GREEN'S CONJECTURE

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1. Koszul cohomology

We begin with a complex vector space V of dimension n and form the symmetric algebra

$$S := S^*V = \bigoplus_{k\geq 0} S^kV.$$

Suppose now we are given a graded S-module

$$B = \bigoplus_{k\geq 0} B^k.$$

We define

$$\mathcal{K}_{p,q}(B,V) = \left(\bigwedge^p V\right)\otimes B^q$$

and define a coboundary operator

$$\delta : \mathcal{K}_{p,q}(B,V) \to \mathcal{K}_{p-1,q+1}(B,V) \tag{1}$$

as the composition

$$\begin{tikzcd}
v_1\wedge\ldots\wedge v_p\otimes b \arrow[d, mapsto] & \left(\bigwedge^p V\right)\otimes B^q \arrow[d] \\
\sum_i (-1)^i v_1\wedge\ldots\wedge v_{i-1}\wedge v_{i+1}\wedge\ldots\wedge v_p\otimes v_i\otimes b \arrow[d, mapsto] & \left(\bigwedge^{p-1} V\right)\otimes V\otimes B^q \arrow[d] \\
\sum_i (-1)^i v_1\wedge\ldots\wedge v_{i-1}\wedge v_{i+1}\wedge\ldots\wedge v_p\otimes (v_i\cdot b) & \left(\bigwedge^{p-1} V\right)\otimes B^{q+1}
\end{tikzcd}.$$

Exercise 1.1.

$$\delta\circ\delta = 0.$$

Definition 1.1. The Koszul cohomology is given by

$$K_{p,q}(B,V) = H^0\left(\mathcal{K}_{p-*,q+*}(B,V),\delta\right).$$

Next consider

$$V^\vee = SpecS$$

as an affine variety with free resolution

$$0\to\bigwedge^n V\otimes\mathcal{O}_{V^\vee}\to\ldots\to\bigwedge^p V\otimes\mathcal{O}_{V^\vee}\to\ldots\to V\otimes\mathcal{O}_{V^\vee}\to\mathcal{O}_{V^\vee}$$

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of the skyscraper sheaf

$$\frac{\mathcal{O}_{V^\vee}}{\mathfrak{m}_{0,V^\vee}}$$

supported at $0 \in V^\vee$. Since $V^\vee$ is affine, Γ is exact, giving us the Koszul resolution

$$0 \to \bigwedge^n V \otimes S \to \ldots \to \bigwedge^p V \otimes S \to \ldots \to V \otimes S \to S \tag{2}$$

of the graded S-module

$$\mathbb{C} = \frac{S}{\mathfrak{m}_0 S}.$$

Tensoring (2) with the S-module B and taking the degree-$(p+q)$ part, we have

$$0 \to \bigwedge^n V \otimes B^{q-n+p} \to \ldots \to \bigwedge^p V \otimes B^q \to \ldots \to V \otimes B^{p+q-1} \to B^{p+q} \tag{3}$$

One checks that the maps in (3) are exactly the maps δ above. Thus

$$K_{p,q}(B,V) = Tor_p^{p+q}(B,\mathbb{C}).$$

On the other hand, if we choose a minimal (graded) free resolution

$$\ldots \to \sum\nolimits_q M_{p,q} \otimes_{\mathbb{C}} S(-q) \to \ldots \to \sum\nolimits_q M_{0,q} \otimes_{\mathbb{C}} S(-q) \to B \to 0$$

of B/S, we can compute $Tor_p^{p+q}(B,\mathbb{C})$ by tensoring this last sequence by $\frac{S}{\mathfrak{m}_0 S}$. Since this reduces all boundary maps to zero by minimality,

$$M_{p,p+q} = Tor_p^{p+q}(B,\mathbb{C}) = K_{p,q}(B,V).$$

2. GEOMETRY

Let X be a projective variety and L be a line bundle on X. We apply the constructions of the previous section to the case

