On stable rank two bundles with canonical determinant and many sections

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1 INTRODUCTION

Let C be a smooth projective curve of genus $g \geq 2$, and consider the moduli space $\mathcal{M}_C^s(2, 2g-2)$ of stable rank two bundles on C of degree $2g-2$. This is a smooth, quasiprojective variety of dimension $(3g-3)+g$, and via the "determinant map"

$$\Delta : \mathcal{M}_C^s(2, 2g-2) \to Pic^{2g-2}(C)$$

sending E to $\wedge^2 E$, is a fiber bundle over the Jacobian, with fibers of dimension $3g-3$. Let $\mathcal{M}_C^s(2, K) := \Delta^{-1}(K_C)$ be the moduli space of stable rank two bundles with canonical determinant.

The moduli space $\mathcal{M}_C^s(2, 2g-2)$ naturally compactifies to $\mathcal{M}_C(2, 2g-2)$, the moduli space of s-equivalence classes of semistable bundles (see [Se]). For our purposes, the potential advantages of the fact that $\mathcal{M}_C(2, 2g-2)$ is projective are outweighed by the disadvantages: (1) that $\mathcal{M}_C(2, 2g-2)$ is singular (for $g \geq 3$) away from $\mathcal{M}_C^s(2, 2g-2)$ and (2) that $\mathcal{M}_C(2, 2g-2)$ does not parametrize bundles away from $\mathcal{M}_C^s(2, 2g-2)$, but merely equivalence classes of bundles. As a result, we will work with $\mathcal{M}_C^s(2, 2g-2)$ in this paper.

Because $\chi(E) = 0$ for any rank two bundle of degree $2g-2$, it follows that the subscheme $\Theta \subset \mathcal{M}_C^s(2, 2g-2)$ parametrizing stable bundles E with $\mathrm{H}^0(C, E) \neq 0$ is a divisor, and following the line bundle precedent, this is called the Riemann theta divisor for rank two bundles. More generally, using Brill-Noether notation, we denote by $W^r_{2,2g-2}$ the locus (with a natural scheme structure) of stable bundles $E \in \mathcal{M}_C^s(2, 2g-2)$ such that $\mathrm{h}^0(C, E) \geq r+1$.

The central theme of this paper may be stated as follows:

The Brill-Noether loci are not fiber bundles over the Jacobian.

Indeed, if we define $W^r_{2,K}$ as the scheme-theoretic intersection of $W^r_{2,2g-2}$ and $\mathcal{M}_C^s(2, K)$, then we will see that the $W^r_{2,K}$ are distinguished among the intersections of $W^r_{2,2g-2}$ with fibers of Δ. We hope to convince the reader that these loci are sufficiently interesting to be worth further study.

Reasoning by analogy with the well-established Brill-Noether theory for line bundles (see e.g. [ACGH]), several authors postulated expected dimensions for the Brill-Noether loci $W^r_{2,d} \subset \mathcal{M}_C^s(2, d)$ (cf. [T] and [Su]). In our case, this gives an expected dimension for $W^r_{2,2g-2}$ of

$$\rho_{2g-2}(r) = 4g - 3 - (r+1)^2.$$

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Thus following the lead of Brill-Noether for line bundles, one might be tempted to make the following predictions (listed in descending order of likelihood) about the $W^r_{2,2g-2}$ on a *general* curve C of genus g:

(a) Non-existence: If $\rho_{2g-2}(r) < 0$, then $W^r_{2,2g-2} = \emptyset$.

(b) Dimension: If $\rho_{2g-2}(r) \geq 0$, then $\dim(W^r_{2,2g-2}) = \rho_{2g-2}(r)$.

(c) Smoothness: $W^r_{2,2g-2}$ is smooth of dimension $\rho(r)$ away from $W^{r+1}_{2,2g-2}$.

However these predictions are too naive. We give counterexamples in genus 5 and 6, which we believe are systematic, and are due to the fact that for genus 5 and above and large r, the loci $W^r_{2,K}$ can contribute whole components of the "wrong" dimension to $W^r_{2,2g-2}$. We will present our analysis of $W^r_{2,K}$ for small genus and r in §3 and §4.

