THE FORMULA $12 = 10 + 2 \times 1$ AND ITS GENERALIZATIONS: COUNTING RATIONAL CURVES ON $\mathbf{F}_2$

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1. The problem

Gromov-Witten invariants are symplectic invariants of compact symplectic manifolds [K-M, R-T]. Thus, they are constant along deformations of compact Kähler manofolds, and in particular along deformations of smooth complex projective varieties. For convex varieties, Gromov-Witten invariants in genus 0 are enumerative, namely they count the number of rational curves satisfying sufficiently general incidence conditions, see [Kon, B-M, F-P, R-T].

In this note we consider an explicit case of a surface - the Hirzebruch surface $\mathbf{F}_2$ - which is not convex, but deforms algebraically to the convex surface $\mathbb{P}^1 \times \mathbb{P}^1$. This enables us to compare the Gromov-Witten invariants of $\mathbf{F}_2$ (calculated as usual on $\mathbb{P}^1 \times \mathbb{P}^1$) with its enumerative invariants (see [C-H]) explicitly, by following the rational curves on $\mathbb{P}^1 \times \mathbb{P}^1$ to their limits on $\mathbf{F}_2$. As a result we get a recursive formula for the enumerative invariants of $\mathbf{F}_2$ involving those of $\mathbb{P}^1 \times \mathbb{P}^1$. The formulas are different from those of Caporaso and Harris, but the resulting numbers coincide.

The results of this note, inspired by a talk of L. Caporaso, were worked out in late 1995 and written up in 1998.

2. Preliminaries

We work over $\mathbb{C}$.

2.1. The deformation. We fix once and for all a one-parameter deformation $\pi : X \to S$ of the surface $\mathbf{F}_2$. Here S is an affine curve (or the spectrum of a discrete valuation ring), $s \in S$ is a closed point, and $\eta \in S$ the generic point (or general point, depending on the reader's preference). The morphism $X \to S$ is flat, projective, of relative dimension 2. We fix isomorphisms $\mathbf{F}_2 \xrightarrow{\sim} X_s$ and $(\mathbb{P}^1 \times \mathbb{P}^1)_\eta \xrightarrow{\sim} X_\eta$. An explicit description of such a deformation space can be found in [Kod].

2.2. Divisor classes on $\mathbf{F}_2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. The Néron-Severi group of $X_s = \mathbf{F}_2$ is generated by two classes. One class corresponds to the fiber of the ruling, denoted F, and one corresponds to a general section, denoted C_2. We denote the exceptional (-2) curve by E - it is linearly equivalent to $C_2 - 2F$.

We will only concern ourselves with **nef** Néron-Severi classes on $\mathbf{F}_2$. The associated linear series of these classes have no base points. These classes are of the form $a \cdot C_2 + b \cdot F$ for $a, b \geq 0$.

The Néron-Severi group of $X_\eta = (\mathbb{P}^1 \times \mathbb{P}^1)_\eta$ is generated by the rulings, which we may denote by F and C_0.

Since the surfaces are rational, their Picard groups are isomorphic to their Néron-Severi groups.

2.3. Divisor classes on X. Looking at the total space, the relative Picard group of $X \to S$ is a trivial rank-2 local system, where the class F on X_s deforms to the class F on X_η (and there is no fear of confusion), and C_2 deforms to $C_0 + F$. We denote by $\mathcal{L}_{a,b}$ the invertible sheaf on X whose restriction to X_s is $\mathcal{O}_{X_s}(a \cdot C_2 + b \cdot F)$. On X_η it restricts to the line bundle $\mathcal{O}_{(\mathbb{P}^1 \times \mathbb{P}^1)_\eta}(a, a+b)$. The formation of the locally free sheaf $\pi_* \mathcal{L}_{a,b}$ commutes with base change whenever $a, b \geq 0$. Denote by $\mathbb{P}_{a,b}$ the associated family of linear series, $\mathbb{P}_{a,b} = \mathbb{P}(\pi_* \mathcal{L}_{a,b})^\vee$.