$$\begin{aligned} V &= H^0(X;L) \\ B^q &= H^0(X;L^q). \end{aligned}$$

We write

$$K_{p,q}(B,V) = K_{p,q}(X,L).$$

Notice

$$\begin{array}{rcl} K_{0,q}(B,V) & = & 0,\ q \geq 2 \\ & \Updownarrow & \\ M_{0,q} & = & 0,\ q \geq 2 \\ & \Updownarrow & \\ S^q H^0(X;L) & \twoheadrightarrow & H^0(X;L^q),\ q \geq 2 \end{array} \tag{4}$$

that is, X is projectively normal with respect to L. Similarly notice that, if (4) is satisfied,

$$\begin{array}{rcl} K_{1,q}(B,V) & = & 0,\ q \geq 2 \\ & \Updownarrow & \\ M_{1,1+q} & = & 0,\ q \geq 2 \\ & \Updownarrow & \\ M_{1,2} \otimes S(-2) & \twoheadrightarrow & I_X \end{array} \tag{5}$$

where

$$I_X = \ker\left(S^* H^0(X;L) \to \bigoplus H^0(X;L^q)\right).$$

This is simply the statement that I_X is generated by quadrics.

Notice also that there is a twisted version of this theory for any vector bundle F/X with

$$B = \bigoplus H^0(X; L^q \otimes F).$$

Our notation for the Koszul cohomology will be

$$K_{p,q}(B,V) = K_{p,q}(X,L;F).$$

3. Computing Koszul cohomology in the geometric case

We continue with the situation of the last section and, for simplicity, we assume that L is generated by global sections. Define the locally free sheaf M_L by the exact sequence

$$0 \to M_L \to H^0(X;L) \otimes \mathcal{O}_X \to L \to 0. \tag{6}$$

This sequence gives rise to the exact sequence

$$0 \to \bigwedge^p M_L \otimes L^q \to \bigwedge^p H^0(X;L) \otimes L^q \to \left(\bigwedge^{p-1} M_L\right) \otimes L^{q+1} \to 0$$

which we will call $(*_{p,q})$, and so to a map

$$\delta' : \bigwedge^p H^0(X;L) \otimes H^0(L^q \otimes F) \to H^0\left(\left(\bigwedge^{p-1} M_L\right) \otimes L^{q+1} \otimes F\right)$$

which composes with the inclusion

$$H^0\left(\left(\bigwedge^{p-1} M_L\right) \otimes L^{q+1} \otimes F\right) \to \bigwedge^{p-1} H^0(X;L) \otimes H^0\left(L^{q+1} \otimes F\right)$$

to induce the map δ given in (1). Thus

$$\ker\delta = H^0\left(\bigwedge^p M_L \otimes L^q \otimes F\right)$$

and therefore

$$K_{p,q}(X,L;F) \cong \frac{H^0\left(\bigwedge^p M_L \otimes L^q \otimes F\right)}{\delta'\left(\bigwedge^{p+1} H^0(X;L) \otimes H^0\left(L^{q-1} \otimes F\right)\right)}. \tag{7}$$

The isomorphism (7) is actually quite useful for computing Koszul cohomology. The first instance is as follows:

Theorem 3.1. *(Duality theorem) If X is a smooth curve and L is generated by global sections,*

$$K_{p,q}(X,L;F)^\vee = K_{r-1-p,2-q}\left(X,L;F^\vee \otimes K_X\right)$$

where $h^0(X;L) = r+1$.

Proof. Consider the exact sequence

$$0 \to \bigwedge^{p+1} M_L \otimes F \otimes L^{q-1} \to \bigwedge^{p+1} H^0(X;L) \otimes F \otimes L^{q-1} \to \bigwedge^{p} M_L \otimes F \otimes L^q \to 0.$$

Then δ' in (7) is simply the induced map on H^0 in this last sequence so that $K_{p,q}(X,L;F)$ is given by

$$\ker\left(H^1\left(\bigwedge^{p+1} M_L \otimes F \otimes L^{q-1}\right) \to \bigwedge^{p+1} H^0(X;L) \otimes H^1\left(F \otimes L^{q-1}\right)\right)$$

and so, by Serre duality, $K_{p,q}(X,L;F)^\vee$ is the cokernel of the map

$$\bigwedge^{r-p} H^0(X;L) \otimes H^0\left(F^\vee \otimes L^{1-q} \otimes K_X\right) \to H^0\left(\bigwedge^{p+1} M_L^\vee \otimes F^\vee \otimes L^{1-q} \otimes K_X\right)$$