The source of our counterexamples in genus 5 and 6 is the fact that for large r, the dimension of $W^r_{2,K}$ itself is greater than $\rho_{2g-2}(r)$. If $W^r_{2,2g-2}$ were a fiber bundle over the Jacobian, then the expected dimension of $W^r_{2,K}$ ought to be $\rho_{2g-2}(r) - g$. In fact, we will give an asymptotically far larger expected dimension for $W^r_{2,K}$ of

$$\rho_K(r) = 3g - 3 - \binom{r+2}{2}$$

in §2. In addition, bolstered by our analysis in §3 and §4, we anticipate that this expected dimension is the "correct" one for $W^r_{2,K}$ and make the following (we hope less naive) conjecture, which is also conjectured by Mukai in [Mu2]:

Conjecture: If C is a *general* curve of genus g, then:

(a) Non-existence: If $\rho_K(r) < 0$ then $W^r_{2,K}$ is empty.

(b) Dimension: If $\rho_K(r) \geq 0$ then $\dim(W^r_{2,K}) = \rho_K(r)$.

(c) Smoothness: $W^r_{2,K}$ is smooth away from $W^{r+1}_{2,K}$.

Note that part (b) of the conjecture logically splits into two parts, namely existence of $W^r_{2,K}$ when $\rho_K(r) \geq 0$ and the dimension of $W^r_{2,K}$ once it is shown to be nonempty. In the line bundle case, it can be shown that for all curves, the Brill-Noether loci exist when their expected dimension is non-negative. However, all proofs of this fact rely on the projectivity of $Pic^d(C)$. In our case, as we already mentioned, the varieties $\mathcal{M}^s_C(2, K)$ are not projective, and we suspect that for special curves $\dim(W^r_{2,K})$ may be empty even though $\rho_K(r) \geq 0$. We will prove existence for general curves of genus $g \leq 12$ in §3. The rest of part (b) and (c), as in the line bundle case, is equivalent to the injectivity of a "Petri" map. We will introduce this map in §2 and investigate it in §4 for general curves of genus ≤ 7. In §3, we will also establish (a) for general curves of small genus.

2 EXPECTED DIMENSIONS

In §1, we introduced the expected dimension $\rho_K(r) = 3g - 3 - \binom{r+2}{2}$ for $W^r_{2,K}$. We will justify this in three ways:

(1) $W^r_{2,K}$ is a skew-isotropic degeneracy locus.

We use the skew-symmetric form $\wedge^2 E \cong K$ just as Mumford uses a symmetric form in [M1] and [M2] to construct Brill-Noether loci in Prym varieties as symmetric-isotropic degeneracy loci.

Suppose X is a quasi-projective scheme and $\mathcal{E}$ is a family of stable rank two bundles with canonical determinant on C parametrized by X. That is, $\mathcal{E}$ is a rank two bundle on $C \times X$ such that for each closed point $x \in X$, $E_x := \mathcal{E}|_{C\times\{x\}}$ is stable with canonical determinant. Let π_C and π_X denote the projections from $C \times X$ to C and X respectively. Let D be an effective divisor on C of degree $d >> 0$. Let $\mathcal{E}(D) := \mathcal{E} \otimes \pi_C^* \mathcal{O}_C(D)$.

Then exactly as in [M1], there is a nondegenerate skew-symmetric form on the rank $4d$ bundle $\pi_{X*}(\mathcal{E}(D)/\mathcal{E}(-D))$ such that the rank $2d$ subbundles $\pi_{X*}(\mathcal{E}(D))$ and $\pi_{X*}(\mathcal{E}/\mathcal{E}(-D))$ are isotropic everywhere. Furthermore, for each closed point $x \in X$, we have an identity on the intersection of the fibers of the subbundles: $\mathrm{H}^0(E_x) = \mathrm{H}^0(C, E_x(D)) \cap\ \mathrm{H}^0(C, E_x/E_x(-D)) \subset \mathrm{H}^0(C, E_x(D)/E_x(-D))$.

Suppose V is a fixed vector space of dimension $2k$ with a non-degenerate skew form $B : \wedge^2 V \to \mathbf{C}$. Let $\Sigma \subset \mathrm{Gr}(k, V)$ be the subvariety of the Grassmannian consisting of the k-dimensional isotropic subspaces of V. Then it is an easy check (as in [H]) that for fixed $W_0 \in \Sigma$, the set

$$\{W \in \Sigma \mid \dim(W \cap W_0) \geq r + 1\}$$

is of codimension $\binom{r+2}{2}$ in Σ. An inequality on dimensions immediately follows: $\dim(\{x \in X \mid \mathrm{H}^0(C, E_x) \geq r+1\}) \geq \dim(X) - \binom{r+2}{2}$ whenever this locus is nonempty. Thus $W^r_{2,K}$, if nonempty, has dimension at least $\rho_K(r)$.