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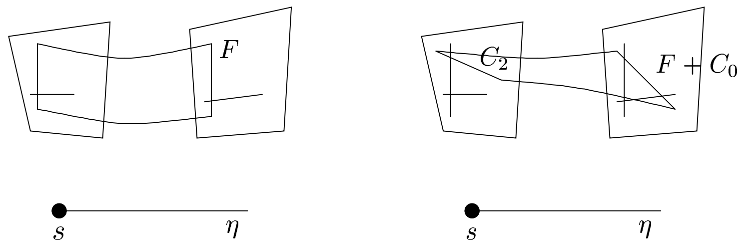

FIGURE 1. The divisor F deforms to F, while C_2 deforms to $F + C_0$.

2.4. **Moduli of stable maps.** It follows from [B-M] that for each (a, b) there is a Deligne - Mumford stack $\overline{\mathcal{M}}_{a,b} = \overline{\mathcal{M}}_0(X/S, \mathcal{L}_{a,b})$ parametrizing genus 0 stable maps, mapping to the fibers of X over S, with image class $c_1(\mathcal{L}_{a,b})$. This stack admits a coarse moduli space $\overline{\mathbf{M}}_{a,b}$, and by [F-P] there is a *projective* morphism $\overline{\mathbf{M}}_{a,b} \to S$.

Since $\mathbb{P}^1 \times \mathbb{P}^1$ is convex, it follows from [Kon] or [F-P] that when $a > 0$ and $b \geq 0$, the generic fiber $(\overline{\mathcal{M}}_{a,b})_\eta$ is smooth, irreducible and of the expected dimension: $\dim(\overline{\mathcal{M}}_{a,b})_\eta = 4a + 2b - 1$; in particular, the coarse moduli space $(\overline{\mathbf{M}}_{a,b})_\eta$ has only finite quotient singularities. Moreover, it contains a dense open set parametrizing immersions of $\mathbb{P}^1$ into $\mathbb{P}^1 \times \mathbb{P}^1$ as a nodal curve.

2.5. **Gromov-Witten invariants.** Fix non-negative integers a and b, to which we have associated an invertible sheaf $\mathcal{L}_{a,b}$ and divisor classes $aC_2 + bF$ on $\mathbf{F}_2$ and $aC_0 + (a + b)F$ on $\mathbb{P}^1 \times \mathbb{P}^1$. The associated Gromov-Witten invariant $N(aC_2 + bF) = N(aC_0 + (a + b)F)$ was originally defined as the number of pseudo-holomorphic rational curves, analytic with respect to a generic almost complex structure, on $\mathbf{F}_2$. Since $\mathbb{P}^1 \times \mathbb{P}^1$ is convex, its holomorphic structure is sufficiently general for this purpose - thus the Gromov-Witten invariant $N(aC_2 + bF) = N(aC_0 + (a + b)F)$ is the number of irreducible nodal rational curves on $\mathbb{P}^1 \times \mathbb{P}^1$ of class $aC_0 + (a + b)F$ passing through $4a + 2b - 1$ general points.

If moreover $\mathcal{L}_{a,b}$ is nontrivial, then there is a natural morphism $\mathbf{Div} : \overline{\mathbf{M}}_{a,b} \to \mathbb{P}_{a,b}$, which to a stable map $f : C \to X$ associates the unique image divisor. Using this fact, we can define the Gromov-Witten number as the intersection number

$$N(aC_2 + bF) = c_1(\mathbf{Div}^* \mathcal{O}_{\mathbb{P}_{a,b}}(1))^{4a+2b-1} \cap [(\overline{\mathbf{M}}_{a,b})_\eta].$$

Remark 2.5.1. In general there is a *virtual fundamental class* $[(\overline{\mathbf{M}}_{a,b})^{\mathrm{virt}}]$ which replaces the fundamental class in the non-convex case. The identification of the virtual fundamental class can be subtle.

2.6. **Enumerative invariants on $\mathbf{F}_2$.** Analogously, denote by $N_2(aC_2 + bF)$ the number of irreducible nodal rational curves of class $aC_2 + bF$ passing through $4a + 2b - 1$ general points on $\mathbf{F}_2$. The following question arises naturally:

> What is $N_2(aC_2 + bF)$, what is its relationship with $N(aC_2 + bF)$, and can the relationship be explained via the deformation space X?