where we use the isomorphism

$$\bigwedge^{r-p} H^0(X;L) \cong \bigwedge^{p+1} H^0(X;L)^\vee.$$

But now

$$\bigwedge^{r} M_L = L^{-1}$$

so that

$$\bigwedge^{p+1} M_L^\vee = \bigwedge^{r-p-1} M_L \otimes L$$

and so $K_{p,q}(X,L;F)^\vee$ is the cokernel of the transposed map

$$\bigwedge^{r-p} H^0(X;L) \otimes H^0\left(F^\vee \otimes L^{1-q} \otimes K_X\right) \to H^0\left(\bigwedge^{r-p-1} M_L \otimes F^\vee \otimes L^{2-q} \otimes K_X\right),$$

which one verifies to be the map δ'. But by (7) this is just

$$K_{r-1-p,2-q}(X,L;F^\vee \otimes K_X).$$

□

Applying Theorem 3.1 to the case $L = K_X$ and $F = \mathcal{O}_X$ we have

$$\begin{aligned} K_{p,q}(X,K_X)^\vee &= K_{g-p-2,2-q}(X,K_X;K_X) \\ &= K_{g-p-2,3-q}(X,K_X). \end{aligned}$$

Thus we have that

$$K_{p,q}(X,K_X) = 0 \tag{8}$$

whenever $q > 3$, so that the ideal of the canonical image of X is always generated by quadrics and cubics. For $q = 3$ we have by (7) and Theorem 3.1 that

$$\begin{aligned} K_{p,3}(X,K_X)^\vee &= K_{g-p-2,0}(X,K_X) \\ &= H^0\left(\bigwedge^{g-p-2} M_{K_X}\right) \end{aligned}$$

so that

$$K_{g-2,3}(X,K_X) = \mathbb{C}$$

and, if $p > g-2$,

$$K_{p,3}(X, K_X) = 0.$$

Furthermore, if $p < g-2$, the exact sequence

$$0 \to \bigwedge^a M_{K_X} \to \bigwedge^a H^0(K_X) \otimes \mathcal{O}_X \to \bigwedge^{a-1} M_{K_X} \otimes K_X \to 0$$

shows that, if $a > 0$, then a non-zero section of $H^0(\bigwedge^a M_{K_X})$ could be wedged with a non-zero section of $\bigwedge^{g-a} H^0(K_X) \otimes \mathcal{O}_X$ to give a non-zero section of $\bigwedge^g H^0(K_X) \otimes \mathcal{O}_X$ which vanishes in $\bigwedge^{g-1} M_{K_X} \otimes K_X = \mathcal{O}_X$ which is absurd. Thus, for $a > 0$,

$$H^0\left(\bigwedge^a M_{K_X}\right) = 0 \tag{9}$$

and so, if $p \neq g-2$,

$$K_{p,3}(X, K_X) = 0.$$

As a second application we have the following proof of a famous theorem of Noether, due to Green and Lazarsfeld:

Theorem 3.2. *If $g \geq 2$ then the multiplication map*

$$\alpha : S^2 H^0(K_X) \to H^0\left(K_X^2\right)$$

if and only if X is not hyperelliptic (i.e. has no g_2^1).

Proof. If X is hyperelliptic its canonical image is a rational normal curve so that $\ker(\alpha)$ is too big to allow surjectivity. Suppose X is not hyperelliptic. By (7) we must exactly show

$$K_{0,2}(X, K_X) = 0.$$

By Theorem 3.1 this is just the assertion

$$K_{g-2,1}(X, K_X) = 0$$

which, by (7) is the assertion that the map

$$\delta' : \bigwedge^{g-1} H^0(K_X) \to H^0\left(\bigwedge^{g-2} M_{K_X} \otimes K_X\right)$$

is surjective. Now from the exact sequence

$$0 \to \bigwedge^{g-1} M_{K_X} \to \bigwedge^{g-1} H^0(K_X) \otimes \mathcal{O}_X \to \bigwedge^{g-2} M_{K_X} \otimes K_X \to 0$$