(2) The Petri map factors.

Consider the Petri map for rank two bundles

$$\mu : \mathrm{H}^0(C, E) \otimes \mathrm{H}^0(C, E^* \otimes K) \to \mathrm{H}^0(C, E \otimes E^* \otimes K).$$

If $\wedge^2 E \cong K$, then $E \cong E^* \otimes K$, and the domain of the Petri map can be thought of as $\mathrm{H}^0(C, E) \otimes \mathrm{H}^0(C, E)$. Then $\mu(\wedge^2 \mathrm{H}^0(E)) \subset \mathrm{H}^0(C, K)$, and the subspace $\mathrm{H}^0(C, \mathrm{Sym}^2 E)^* \subset \mathrm{H}^0(C, E \otimes E)^*$ is the Zariski tangent space to $\mathcal{M}^s_C(2, K)$ at E. As in the line bundle case (see [ACGH]), the Zariski tangent space to $W^r_{2,2g-2}$ at E is $(\mathrm{coker}\mu)^*$. Thus, the Zariski tangent space to $W^r_{2,K}$ at E is an intersection:

$$T_{W^r_{2,K}}(E) = (\mathrm{coker}\mu)^* \cap \mathrm{H}^0(C, \mathrm{Sym}^2 E)^*$$

or equivalently, the dual of the cokernel of the factored Petri map:

$$\tilde{\mu} : \mathrm{Sym}^2 \mathrm{H}^0(C, E) \to \mathrm{H}^0(C, \mathrm{Sym}^2 E)$$

and we obtain the same lower bound as in (1) (indeed, (2) is an infinitesimal version of (1)...see [Mu2]).

Moreover, if the factored Petri map is injective for $E \in W^r_{2,K} - W^{r+1}_{2,K}$, then in a neighborhood of E, $W^r_{2,K}$ is smooth of the expected dimension. It is always injective if and only if (c) of the conjecture holds. Notice that as soon as $\binom{r+1}{2} > g$, the full

Petri map has no chance of being injective at $E \in W^r_{2,2g-2}$ since $\dim(\wedge^2 \mathrm{H}^0(C,E)) > \dim(\mathrm{H}^0(C,K))$. We will show that this occurs already for $g=5$.

(3) An open subset of $W^r_{2,K}$ is the quotient of a symmetric degeneracy locus.

Let $\mathbf{P}(\mathrm{Ext}^1(K,\mathcal{O}_C))$ be the projective space parametrizing extensions of the form $0 \to \mathcal{O}_C \to E \to K \to 0$ modulo equivalence and scalars. If $E \in \Theta$ has a nowhere vanishing section, then E is in the image of the natural birational "extension map" carrying extensions to the bundle they define.

$$\phi : \mathbf{P}(\mathrm{Ext}^1(K,\mathcal{O})) - -- > \Theta$$

Now the locus of points in $\mathbf{P}(\mathrm{Ext}^1(K,\mathcal{O}))$ such that the associated bundle E has $r+1$ independent sections is easily identified as a symmetric degeneracy locus. Namely, the short exact sequence $0 \to \mathcal{O} \to E \to K \to 0$ gives rise to a long exact sequence on cohomology:

$$\ldots \to \mathrm{H}^0(C,E) \to \mathrm{H}^0(C,K) \xrightarrow{\delta} \mathrm{H}^1(C,\mathcal{O}) \to \ldots$$

By Serre duality, $\mathrm{H}^1(C,\mathcal{O}_C)^* \cong \mathrm{H}^0(C,K)$, and the map δ is symmetric. Also by Serre duality, $\mathbf{P}(\mathrm{Ext}^1(K,\mathcal{O})) = \mathbf{P}(\mathrm{H}^0(C,K^2)^*)$. As long as C is not hyperelliptic, Noether's theorem tells us that the multiplication map $\mathrm{Sym}^2(\mathrm{H}^0(C,K)) \to \mathrm{H}^0(C,K^{\otimes 2})$ is surjective, so $\mathbf{P}(\mathrm{Ext}^1(K,\mathcal{O}))$ is a subspace of $\mathbf{P}(\mathrm{Sym}^2\mathrm{H}^0(C,K)^*)$.