A direct recursive formula for $N_2(aC_2 + bF)$ has been given by Caporaso and Harris in [C-H]. For example, $N_2(2C_2) = 10$ while $N(2C_2) = 12$. Caporaso and Harris gave an interpretation for the difference $N(2C_2) - N_2(2C_2) = 2$, but not in terms of a deformation of $\mathbf{F}_2$.

3. THE FORMULA AND ITS PROOF

3.1. **Statement of enumerative result.**

Theorem 3.1.1. *For a nef divisor class $D = aC_2 + bF$ we have*

$$N(D) = \sum_{k=0}^{a-1} \binom{b+2k}{k} N_2(D - kE).$$

Remark 3.1.2. Note that $D - kE = (a - k)C_2 + (b + 2k)F$. When $D = 2C_2$ we get $N(D) = N_2(D) + 2 \times N_2(D - E) = N_2(D) + 2 \times N_2(C_2 + 2F)$. Now it is well known (see, e.g., [C-H]) that $N(D) = N(2C_0 + 2F) = 12$, and $N_2(C_2 + 2F) = 1$ since $C_2 + 2F$ has arithmetic genus 0. Thus the number $N_2(D) = 10$ fits in the formula $12 = 10 + 2 \times 1$, whence the title of this note.

3.2. **Relevant components of $(\overline{\mathbf{M}}_{a,b})_s$.** Consider the Zariski closure of $(\overline{\mathbf{M}}_{a,b})_\eta$ in $\overline{\mathbf{M}}_{a,b}$. This is a flat scheme over S, therefore the degree of the invertible sheaf $\mathbf{Div}^* \mathcal{O}_{\mathbb{P}_{a,b}}(1)$ is constant along the fibers. Calculating this degree on the special fiber boils down to the following issues:

(1) Among the irreducible components of $(\overline{\mathbf{M}}_{a,b})_s$, which ones have dimension $\geq 4a + 2b - 1$?
(2) Among these irreducible components, for which ones does the image in $(\mathbb{P}_{a,b})_s$ have dimension $\geq 4a + 2b - 1$?
(3) Among the latter components, which ones contain a $4a+2b-1$ dimensional subscheme M_0 of the Zariski closure $\overline{(\overline{\mathbf{M}}_{a,b})_\eta}$, such that the image of M_0 in $(\mathbb{P}_{a,b})_s$ has dimension $4a + 2b - 1$?
(4) What are the degrees of these subschemes M_0 with respect to $\mathbf{Div}^* \mathcal{O}_{\mathbb{P}_{a,b}}(1)$?

An irreducible component which contains a $4a + 2b - 1$ dimensional subscheme M_0 of the Zariski closure $\overline{(\overline{\mathbf{M}}_{a,b})_\eta}$ whose image in $(\mathbb{P}_{a,b})_s$ has dimension $4a+2b-1$, is called **relevant**. Ideally, one would hope that every relevant irreducible component M has dimension exactly $4a + 2b - 1$, is reduced, and has an easily computable degree; in particular $M = M_0$. The following propositions state that this is indeed the case:

Proposition 3.2.1. *Let $M \subset (\overline{\mathbf{M}}_{a,b})_s$ be a relevant irreducible component. Then the general point of M corresponds to a genus 0 stable map $f : C \to \mathbf{F}_2$ of the following form:*

(1) *the curve C decomposes into irreducible components $C = C' \cup E_1 \cup \ldots \cup E_k$ for some $0 \leq k \leq a - 1$;*
(2) *the morphism f maps each of the components E_i isomorphically to the exceptional (-2) curve E;*
(3) *the morphism f maps C' birationally onto a nodal rational curve of class $D - kE$ in $\mathbf{F}_2$ meeting E transversally;*
(4) *the intersection $C' \cap E_i$ consists of a single node. The intersection $E_i \cap E_j$ is empty for $i \neq j$*

Remark 3.2.2. We call such a stable maps *an antenna*. The component C' is its *spine* and the components E_i are the *cross-pieces*.