and the fact that

$$\bigwedge^{g-1} M_{K_X} = -K_X$$

we see that it suffices to show that, if X is not hyperelliptic,

$$h^0\left(\bigwedge^{g-2} M_{K_X} \otimes K_X\right) \leq g.$$

Now

$$\bigwedge^{g-2} M_{K_X} \otimes K_X = M_{K_X}^{\vee}.$$

Since X is not hyperelliptic, X is canonically embedded in $\mathbb{P}^{g-1}$ and so we have that, for generic $x_1, \ldots, x_{g-2} \in X$, the bundles

$$K_X(-x_1 - \ldots - x_m)$$

are globally generated for all $m = 1, \dots, g-2$ and

$$h^0\left(K_X\left(-x_1 - \dots - x_{g-2}\right)\right) = 2.$$

So from the exact diagram

$$\begin{tikzcd}
M_{K_X(-x_1-\dots-x_m)} \arrow[r] \arrow[d] & M_{K_X(-x_1-\dots-x_{m-1})} \arrow[r] \arrow[d] & \mathcal{O}_X\left(-x_m\right) \arrow[d] \\
H^0\left(K_X\left(-x_1 - \dots - x_m\right)\right) \otimes \mathcal{O}_X \arrow[r] \arrow[d] & H^0\left(K_X\left(-x_1 - \dots - x_{m-1}\right)\right) \otimes \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[d] \\
K_X\left(-x_1 - \dots - x_m\right) \arrow[r] & K_X\left(-x_1 - \dots - x_{m-1}\right) \arrow[r] & \mathbb{C}_{x_m}
\end{tikzcd}$$

for $m \leq g-2$ we obtain exact sequences

$$0 \to \mathcal{O}_X\left(x_m\right) \to M^{\vee}_{K_X(-x_1-\dots-x_{m-1})} \to M^{\vee}_{K_X(-x_1-\dots-x_m)} \to 0.$$

Also

$$M^{\vee}_{K_X(-x_1-\dots-x_{g-2})} = K_X\left(-x_1 - \dots - x_{g-2}\right).$$

Thus

$$h^0\left(M^{\vee}_{K_X}\right) \leq \sum\nolimits_{m=1}^{g-2} h^0\left(\mathcal{O}_X\left(x_m\right)\right) + h^0\left(K_X\left(-x_1 - \dots - x_{g-2}\right)\right) = g.$$

□

A similar but more elaborate argument gives Green and Lazarsfeld proof of Petri's theorem:

Theorem 3.3. *Let X be a non-hyperelliptic curve. The ideal I_X of the canonically embedded X is generated by quadrics if and only if X has no g^1_3 nor g^2_5.*

Proof. If X has a g^1_3, the canonical curve has a trisecant line so I_X cannot be generated by quadrics. If X has a g^2_5, the canonical curve has a 5-secant plane and so cannot be generated by quadrics. In the other direction, by (5) we must show that

$$K_{1,q}\left(X, K_X\right) = 0,\ q \geq 2,$$

that is, that

$$K_{1,2}\left(X, K_X\right) = 0$$

since $g > 3$. By Theorem 3.1 we must show that

$$K_{g-3,1}\left(X, K_X\right) = 0,$$

which by (7) translates to the surjectivity of the map

$$\delta' : \bigwedge\nolimits^{g-2} H^0\left(X; K_X\right) \to H^0\left(\bigwedge\nolimits^{g-3} M_{K_X} \otimes K_X\right).$$

As before, the kernel of this map is $H^0\left(\bigwedge^{g-2} M_{K_X}\right)$ which, as we saw in (9) to be zero. So one must show that, for X with no g^1_3 and no g^2_5,

$$h^0\left(\bigwedge\nolimits^{g-3} M_{K_X} \otimes K_X\right) \leq \binom{g}{2}.$$

