcal{O}_{\mathbf{P}^n}(l_i)$ on $\overline{M}_{0,1}(\mathbf{P}^n, 1)$. For all i, we have $f_i^*(D_i) = -\tau_i^* \psi - \rho_i^* \psi$, the second term of which does not contribute to the push-forward, and treating the rest of the expression as in the previous paragraph, we obtain

$$\pi_{1*} c_1 = \pi_{1*} \tau_i^* \left((-\psi)^{m-1} \prod_{i=1}^m (-l_i! \psi^{l_i - 1}) \right) = -\prod_{i=1}^m l_i!$$

As in the hypersurface case, the desired form of $\pi_{\mu_*} c_\mu$ follows from the inductive description of the c_μ.

5. The Calabi-Yau Case. The decomposition of Lemma 4.4 together with the push-pull formula of Lemma 4.3 always yield:

$$e_* \left(\frac{[S]_d}{t(t-\psi)} \right) = \sum_\mu \frac{\phi_{d_1} \cup {j'_\mu}^* \pi_{\mu_*} c_\mu}{r! t^r}$$

so that the challenge is to find a method for computing the ${j'_\mu}^* \pi_{\mu_*} c_\mu$. This seems to be very difficult in the general-type case, but in the Calabi-Yau case there is a wonderful simplification:

Lemma 5.1: When S is Calabi-Yau, the decomposition of Lemma 4.4 has the following additional properties:

(a) The equivariant chern classes $\lambda_\mu(h,t) := j_\mu'^* \pi_{\mu *} c_\mu$ satisfy

$$\lambda_\mu(h,t) = \lambda_{d_1,d_2}(h,t)\lambda_{d_2,d_3}(h,t)\cdots\lambda_{d_r,d}(h,t)$$

where $\lambda_{d-i,d}(h,t) = j_i'^* \pi_{i*} c_i$.

(b) The "simple" classes $\lambda_{d-i,d}(h,t)$ are linear and satisfy:

$$\lambda_{d-i,d}(h,t) = \lambda_{0,i}(h+(d-i)t,t)$$

We immediately obtain the following:

A Formula for the Calabi-Yau Case: If $l_1 + ... + l_m = n+1$, then there are linear forms $\lambda_i(h,t)$ only depending upon $(l_1,...,l_m)$ such that:

$$e_*\left(\frac{[S]_d}{t(t-\psi)}\right) = \sum_\mu \frac{\phi_{d_1} \cup \prod_{i=1}^r \lambda_{\Delta_i}(h+d_i t,t)}{r! t^r}$$

Proof: Set $\lambda_i(h,t) = \lambda_{0,i}(h,t)$ and apply the lemma.

Proof of Lemma 5.1: Recall that Lemma 4.4 gave us:

$$c_\mu = \prod_{i=1}^r \sum_{k_i=1}^{\Delta_i l} (-k_i)\tau_{\Delta_i}^* c_{\Delta_i l - k_i}(E_{\Delta_i}^{k_i+1}) \cup \pi_\mu^* c_{k_i-1}(F_{d_i}^{k_i}/F_{d_i}^1)$$

(again we will start with the hypersurface case) and in particular,

$$c_i = \sum_{k=1}^{il} (-k)\tau_i^* c_{il-k}(E_i^{k+1}) \cup \pi_i^* c_{k-1}(F_{d-i}^k/F_{d-i}^1)$$

Recall also the construction of the map π_i:

$$\pi_i : \overline{N}_{0,1}(\mathbf{P}^n, d-i) \times_{\mathbf{P}^n} \overline{M}_{0,1}(\mathbf{P}^n, i) \xrightarrow{\rho_i} \overline{N}_{0,1}(\mathbf{P}^n, d-i) \xrightarrow{v} \mathbf{P}^1 \times \mathbf{P}^n_{d-i}$$