Since $\dim(\mathrm{H}^0(C,E)) = \dim(\ker(\delta))+1$, we see that the locus of extensions such that $\dim(\mathrm{H}^0(C,E)) \geq r+1$ is the restriction of the "universal" degeneracy locus $D_r \subset \mathbf{P}(\mathrm{Sym}^2\mathrm{H}^0(C,K)^*)$ of symmetric maps with kernel of dimension at least r.

If E is a stable bundle with a nowhere-vanishing section, then it turns out that the pre-image of E in $\mathbf{P}(\mathrm{Ext}^1(K,\mathcal{O}))$ is isomorphic to an open subset of $\mathbf{P}(\mathrm{H}^0(C,E))$ (see [B]). Thus the expected dimension of any component of $W^r_{2,K}$ where the general bundle admits a nowhere-vanishing section is:

$$\dim(\mathbf{P}(\mathrm{Ext}^1(K,\mathcal{O}))) - \operatorname{codim}(D_r) - r = 3g-4-\binom{r+1}{2} - r = \rho_K(r)$$

3 EXISTENCE IN LOW GENERA

We will use the third approach of §2 to establish the existence of $W^r_{2,K}$ when $\rho_K(r) \geq 0$ and C is general. Namely, consider again the map

$$\phi : \mathbf{P}(\mathrm{Ext}^1(K,\mathcal{O})) - - > \Theta.$$

As we observed in §2, if C is not a hyperelliptic curve, then $\mathbf{P}(\mathrm{Ext}^1(K,\mathcal{O})) \cong \mathbf{P}(\mathrm{H}^0(C,K^2)^*)$ may be thought of as a projective linear subspace in the universal projective space of symmetric maps $q : \mathrm{H}^0(C,K) \to \mathrm{H}^0(C,K)^*$, and the intersection $D_r \cap \mathbf{P}(\mathrm{H}^0(C,K^2)^*)$ with the universal degeneracy loci give the loci of extensions $0 \to \mathcal{O} \to E \to K \to 0$ such that $\dim(\mathrm{H}^0(C,E)) \geq r+1$. This intersection is always nonempty of dimension at least $\rho_K(r)+r$ whenever $\rho_K(r) \geq 0$.

On the other hand, let $\psi : C \to \mathbf{P}(\mathrm{H}^0(C,K^2)^*)$ be the bicanonical embedding (we assume $g \geq 3$). Let $\mathrm{Sec}^d(C)$ be the secant variety defined as the closure of the union of the spans of all d-tuples of points on C. Then $\mathrm{Sec}^{g-1}(C) \subset \mathbf{P}(\mathrm{H}^0(C,K^2)^*)$ is an irreducible subvariety of dimension $(g-1)+(g-2) = 2g-3$. This is the locus of all extensions giving rise to bundles that are **not** stable. Indeed, if an extension belongs to the span of a divisor D, then there is a map $E \to \mathcal{O}_C(D)$. If $\deg(D) \le g-1$, then by definition, E is not stable. (See [B] for more details.) Thus one way of establishing the existence of $W^r_{2,K}$ is to show that $D_r \cap \mathbf{P}(\mathrm{H}^0(C,K^2)^*)$ is not contained in $\mathrm{Sec}^{g-1}(C)$. This approach works well enough in low genera, as we will see, but seems difficult to implement in general, mainly because the dimension of $\mathrm{Sec}^{g-1}(C)$ grows linearly with g, while the dimensions of $D_r \cap \mathbf{P}(\mathrm{H}^0(C,K^2)^*)$ for large r may remain small.

genus 3: To show: $W^2_{2,K} \neq \emptyset$ ($\rho_K(2) = 0$ and $\rho_K(3) = -4$).