The rest of the proof is a (more complicated) geometric analysis of the canonical curve which leads to a proof analogous to the proof of Noether's theorem above, namely to a good filtration of

$$\bigwedge^{g-3} M_{K_X} \otimes K_X \cong \bigwedge^{2} M_{K_X}^{\vee}.$$

□

4. The Clifford index

Definition 4.1. The Clifford index $Cliff\,(D)$ of a divisor D on a curve X is

$$d - 2r$$

where $d = \deg D$ and $r = h^0\left(\mathcal{O}_X\left(D\right)\right) - 1$. By the Riemann-Roch theorem,

$$h^0\left(\mathcal{O}_X\left(D\right)\right) - h^0\left(K_X\left(-D\right)\right) = d + 1 - g$$

so that

$$Cliff\,(D) = g + 1 - \left(h^0\left(\mathcal{O}_X\left(D\right)\right) + h^0\left(K_X\left(-D\right)\right)\right). \tag{10}$$

The *Clifford index* of X, which we denote as $Cliff\,(X)$, is

$$\min_{h^0(D),h^0(K_X(-D))>1} \left\{Cliff\,(D)\right\}.$$

We begin with Clifford's famous theorem:

Theorem 4.1.

$$Cliff\,(X) \geq 0$$

and equality holds if and only if X is hyperelliptic and

$$|D| = a \cdot g_2^1.$$

Proof. The map

$$\mathbb{P}\left(H^0\left(\mathcal{O}_X\left(D\right)\right)\right) \times \mathbb{P}\left(H^0\left(K_X\left(-D\right)\right)\right) \to \mathbb{P}\left(H^0\left(K_X\right)\right)$$

given by multiplication is finite so that

$$h^0\left(\mathcal{O}_X\left(D\right)\right) - 1 + h^0\left(K_X\left(-D\right)\right) - 1 \leq h^0\left(K_X\right) - 1.$$

Thus by (10) $Cliff\,(X) \geq 0$. If equality holds then every canonical divisor is the sum of an element of $|D|$ and an element of $|K_X\left(-D\right)|$. Suppose $|D|$ is basepoint-free. If the curve is canonically embedded, the monodromy acts as the full symmetric group on the points of a fixed canonical divisor. So we conclude that every subset of degree d of every canonical divisor belongs to $|D|$ which implies $\dim|D| \geq g-1$. But this is absurd. So the curve is not canonically embedded. Therefore it is hyperelliptic. Since, in the hyperelliptic case, the monodromy group of a fixed canonical divisor interchanges elements of the g_2^1, we conclude that $|D| = a \cdot g_2^1$.

Conversely, if X is hyperelliptic,

$$K_X = (g-1) \cdot g_2^1$$

and so, for

$$|D| = a \cdot g_2^1$$

we have

$$r\left(D\right) \geq a$$

but

$$\deg D = 2a.$$

□

Theorem 4.2. *Suppose X not hyperelliptic.*

$$Cliff\,(X) = 1 \Leftrightarrow \left(\exists\ g_3^1 \text{ or } g_5^2\right).$$

Proof. Suppose there exists a D with

$$\begin{aligned} h^0\left(\mathcal{O}_X(D)\right), h^0\left(K_X(-D)\right) &> 1 \\ h^0\left(\mathcal{O}_X(D)\right) + h^0\left(K_X(-D)\right) &= g, \end{aligned}$$

or equivalently

$$d = 2r + 1.$$

Choose D of minimal degree with this property. If $r = 1$, the $|D| = g_3^1$, and if $r = 2$, $|D| = g_5^2$. So we must show that $r \leq 2$. Suppose $r \geq 3$. Notice that

$$\begin{aligned} \deg\left(K_X(-D)\right) &= 2g - 2 - d \\ &= 2g - 2 - 2r - 1 \\ &= 2\left(h^0\left(K_X(-D)\right) - 1\right) + 1. \end{aligned}$$

So $D' \in |K_X(-D)|$ is maximal with respect to the above property. So

$$\begin{aligned} d' &\geq g - 1 \\ r' &= h^0\left(K_X(-D)\right) - 1 \geq r \geq 3. \end{aligned}$$

Now consider the morphism

$$|D| : X \to \mathbb{P}^r$$

with image X_0. It is generically injective since, if not, then D is not minimal. Consider the sequence

$$0 \to \mathcal{O}_{X_0}(mD) \to \mathcal{O}_{X_0}((m+1)D) \to \sum_{i=1}^{2r+1} \mathbb{C}_i \to 0, \tag{11}$$

and, referring to (11) for $m = 1$, let b denote the rank of the cokernel of

$$S^2 H^0\left(\mathcal{O}_X(D)\right) \to H^0\left(\mathcal{O}_X(2D)\right) \to \sum_{i=1}^{2r+1} \mathbb{C}_i.$$