Let ξ^T be the equivariant hyperplane class on $\mathbf{P}^1$. Then one computes:

$$c_{k-1}^T(F_{d-i}^k/F_{d-i}^1) = \prod_{j=1}^{k-1}(l(h^T + (d-i)t) + j\xi^T)$$

On the other hand, by the Künneth decomposition of in $\Delta \subset \mathbf{P}^n \times \mathbf{P}^n$, the class $\rho_{i*}\tau_i^* c_{il-k}^T(E_i^{k+1})$ is of the form $\kappa_a^i e^*(h^a)$ for a rational number κ_a^i that is clearly independent of d. Putting these together, we see that if S is Calabi-Yau, then $\pi_{i*}c_i$ is a codimension one class, hence $a = 1$ or $a = 0$, and:

$$\pi_{i*}c_i = -v_*\kappa_1^i e^*(h) - 2\kappa_0^i(l(h^T + (d-i)t) + \xi^T)$$

But $e^*(h) = v^*(h^T + (d-i)\xi^T) - \epsilon$ on $\overline{N}_{0,1}(\mathbf{P}^n, d-i)$ for a v-exceptional divisor ϵ (from the explicit description of Φ in Lemma 4.4). Hence:

$$j_i'^*\pi_{i*}c_i = -\kappa_1^i(h + (d-i)t) - 2\kappa_0^i(l(h + (d-i)t) + t)$$

proving part (b) of the lemma with $\lambda_{0,i}(h,t) = (-\kappa_1^i - 2l\kappa_0^i)h - 2\kappa_0^i t$.

Part (a) is proved similarly. It follows from the projection formula that

$$\rho_{\mu_*}c_\mu = \prod_{i=1}^r \sum_{k_i}(-k_i)\rho_{\mu_*}\tau_{\Delta_i}^* c_{\Delta_i l-k_i}^T(E_{\Delta_i}^{k_i+1}) \cup v^* c_{k_i-1}^T(F_{d_i}^{k_i}/F_{d_i}^1)$$

If we let ξ_i^T be the hyperplane class of the ith copy of $\mathbf{P}^1$, then:

$$c_{k_i-1}^T(F_{d_i}^{k_i}/F_{d_i}^1) = \prod_{j=1}^{k_i-1}(l(h^T + d_1 t + \Delta_1\xi_1^T + ... + \Delta_{i-1}\xi_{i-1}^T) + j\xi_i^T)$$

and as in the proof of (b), $\rho_{\mu_*}\tau_{\Delta_i}^* c_{\Delta_i l+1}(E_{\Delta_i}^2) = \kappa_1^i(v^*(h^T + d_i t) - \epsilon_i)$ and $\rho_{\mu_*}\tau_{\Delta_i}^* c_{\Delta_i l+2}(E_{\Delta_i}^3) = \kappa_0^i$. Different ϵ_i divisors multiply together to yield v-exceptional classes, so putting all this together, we get:

$$j_\mu'^*\pi_{\mu_*}c_\mu = \prod_{i=1}^r(-\kappa_1^i(h + d_i t) - 2\kappa_0^i(l(h + d_i t) + t))$$

proving the lemma in the hypersurface case.

The complete intersection case is proved similarly. The only wrinkle in that case is the presence of normal classes to the strata in the decomposition. But $f_i^*(D_i) = \tau_i^*\psi + \psi_{\mathbf{P}^1} + \epsilon$ where ϵ is exceptional for the v map, and thus (modulo ϵ) does not depend upon d. Once again, the lemma now follows from the linearity of the $\rho_{i*}c_i$ and the inductive description of the c_μ.

6. Using the Formula. The real beauty of the Calabi-Yau formula:

$$e_*\left(\frac{[S]_d}{t(t-\psi)}\right) = \sum_\mu \frac{\phi_{d_1} \cup \prod_{i=1}^r \lambda_{\Delta_i}(h+d_i t, t)}{r! t^r}$$

is that it recursively computes itself! This is well-known, but I include the computation here for the reader's enjoyment.

Namely, consider the simple comb type $\{0 \leq 0 < d\}$ which contributes:

$$\frac{\phi_0 \cup \lambda_d(h,t)}{t} = \frac{\prod_{i=1}^m l_i h \cup \lambda_d(h,t)}{t}$$

to right side of the formula. This term is irrelevant to the left side of the formula, which only involves the powers $t^{-2}, t^{-3}, ...$ so the "error" coefficients of t^{-1} and t^0 (there are no more when S is Calabi-Yau) in the rest of the right side of the formula determine this term. Since these only depend upon $\lambda_1, ..., \lambda_{d-1}$, we therefore have an inductive construction of the λ_d.