Here, $\mathbf{P}(\mathrm{H}^0(C,K^2)^*) = \mathbf{P}(\mathrm{Sym}^2\mathrm{H}^0(C,K)^*) \cong \mathbf{P}^5$ and D_2 is the Veronese surface in $\mathbf{P}^5$. So we need to show that $D_2 \not\subset \mathrm{Sec}^2(C)$. Clearly, it suffices to show that $D_2 \cap (\mathrm{Sec}^2(C) - C) = \emptyset$. But suppose not. Then there must be a bundle E (arising from an extension in the intersection) admitting a surjective map $E \to \mathcal{O}_C(p+q)$ and such that $\dim(\mathrm{H}^0(C,E)) \ge 3$. We may then write:

$$0 \to \mathcal{O}(p'+q') \to E \to \mathcal{O}(p+q) \to 0$$

Since C is not hyperelliptic, each of $\mathcal{O}(p+q)$ and $\mathcal{O}(p'+q')$ has only one section, and we obtain a contradiction. Note in this example that D_2 is irreducible of dimension 2, so in fact $W^2_{2,K}$ consists of a single point.

genus 4: To show: $W^2_{2,K} \neq \emptyset$ ($\rho_K(2) = 3$, so $\dim(D_2 \cap \mathbf{P}(\mathrm{H}^0(C,K^2)^*) \ge 5)$.

Because $\dim(D_2 \cap \mathbf{P}(\mathrm{H}^0(C,K^2)^*) \ge \dim(\mathrm{Sec}^3(C))$ and the latter variety is irreducible, it suffices to find a single extension in $\mathrm{Sec}^3(C)$ such that the associated bundle E has fewer than 3 independent sections. But if $p+q+r$ is a general effective divisor of degree 3, then $\mathcal{O}(p+q+r)$ and $K(-p-q-r)$ each have just one section, and any extension:

$$0 \to K(-p-q-r) \to E \to \mathcal{O}(p+q+r) \to 0$$

gives an E with at most 2 independent sections. Such E are associated to the general extension in $\mathrm{Sec}^3(C)$, so we are done.

Note: The hyperplane $\mathbf{P}(\mathrm{H}^0(C,K^2)^*) \subset \mathbf{P}(\mathrm{Sym}^2(\mathrm{H}^0(C,K)^*))$ is general if C is a general curve of genus 4, and it follows that $W^2_{2,K}$ is irreducible, of dimension 3.

The genus 3 and 4 examples were also considered by Laszlo [L].

genus 5: To show: $W^3_{2,K} \neq \emptyset$ ($\rho_K(3) = 2$, so $\dim(D_3 \cap \mathbf{P}(\mathrm{H}^0(C,K^2)^*) \ge 5)$

Here, $\dim(\mathrm{Sec}^4(C)) = 7$ but the intersection of $\mathrm{Sec}^4(C) - \mathrm{Sec}^3(C)$ with the degeneracy locus $D_3 \cap \mathbf{P}(\mathrm{H}^0(C,K^2)^*)$ is nonempty. We will parametrize this intersection, and show that its dimension is small enough that its closure cannot contain $D_3 \cap \mathbf{P}(\mathrm{H}^0(C,K^2)^*)$. This proves existence.

Suppose E comes from an extension which lies in the intersection. Then E fits into the following exact sequence:

$$(*) \quad 0 \to K(-p-q-r-s) \to E \to \mathcal{O}(p+q+r+s) \to 0$$

for some $p,q,r,s \in C$. Because $\mathrm{h}^0(C,E) \ge 4$, the line bundle $\mathcal{O}(p+q+r+s)$ must have two independent sections and the connecting homomorphism $\delta : \mathrm{H}^0(C,\mathcal{O}(p+q+r+s)) \to \mathrm{H}^1(C,K(-p-q-r-s))$ must be zero.

Since C is general, $G^1_4 \subset \mathrm{Pic}^4(C)$ is a smooth curve and for all $L \in G^1_4$, $K \otimes L^{-1} \not\cong L$. The space of extensions $(*)$ for a given g^1_4 is parametrized by $\mathrm{H}^1(C, K(-2p-2q-2r-2s)) \cong \mathrm{H}^0(C, \mathcal{O}(2p+2q+2r+2s))^*$ which therefore has dimension 4. On the other hand, the map from extensions to connecting homomorphisms: $\mathrm{H}^0(C, \mathcal{O}(2p+2q+2r+2s))^* \to \mathrm{Sym}^2(\mathrm{H}^0(C, \mathcal{O}(p+q+r+s))^*$ is dual to the multiplication map, so is clearly surjective, and has a kernel of dimension one. This means that there are one non-split bundle E and one split bundle E' fitting into each sequence $(*)$, such that $\dim(\mathrm{H}^0(C,E)) = 4$.