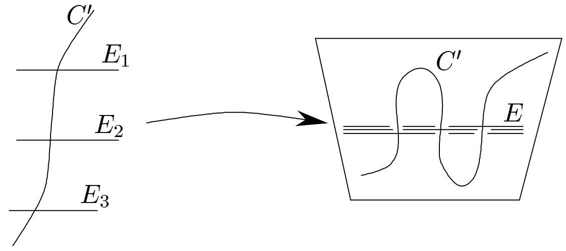

FIGURE 2. An antenna.

Proposition 3.2.3. *For each integer $0 \leq k \leq a - 1$ there is a unique relevant irreducible component $M_k(D) \subset (\overline{\mathbf{M}}_{a,b})_s$ whose generic point corresponds to an antenna with k cross-pieces. This component $M_k(D)$ is reduced, and has degree $\binom{b+2k}{k} N_2(D - kE)$ with respect to $\mathbf{Div}^* \mathcal{O}_{\mathbb{P}_{a,b}}(1)$.*

3.3. **Relevant components: sufficiency of conditions.** Let $f : C \to \mathbf{F}_2$ be an antenna with k cross-pieces. The morphism $f^*\Omega^1_{\mathbf{F}_2} \to \Omega^1_C$ is surjective. Its kernel is the conormal sheaf N^* which is invertible. The infinitesimal deformation space of f (keeping the target $\mathbf{F}_2$ fixed) has tangent space $\mathrm{Hom}(N^*, \mathcal{O}_C) = H^0(C, N)$, and the obstructions lie in $\mathrm{Ext}^1(N^*, \mathcal{O}_C) = H^1(C, N)$. It is easy to see that $\deg_{E_i} N = -1$ and $\deg_{C'} N = 4a + 2b - 2 + k$. A simple calculation reveals that $H^1(C, N) = 0$, namely there are no obstructions, and $\dim \mathrm{Hom}(N^*, \mathcal{O}_C) = 4a + 2b - 1$.

In particular, since $\deg_{E_i} N = -1$ every section of N vanishes on each E_i. Therefore any deformation of f in $\mathbf{F}_2$ remains an antenna with k cross-pieces. Since there are no obstructions, the space of stable maps $(\overline{\mathbf{M}}_{a,b})_s$ is smooth at f, with dimension $4a + 2b - 1$. For the same reason (see [Kol]) f deforms freely in the family $X \to S$, therefore any point in $(\overline{\mathbf{M}}_{a,b})_s$ in a neighborhood of f deforms to $(\overline{\mathbf{M}}_{a,b})_\eta$. This shows that f belongs to a relevant component of $(\overline{\mathbf{M}}_{a,b})_s$.

An easy monodromy and degeneration argument following [Har] or [C-H] shows that the space of antennas with k cross-pieces is irreducible. The morphism $\mathbf{Div} : M_k(D) \to \mathbb{P}_{a,b}$ has degree $\binom{b+2k}{k}$ onto the image, since for a general antenna with k cross-pieces there are $b + 2k$ intersection points of $f(C')$ with E, and the antenna is formed by choosing k of these points as attaching points with E_i. Also, the explicit description shows that the degree of the image $\mathbf{Div}(M_k(D))$ coincides with the degree of $\mathbf{Div}(M_0(D - kE))$, which is by definition the enumerative invariant $N_2(D - kE)$.

Thus, in order to complete the proof of the propositions and the theorem, it suffices to show that there are no other relevant components.

3.4. **Relevant components: necessity of conditions.** Consider a relevant component M, containing a subscheme M_0 of dimension $4a+2b-1$ in the closure of $(\overline{\mathbf{M}}_{a,b})_\eta$, and a general point on M_0 corresponding to a stable map $f: C \to \mathbf{F}_2$. Let $C_E = f^{-1}E$ and $C' = \overline{C \smallsetminus C_E}$. The curve C' consists of the components of C whose image does not lie in E.