We may read off the coefficients of $\lambda_d(h,t) = \alpha_d h + \beta_d t$ via:

$$\alpha_d t^{-1} = \frac{1}{\prod_{i=1}^m l_i} \int_{\mathbf{P}^n} \frac{h^{n-m-1} \cup \phi_0 \cup \lambda_d(h,t)}{t}$$

$$\beta_d = \frac{1}{\prod_{i=1}^m l_i} \int_{\mathbf{P}^n} \frac{h^{n-m} \cup \phi_0 \cup \lambda_d(h,t)}{t}$$

In other words, not only does the formula in degree d compute the one-point Gromov-Witten invariants, but via the "error" coefficients, it computes the form λ_d which is to be used in higher degrees!

Let S be the quintic threefold in $\mathbf{P}^4$. Then

$$e_*\left(\frac{[S]_d}{t(t-\psi)}\right) = n_d h^3 t^{-2} + m_d h^4 t^{-3}$$

and it follows from the projection formula that the "actual" physical number of rational curves of degree d on S is $n_d/d = -m_d/2$. For the reader who is unused to this and nervous about the fact that these are not typically integers, the Aspinwall-Morrison formula (see [3]) translates these numbers into the expected numbers of immersed rational curves of degree d. The formula now produces:

$$n_1 = 2875,\ \lambda_1 = -(770)h - (120)t$$
$$n_2 = 4876875/4,\ \lambda_2 = -(421375)h - (60000)t$$
$$n_3 = 8564575000/9,\ \lambda_3 = -(436236875)h - (59937500)t$$
$$n_4 = 15517926796875/16,\ \lambda_4 = -(17351562078125/6)h - (390555125000)t$$

Via the Aspinwall-Morrison formula, this translates into:

$$\frac{n_d}{d} = \sum_{e|d} \frac{N_e}{e^3}$$

where N_e is the expected number of immersed curves of degree e. This gives:

$$N_1 = 2875, N_2 = 609250, N_3 = 317206375 \text{ and } N_4 = 242467530000$$

the well-known numbers of rational curves of degree ≤ 4 on the quintic.

7. A Relative Version. Given a projectivized vector bundle:

$$\pi : \mathbf{P}(V) \to X$$

over a projective manifold X, there are relative moduli spaces:

$$\pi_M : \overline{M}_{0,k}(\mathbf{P}(V), d) \to X \text{ and } \pi_N : \overline{N}_{0,k}(\mathbf{P}(V), d) \to X$$

for stable maps to the fibers of π. The fibers of π_M and π_N are the moduli spaces we studied previously, and these moduli spaces are equipped with evaluation maps (to $\mathbf{P}(V)$) and forgetful maps with the usual properties. The definition of $\mathbf{P}^n_d$ is readily generalized to:

$$\mathbf{P}(V)_d := \mathbf{P}(\mathrm{Sym}^d(W^*) \otimes V)$$

and the basic diagram in the relative setting is:

$$\begin{array}{ccc} \overline{N}_{0,0}(\mathbf{P}(V), d) & \stackrel{u}{\to} & \mathbf{P}(V)_d \\ i \uparrow & & j \uparrow \\ \overline{M}_{0,1}(\mathbf{P}(V), d) & \stackrel{e}{\to} & \mathbf{P}(V) \end{array}$$

All the results of the paper carry through unchanged, with one exception. Proposition 3.1(3) now needs to take into account the chern classes of V. Namely, if $\alpha_1, ..., \alpha_{n+1}$ are the chern roots of $\pi^* V$, then:

$$\epsilon^T(N_{\mathbf{P}(V)/\mathbf{P}(V)_d}) = \prod_{k=1}^{d} \prod_{j=1}^{n+1} (h + \alpha_j + kt)$$

which follows from the diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{\mathbf{P}(V)} & = & \mathcal{O}_{\mathbf{P}(V)} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \pi^* V(1) & \to & \mathrm{Sym}^d(W^*) \otimes \pi^* V(1)|_{\mathbf{P}(V)} & \to & N_{\mathbf{P}(V)/\mathbf{P}(V)_d} & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & T_{\mathbf{P}(V)/X} & \to & T_{\mathbf{P}(V)_d/X}|_{\mathbf{P}(V)} & \to & N_{\mathbf{P}(V)/\mathbf{P}(V)_d} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