The locus of extensions $0 \to \mathcal{O} \to E \to K \to 0$ so that $\dim(\mathrm{H}^0(C,E)) = 4$ and E has a subbundle of degree four is therefore of dimension at most equal to $\dim(G^1_4) + \dim(\mathrm{H}^0(C,E)) - 1 = 4$. (For the split extensions, the dimension is actually 3.) In any case, this is strictly less than the dimension of the degeneracy locus, and the proof is completed.

Note: Since a general canonical curve of genus 5 is a complete intersection of 3 general quadrics in $\mathbf{P}^4$, the subspace $\mathbf{P}(\mathrm{H}^0(C,K^2)^*) \subset \mathbf{P}(\mathrm{Sym}^2\mathrm{H}^0(C,K)^*)$ of codimension 3 is also general, and it follows that $W^3_{2,K}$ is irreducible of dimension 2. Note that this is larger than $\rho_{2g-2}(3) = 1$. For higher genus, this argument for irreducibility and dimension no longer works.

genus 6: To show: $W^4_{2,K} \neq \emptyset$ ($\rho_K(4) = 0$, so $\dim(D_4 \cap P(\mathrm{H}^0(C,K^2)^*) \geq 4$)

(1) $(D_4 \cap P(\mathrm{H}^0(C,K^2)^*) \cap (\mathrm{Sec}^5(C) - \mathrm{Sec}^4(C)) = \emptyset$. There are no g^2_5's on a general curve of genus 6.

(2) Suppose E is associated to an extension in $(D_4 \cap P(\mathrm{H}^0(C,K^2)^*) \cap (\mathrm{Sec}^4(C) - \mathrm{Sec}^3(C))$. Then as for genus 5, E must fit into an exact sequence:

$$(*) \quad 0 \to K(-p-q-r-s) \to E \to \mathcal{O}(p+q+r+s) \to 0$$

where the connecting homomorphism is zero. ($K(-p-q-r-s)$ is a g^2_6.) This time, though, the number of g^1_4's is finite, and for each such g^1_4, the map from extensions to connecting homomorphisms is an isomorphism, so an E in $(*)$ with 5 independent sections must split. But such E's have a two-dimensional family of endomorphisms, from which it follows that the dimension of the locus of extensions in the intersection is at most $\dim(G^1_4) + 5 - 2 = 3$, and as in the genus 5 case, this implies existence.

Note: For genus six, the naive $\rho_{2g-2}(4)$ is negative, so the nonemptiness of $W^4_{2,K}$ contradicts the naive non-existence conjecture.

Note: For a general curve of genus 6 (as we have seen) and greater, it is easy to see that with the exception of genus 9 and 10, a strictly semistable bundle (i.e. a bundle with a line subbundle of degree $g-1$) can never have the maximal number $r+1$ of independent sections so that $\rho_K(r) \geq 0$. This may mean that one could try to prove existence of $W^r_{2,K}$ for largest r by "standard" techniques (i.e. Porteous formula or ampleness) on the compactification $\mathcal{M}_C(2, 2g-2)$, arguing that any nonempty class so obtained must lie in $\mathcal{M}^s_C(2, 2g-2)$.

The argument for genus 6 can be generalized as follows. Find the first $d \leq g-1$ so that $(D_r \cap \mathbf{P}(\mathrm{H}^0(C,K^2)^*)) \cap (\mathrm{Sec}^d(C) - \mathrm{Sec}^{d-1}(C)) \neq \emptyset$. Then argue that for this choice of d, the dimension of the intersection is less than the expected dimension of $D_r \cap \mathbf{P}(\mathrm{H}^0(C,K^2)^*)$. Conclude that $W^r_{2,K} \neq \emptyset$.

This argument works well at least up to genus 12. However, it seems not to work beyond that point, the reason being, we suspect, that the locus of extensions parametrizing unstable bundles with a large number of sections begins to have larger than expected dimension. In particular, genus 13 seems to be a very delicate case.

As additional evidence for $\rho_K(r)$, we show nonexistence for small genus and $\rho_K(r) < 0$. (This is part (a) of the conjecture.)

genus 3: To show: $W^3_{2,K} = \emptyset$.