An easy dimension count, using the fact that $\mathbf{Div}(M)$ has dimension $4a+2b-1$, shows that C' is irreducible. This means that there is a decomposition into connected curves $C = C' \cup E_1 \ldots \cup E_l$, where E_i map generically finitely to E with degree d_i, and C' maps to a general irreducible nodal rational curve of class $D-kE$ where $k = \sum_{i=1}^l d_i$. The curves E_i are trees of rational curves, each meeting C' at a single node. The picture is similar to that of an antenna, but where cross-pieces can be more elaborate.

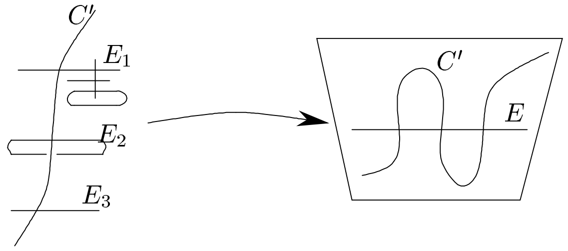

FIGURE 3. A generalized antenna.

We need to show that $d_i = 1$ for all i. This will be derived, in a manner similar to that in [C-H], from the fact that f deforms to X_η.

For any choice of a fiber F on $\mathbf{F}_2$, the space X contains a Cartier divisor H such that $H_\eta = C_0$ and $H_s = E+F$. Indeed, the linear series of C_0, which has dimension 1, has as its limit a subscheme of the linear series of $E+F$, which also has dimension 1, therefore every element $E+F$ is in this limit.

Looking at our proposed general stable map $f: C \to \mathbf{F}_2$, note that $f(C')$ meets $E+F$ transversely in $(D-kE)\cdot(E+F) = a+b+k$ points.

Consider now a finite surjective base change $S' \to S$, a deformation $\tilde{C} \to S'$ of C and a morphism $\tilde{f}: \tilde{C} \to X_{S'}$ (where $X_{S'} = X \times_S S'$). Assume $\tilde{f}$ is a general deformation of f into X, namely that $\tilde{C}_{\eta'}$ is a smooth rational curve immersed by $\tilde{f}_{\eta'}$ in $X_{\eta'}$. Consider $\tilde{f}^*H$. This is a Cartier divisor on $\tilde{C}$. Its restriction to $\tilde{C}_{\eta'}$ has degree $D \cdot H_s = a+b$. Thus we can write $\tilde{f}^*H = H_h + \sum_{i=1}^l E_i'$, where $H_h \to S'$ is finite of degree $a+b$, and E_i' are Weil divisors supported along E_i. In particular, the *total number* of points in $C' \cap \tilde{f}^*H$, counted *without multiplicities,* is at most $a+b+l$. Since $l \le k$ an equality must hold, in particular $d_i = 1$ for all i. $\square$

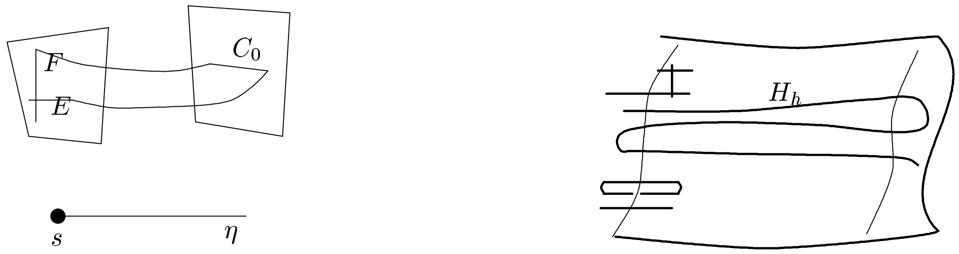

FIGURE 4. The divisor H (left) and its pullback (right).

Remark 3.4.1. The virtual fundamental class $[(\overline{\mathbf{M}}_{a,b})^{\mathrm{virt}}]$ does have nontrivial support on all the components of $(\overline{\mathbf{M}}_{a,b})_s$ described above, even when $d_i > 1$. Indeed the virtual fundamental class is the direct image of the fundamental class of the closure of $(\overline{\mathbf{M}}_{a,b})_\eta$. What we have shown above is that the components where $d_i > 1$ do not contribute to Gromov-Witten invariants.

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