We get the following interesting formula already in degree one:

Relative Schubert Calculus: Chern classes on the relative flag bundle $\mathrm{Fl}(1, 2, V)$ over X push forward to $\mathbf{P}(V)$ via:

$$e_* \left(\frac{\sigma(q_1, q_2)}{t(t - \psi)} \right) = \frac{\sigma(h, h + t)}{\prod_{j=1}^{n+1} (h + \alpha_j + t)} + O(t^{-1})$$

where $\sigma(q_1, q_2)$ is any chern class pulled back from the relative Grassmann bundle $G(2, V)$ and expressed as a symmetic polynomial in the chern roots of the (dual of the) universal sub-bundle S^*.

Proof: S^* pulls back to $\pi_* e^* \mathcal{O}_{\mathbf{P}(V)}(1)$ on $\mathrm{Fl}(1, 2, V) = \overline{M}_{0,1}(\mathbf{P}(V), 1)$. By Proposition 4.1, the map $\Phi : \pi_* e^* \mathcal{O}_{\mathbf{P}(V)}(1) \to W^* \otimes u^* \mathcal{O}_{\mathbf{P}(V)_d}(1)$ of bundles over $\overline{N}_{0,0}(\mathbf{P}(V), 1)$ is an isomorphism off the unique boundary stratum D_1. It follows from the argument of §3 (and Lemma 4.3) that:

$$e_* \left(\frac{\sigma(q_1, q_2)}{t(t - \psi)} \right) = \frac{\sigma(h, h + t)}{\prod_{j=1}^{n+1} (h + \alpha_j + t)} + \frac{{j'_1}^* \pi_{1*} c_1}{t}$$

for some equivariant cohomology class c_1 supported on D_1.

We can express the denominator in terms of Segre classes:

$$\frac{1}{\prod_{j=1}^{n+1}(h+\alpha_j+t)} = \frac{1}{(h+t)^{n+1}} s_{(h+t)^{-1}}(\pi^* V)$$

where $s_{(h+t)^{-1}}(\pi^* V) = 1 + (h+t)^{-1} s_1(\pi^* V) + (h+t)^{-2} s_2(\pi^* V) + ...$

In the case of a "linear" complete intersection $S \subset \mathbf{P}(V)$ defined by m sections of $\mathcal{O}_{\mathbf{P}(V)}(1)$, then $[S]_1 = (q_1 q_2)^m$ and we get a generalized "Porteous" formula for lines:

$$e_* \left(\frac{[S]_1}{t(t-\psi)} \right) = h^m (h+t)^{m-n-1} s_{(h+t)^{-1}}(\pi^* V) + O(t^{-1})$$

The t^{-2} coefficient is $h^m(s_{m-n+1}(\pi^* V) - s_{m-n}(\pi^* V)h)$. When we multiply this by h and push forward to X, we get the formula:

$$s^2_{m-n+1}(V) - s_{m-n}(V) s_{m-n+2}(V)$$

for the push-forward to X of the class $[S]_1$ on the Grassmann bundle. This agrees with the classical Porteous formula for lines (see e.g. [16]).

For general degree and a general complete intersection $S \subset \mathbf{P}(V)$, let:

$$\phi_d = \frac{\prod_{i=1}^{m} \prod_{k=0}^{dl_i} (l_i h + kt)}{\prod_{k=1}^{d} \prod_{j=1}^{n+1} (h + \alpha_j + kt)}$$

Then the exact analogues of the Fano formulas hold with this ϕ_d, and for Calabi-Yau's we have the analogous:

Relative Calabi-Yau Formula: If $\sum_{i=1}^{m} l_i \leq n+1$, then there are linear equivariant classes $\lambda_e(h,t) \in \mathrm{H}^*(\mathbf{P}(V), \mathbf{Q})[t]$ such that:

$$e_* \left(\frac{[S]_d}{t(t-\psi)} \right) = \sum_{\mu} \frac{\phi_{d_1} \cup \prod_{i=1}^{r} \lambda_{\Delta_i}(h + d_i t, t)}{r! t^r}$$