If a rank two bundle has $2n$ sections, it is immediate that it has a section vanishing at n points. Thus, if $E \in W^3_{2,K}$, then there is a section of E vanishing at 2 points. This contradicts stability.

genus 4: To show: $W^3_{2,K} = \emptyset$.

If $E \in W^3_{2,K}$, then E must fit into an exact sequence:

$$0 \to \mathcal{O}(p+q) \to E \to K(-p-q) \to 0$$

But this cannot produce $h^0(C,E) \geq 4$ since C is not hyperelliptic.

genus 5: To show: $W^4_{2,K} = \emptyset$.

If $E \in W^4_{2,K}$, then either E fits into an exact sequence:

$$0 \to \mathcal{O}(p+q) \to E \to K(-p-q) \to 0$$

and as in genus 4, not enough sections are produced, or E fits into an exact sequence:

$$0 \to \mathcal{O}(p+q+r) \to E \to K(-p-q-r) \to 0$$

which is even worse, since there are no g^1_3's on a general curve of genus 5.

genus 6: To show: $W^5_{2,K} = \emptyset$.

Here, $E \in W^5_{2,K}$ would have a section vanishing at 3 or 4 points. Both of these are out because there are no g^1_3's and a g^1_4 sub-line bundle wouldn't give enough sections.

These arguments also can be extended out to at least genus 8. If we additionally use Segre's theorem, which states that any rank two bundle E with canonical determinant must have a subbundle of degree at least $\left[\frac{g-1}{2}\right]$ (cf. [L1]), then we can also prove nonexistence for genus $9, 11$ and 13 by these arguments.

However, genus 10 and 12 are more subtle. Mukai (cf. [Mu1] or [V]) has shown that if C is a curve of genus $2s$, $s \geq 5$, general among the generators of Pic of a $K3$ surface, then there is always a stable rank 2 bundle on C with canonical determinant and $s+2$ independent sections. This is exactly where we need to prove non-existence for genus 10 and 12. In light of Lazarsfeld's theorem (cf. [L2]) that such curves are Petri general for line bundles, it follows that unlike the other low genera cases, the locus of curves of genus 10 or 12 that satisfy non-existence in our conjecture will not contain the Petri general locus, if indeed it is non-empty.

4 INJECTIVITY OF THE PETRI MAP

In this section, we show that the modified Petri map is injective for small values of r. This will complete the picture for several of the low genus examples studied in §3. In particular, we prove:

Proposition 4.1: If C is general of genus $g \geq 2$, E is a stable bundle with canonical determinant generically generated by its global sections and $\dim(\mathrm{H}^0(C, E)) \leq 5$, then

$$\mu : \mathrm{Sym}^2(\mathrm{H}^0(C, E)) \to \mathrm{H}^0(C, \mathrm{Sym}^2(E))$$

is injective.

We note in passing that if E is not generically generated by its global sections, there is a subbundle L of E of degree less than $g-1$, such that $\mathrm{H}^0(C, L) = \mathrm{H}^0(C, E)$. It is not hard to show that if $h^0(C, E) \leq 4$, the map $\mu : \mathrm{Sym}^2(\mathrm{H}^0(C, L)) \to \mathrm{H}^0(C, L^2)$ is injective. A general curve can only admit a line bundle of degree less than $g-1$ with 5 sections if $g \geq 35$, so for now we can safely ignore the bundles not generated by their global sections, and conclude from the Proposition and §3 that:

The conjecture holds for general curves of genus ≤ 7

Proof of the Proposition: For $h^0(C, E) \leq 2$, the proposition is easily verified. For $h^0(C, E) \geq 3$, we begin by giving a geometric description of the injectivity of μ. The evaluation map:

$$\varepsilon : \mathrm{H}^0(C, E) \otimes \mathcal{O}_C \to E$$

induces the morphism:

$$\Phi_E : C \to \ \mathrm{Gr}(2, \mathrm{H}^0(C, E)^*), \quad p \mapsto E_p^*$$

Let q be a quadric hypersurface in $\mathbf{P}(\mathrm{H}^0(C, E)^*)$ and denote by $\mathrm{Gr}(1, q)$ the subvariety in $\mathrm{Gr}(2, \mathrm{H}^0(C, E)^*)$ of projective lines in q. Then

Claim: The map μ is injective if and only if $\Phi_E(C)$ is not contained in $\mathrm{Gr}(1, q)$ for any quadric q in $\mathbf{P}(\mathrm{H}^0(C, E)^*)$.