To bound b take a general hyperplane H in $\mathbb{P}^r$. By monodromy all sets of $\leq (r+1)$ points of $(X_0 \cap H)$ are linearly independent. Thus I can choose two independent hyperplanes H_1 and H_2 containing two disjoint subsets of cardinality $r - 1$ so that exactly 3 points are missed by

$$(H_1 \cup H_2) \cap (X_0 \cap H).$$

Since $r > 2$ this argument shows that any three points of $(X_0 \cap H)$ are omitted in this way. But if $b \geq 4$, there must be four points such that

$$\sum_{j=1}^{4} \mathbb{C}_{i(j)} \to \frac{\sum_{i=1}^{2r+1} \mathbb{C}_i}{image\left(S^2 H^0\left(\mathcal{O}_X(D)\right)\right)}$$

is injective which would imply

$$\sum_{j=1}^{4} \mathbb{C}_{i(j)} \cap image\left(S^2 H^0\left(\mathcal{O}_X(D)\right)\right) = 0.$$

So $b \leq 3$. Now referring to (11) for $m \geq 2$, the same argument shows that the composition

$$S^{m+1}H^0\left(\mathcal{O}_X\left(D\right)\right) \to H^0\left(\mathcal{O}_X\left(\left(m+1\right)D\right)\right) \to \sum\nolimits_{i=1}^{2r+1} \mathbb{C}_i$$

is surjective, so recursively we conclude

$$h^1\left(\mathcal{O}_X\left(2D\right)\right) = 0.$$

Thus

$$h^0\left(K_X\left(-D\right)\right) = h^1\left(\mathcal{O}_X\left(D\right)\right) = b \leq 3.$$

This contradicts our assumption that

$$3 \leq r'.$$

Alternatively Castelnuovo's bound

$$d-1 = m\left(r-1\right)+e \Rightarrow g \leq \left(\begin{array}{c} m \\ 2 \end{array}\right)\left(r-1\right)+me$$

applied in our case gives

$$\begin{aligned} d-1 &= 2\left(r-1\right)+2,\ r>3 \\ d-1 &= 3\left(r-1\right),\ r=3. \end{aligned}$$

So

$$\begin{aligned} g &\leq 1\cdot\left(r-1\right)+4 = r+3,\ r>3 \\ g &\leq 3\cdot 2,\ r=3. \end{aligned}$$

But we have seen above that $h^0\left(K_X\left(-D\right)\right) \geq 4$. Thus

$$g \geq h^0\left(\mathcal{O}_X\left(D\right)\right)+4 = r+5 \geq 8.$$

□

So we are finally ready to introduce:

Conjecture 4.3. *(M. Green)*

$$K_{l,2}\left(X,K_X\right)=0,\ l \leq a \Leftrightarrow Cliff\left(X\right) > a.$$

Combining Theorem 3.2 and Theorem 4.1, Theorem 3.3 and Theorem 4.2, we see that the theorems of Noether and Petri above are just the cases $c=0$ and $c=1$ of this conjecture.

Remark 4.1. One can show that, for $c \leq g-4$,

$$\left(K_{c,2}\left(X,K_X\right)=0\right) \Rightarrow \left(K_{l,2}\left(X,K_X\right)=0,\ l \leq c\right).$$

The conjecture is proved in one direction by the following:

Theorem 4.4. *(Green-Lazarsfeld)*

$$K_{c,2}\left(X,K_X\right) \neq 0 \Leftarrow Cliff\left(X\right)=c.$$

Proof.

$$K_{c,2}\left(X,K_X\right)^{\vee}=K_{g-2-c,1}\left(X,K_X\right).$$

On the other hand there exists a D such that

$$c=g+1-h^0\left(D\right)-h^0\left(K_X-D\right).$$

So

$$g-c-2=r\left(D\right)+r\left(K_X-D\right)-1.$$

Furthermore we may assume that $\mathcal{O}_X\left(D\right)$ and $K_X\left(-D\right)$ are generated by sections since they have minimal Clifford index. Then theorem follows from a special case of the following theorem. ☐