As an application, consider the "linear relative Calabi-Yau's", i.e. the $S \subset \mathbf{P}(V)$ that are cut out by $n+1$ transverse sections of $\mathcal{O}_{\mathbf{P}(V)}(1)$. The following was first proved in [5] in the context of symmetric products of a smooth curve C, where the $g-1$st symmetric product C_{g-1} is an example of a linear relative Calabi-Yau over the Jacobian of C:

Proposition 7.1: The three-point Gromov-Witten invariants:

$$\int_{\overline{M}_{0,3}(\mathbf{P}(V),d)} e_1^* h \cup e_2^* h \cup e_3^* c \cup [S]_d$$

of linear relative Calabi-Yau's are independent of $d \geq 1$.

Remark: When X admits no rational curves, this says that the quantum product of h with itself in the quantum cohomology ring of S is of the form:

$$h * h = h^2 + bq + bq^2 + bq^3 + ...$$

Proof: By the projection formula and the relative Schubert calculus, the proposition is equivalent to $e_*([S]_d) = (1/d^2)e_*([S]_1) = (1/d^2)h^{n+1}(s_2 - s_1 h)$.

In this case, $\phi_d = h^{n+1} s_{(h+t)^{-1}}(\pi^* V) s_{(h+2t)^{-1}}(\pi^* V) ... s_{(h+dt)^{-1}}(\pi^* V)$

$$= h^{n+1}\left(1 + \frac{s_1}{t}\sum_{k=1}^{d}\frac{1}{k} + \frac{s_1^2}{t^2}\sum_{1\leq j<k\leq d}\frac{1}{jk} + \frac{s_2 - s_1 h}{t^2}\sum_{k=1}^{d}\frac{1}{k^2} + ...\right)$$

We make the following (only valid for linear Calabi-Yau's):

Assumption: The $\lambda_e(h,t)$ of the Calabi-Yau formula are independent of h.

We can separate variables in the formula:

$$e_*\left(\frac{[S]_d}{t(t-\psi)}\right) = \sum_{d_1}\phi_{d_1}\sum_{0<e_1<...<e_r=d-d_1}\frac{\prod_{i=1}^{r}(\lambda_{e_i - e_{i-1}}(t)/t)}{r!}$$

and if we express this in terms of generating functions, we get:

$$h^{n+1} + \sum_{d>0} q^d e_*\left(\frac{[S]_d}{t(t-\psi)}\right) = (\sum_{d\geq 0} q^d \phi_d)\exp(\sum_{e>0} q^e \lambda_e(t)/t)$$

Let $\lambda_e(t) = a_e t + b_e$ where $a_e \in \mathbf{Q}$ and b_e is a cohomology class of degree one coming from X. Then equating coefficients of t^0 gives:

$$1 = (1 + q + q^2 + ...)\exp(\sum_e a_e q^e)$$

so that $\sum_e a_e q^e = \log(1-q)$ and hence $a_e = -\frac{1}{e}$.

Similarly, the t^{-1} coefficients give $0 = \frac{s_1}{t}\sum_{k=1}^{d}\frac{1}{k} + \sum_{k=1}^{d}\frac{b_k}{t}$ for each d, so that by induction, $b_e = -\frac{s_1}{e}$, and we get $\lambda_e(t) = -\frac{1}{e}(t+s_1)$ which is indeed independent of h. Plug these $\lambda_e(t)$ in the generating function:

$$h^{n+1} + \sum q^d e_*[S]_d t^{-2} + ... =$$

$$(\sum_e q^e \phi_e)(1-q)(1 + \frac{s_1}{t}\log(1-q) + \frac{s_1^2}{2!t^2}\log(1-q)^2 + ...)$$

and the t^{-2} term on the right is $(1/d^2)h^{n+1}(s_2 - s_1 h)$, as desired.