Proof of Claim: Let S be the universal subbundle over the Grassmannian $G := \mathrm{Gr}(2, \mathrm{H}^0(C, E)^*)$, and let $i : S \to \mathrm{H}^0(C, E)^* \otimes \mathcal{O}_G$ be the natural inclusion map. Consider the (injective) map:

$$\Psi : \mathrm{Sym}^2(\mathrm{H}^0(C, E)) \to \mathrm{H}^0(G, \mathrm{Sym}^2 S^*), \phi \mapsto i^* \circ \phi \circ i.$$

Let q_ϕ be the quadric hypersurface in $\mathbf{P}(\mathrm{H}^0(C, E)^*)$ associated to the quadratic form $\phi \in \mathrm{Sym}^2(\mathrm{H}^0(C, E))$. Then $\mu(\phi) = 0$ if and only if $\Phi_E(C)$ is contained in the zero locus of $\Psi(\phi)$, which is just $\mathrm{Gr}(1, q_\phi)$.

If $\dim(\mathrm{H}^0(C, E)) = 3$, then $\mathrm{Gr}(1, q)$ consists of at most two points. By the preceeding claim, μ is injective.

If $\dim(\mathrm{H}^0(C, E)) = 4$ and q_ϕ is smooth, then the image of $\mathrm{Gr}(1, q_\phi)$ under the Plücker embedding, $\rho : \mathrm{Gr}(2, \mathrm{H}^0(C, E)^*) \to \mathbf{P}(\wedge^2 \mathrm{H}^0(C, E)^*)$, is equal to a disjoint pair of smooth plane conics (see [GH], p. 199.) It follows that if $\Phi_E(C)$

is contained in $\mathrm{Gr}(1, q_\phi)$, then there exists a line bundle L in W^1_{g-1} such that $L^2 \subset (\rho \circ \Phi_E)^* \mathcal{O}(1) \cong K_C$ which contradicts the injectivity of the ordinary Petri map for line bundles.

If q_ϕ is of rank three with vertex p and $\Phi_E(C)$ is contained in q_ϕ, then $\Phi_E(C)$ is contained in the plane:

$$\Gamma = \{\Lambda : \langle p \rangle \subset \Lambda \subset \mathrm{H}^0(C, E)^*\}.$$

But this is the zero locus of a section of the universal quotient bundle Q over G. Consequently, the pull-back to C of the kernel of the map

$$\mathrm{H}^0(C, E)^* \otimes \mathcal{O}_G \to Q$$

has a section. That is, E^* has a section, which contradicts the stability of E.

If q_ϕ is of rank two, then $\mathrm{Gr}(1, q_\phi)$ is a pair of disjoint planes R_1 and R_2. These planes are the zero loci of sections of S^*, so $\Phi_E(C)$ cannot be contained in either of them. The case where q_ϕ is of rank one is the same.

If $\dim(\mathrm{H}^0(C, E)) = 5$ and q_ϕ is smooth, then the image of $\mathrm{Gr}(1, q_\phi)$ under the Plücker embedding is given by the image of $\mathbf{P}^3$ under the Veronese embedding:

$$\nu : \mathbf{P}^3 \to \mathbf{P}^9$$

Therefore, once again, if $\Phi_E(C)$ is contained in $\mathrm{Gr}(1, q_\phi)$, then there exists a line bundle L in W^1_d with $d \leq g - 1$ and

$$L^2 \subseteq (\rho \circ \Phi_E)^* \mathcal{O}(1) \cong K_C$$

But no such L exists on a general curve.

The cases where q_ϕ is not smooth may be disposed of by arguments similar to those when $\dim(\mathrm{H}^0(C, E)) = 4$.

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Update: Independently of the authors (and at around the same time), Shigeru Mukai formulated the same conjecture for Brill-Noether loci of rank two bundles with canonical determinant ([Mu2]). In addition, Mukai has exhibited some breathtakingly beautiful examples in small genus, where the geometry of the lowest dimensional (smooth!) loci is worked out in detail. The interested reader is urged to consult [Mu2] and [Mu3]

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