8. The Relationship with the Mirror Conjecture. The astute reader will have noticed that the Calabi-Yau formula in §6 does not quite match with the mirror conjecture for Calabi-Yau complete intersections from the introduction! We will establish the latter with the aid of the following:

Proposition 8.1: Let

$$F(q) = \sum_{d=0}^{\infty} \sum_{0<d_1<...<d_r=d} \frac{\prod_{i=1}^{r}(y_{d_i-d_{i-1}} + x_{d_i-d_{i-1}}d_{i-1})}{r!} q^d$$

Then $\log(F(q))$ is a linear function of $y_1, y_2, ...$

Proof: (Pavel Etinghof) Let E be the Euler vector field:

$$E = \sum_{k=1}^{\infty} k x_k \frac{\partial}{\partial x_k} + k y_k \frac{\partial}{\partial y_k}$$

Then we may rewrite $F(q)$ as:

$$F(q) = \exp(\sum_{d=1}^{\infty} q^d y_d + q^d x_d E)\cdot 1 = \sum_{r=0}^{\infty} \frac{(\sum q^d y_d + q^d x_d E)^r}{r!}\cdot 1$$

Consider the function $G(t,q) = \exp(t\sum_{d=1}^{\infty} q^d y_d + q^d x_d E)\cdot 1$ satisfying $G(1,q) = F$, $G(0,q) = 1$ and $q\frac{\partial G}{\partial q} = E\cdot G$ (because the Euler vector field is homogeneous of degree zero). Thus:

$$\frac{\partial G}{\partial t} = \sum_{d=1}^{\infty} q^d y_d G + \sum_{d=1}^{\infty} q^d x_d E\cdot G$$

hence

$$\frac{\partial \log(G)}{\partial t} = \frac{1}{G}\frac{\partial G}{\partial t} = \sum_{d=1}^{\infty} q^d y_d + \sum_{d=1}^{\infty} q^{d+1} x_d \frac{\partial \log(G)}{\partial q}.$$

By the fundamental theorem of calculus(!) this gives:

$$\log(G)(t,q) = \sum_{d=1}^{\infty} t q^d y_d + \sum_{d=1}^{\infty} q^{d+1} x_d \int_0^t \frac{\partial \log(G(s,q))}{\partial q}\, ds$$

which proves the desired linearity of $\log(F) = \log(G)(1,q)$ in the y variables by induction on the power of q.

Corollary: If we let $F(q) = \exp(\sum_{d=1}^{\infty} y'_d q^d)$ then

$$y'_d = \sum_{0<d_1<\ldots<d_r=d} \frac{y_{d_1} \prod_{i=2}^{r} (x_{d_i - d_{i-1}} d_{i-1})}{r!}$$

Proof: Cast out the non-linear terms (in the y_k) from the identity:

$$\sum_{0<d_1<\ldots<d_r=d} \frac{\prod_{i=1}^{r} y'_{d_i - d_{i-1}}}{r!} = \sum_{0<d_1<\ldots<d_r=d} \frac{\prod_{i=1}^{r} (y_{d_i - d_{i-1}} + x_{d_i - d_{i-1}} d_{i-1})}{r!}$$

Finally, suppose S is Calabi-Yau and let $\lambda_d = \alpha_d h + \beta_d t$. Then our formula may be written as follows:

$$\Sigma(q) = \sum_d \phi_d \sum_{0<e_1<\ldots<e_r=e} \frac{\prod_{i=1}^{r} (\alpha_{e_i - e_{i-1}}(d + \frac{h}{t}) + \beta_{e_i - e_{i-1}}) + \alpha_{e_i - e_{i-1}} e_{i-1}}{r!} q^{d+e}$$

The proposition applies to give us the new coordinates y'_e which are linear in $\alpha_e(d + \frac{h}{t}) + \beta_e$ (and polynomial in the α_k's), hence of the form:

$$y'_e = a_e(d + \frac{h}{t}) + b_e$$

where a_e and b_e are independent of d. This gives:

$$\Sigma(q) = \sum_d \phi_d \sum_{0<e_1<\ldots<e_r=e} \frac{\prod_{i=1}^{r} (a_{e_i - e_{i-1}}(d + \frac{h}{t}) + b_{e_i - e_{i-1}})}{r!} q^{d+e}$$

which proves the mirror conjecture.

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