Theorem 4.5. *Let X be a projective variety with L_1 and L_2 line bundles on X such that, for $i=1,2$,*

$$h^0\left(L_i\right)=r_i+1\geq 2.$$

Suppose both L_i are generated by global sections. Then

$$K_{r_1+r_2-1,1}\left(X,L_1\otimes L_2\right)\neq 0.$$

Proof. Let L denote $L_1\otimes L_2$. By (7)

$$K_{r_1+r_2-1,1}\left(X,L\right)=\frac{H^0\left(\bigwedge^{r_1+r_2-1}M_L\otimes L\right)}{\delta'\left(\bigwedge^{r_1+r_2}H^0\left(X;L\right)\right)}.$$

Let σ and τ respectively be sections of L_1 and L_2 whose common zeros have codimension 2. The kernel of the map

$$\tau+\sigma:H^0\left(L_1\right)\oplus H^0\left(L_2\right)\to H^0\left(L\right)$$

is given by (α,β) such that

$$\alpha\tau=-\beta\sigma.$$

The basepoint-free pencil trick says that the kernel is generated by $(-\sigma,\tau)$ so the image of this map has rank r_1+r_2+1. Let

$$\overline{H}=\left(\tau+\sigma\right)\left(H^0\left(L_1\right)\oplus H^0\left(L_2\right)\right).$$

Now form the diagram

$$\begin{tikzcd}
M_{L_1}\oplus M_{L_2} \arrow[r, dashed, "j"] \arrow[d] & M_L \arrow[d] \\
H^0\left(L_1\right)\otimes\mathcal{O}_X\oplus H^0\left(L_2\right)\otimes\mathcal{O}_X \arrow[r, "\tau+\sigma"] \arrow[d] & H^0\left(L\right)\otimes\mathcal{O}_X \arrow[d] \\
L_1\oplus L_2 \arrow[r, "\tau+\sigma"] & L
\end{tikzcd} \tag{12}$$

The Snake Lemma implies that j is generically injective giving rise to an injection

$$\bigwedge^{r_1+r_2-1}\left(M_{L_1}\oplus M_{L_2}\right)\to\bigwedge^{r_1+r_2-1}M_L.$$

Since

$$rank\left(\bigwedge^{r_1+r_2-1}\left(M_{L_1}\oplus M_{L_2}\right)\right)=r_1+r_2$$

and

$$\det\left(M_{L_1}\oplus M_{L_2}\right)=-L$$

we conclude that

$$\bigwedge^{r_1+r_2-1}\left(M_{L_1}\oplus M_{L_2}\right)\otimes L=M_{L_1}^{\vee}\oplus M_{L_2}^{\vee}.$$

So we have an inclusion

$$H^0\left(M_{L_1}^{\vee}\right)\oplus H^0\left(M_{L_2}^{\vee}\right)\to H^0\left(\bigwedge^{r_1+r_2-1}M_L\otimes L\right).$$

On the other hand dualizing the injections

$$M_{L_i}\to H^0\left(L_i\right)\otimes\mathcal{O}_X$$

and taking H^0 we obtain injections

$$H^0\left(L_i\right)^{\vee}\to H^0\left(M_{L_i}^{\vee}\right)$$

and therefore produce a space

$$\widetilde{H}\subseteq\ker\delta\subseteq\left(\bigwedge^{r_1+r_2-1}H^0\left(L\right)\right)\otimes H^0\left(L\right)$$

spanned by r_1+1+r_2+1 independent sections of

$$H^0\left(\bigwedge^{r_1+r_2-1}M_L\otimes L\right)\subseteq\left(\bigwedge^{r_1+r_2-1}H^0\left(L\right)\right)\otimes H^0\left(L\right).$$

Now by construction of the mapping j in (12),

$$\widetilde{H}\subseteq\left(\bigwedge^{r_1+r_2-1}\overline{H}\right)\otimes H^0\left(L\right)\subseteq\left(\bigwedge^{r_1+r_2-1}H^0\left(L\right)\right)\otimes H^0\left(L\right).$$

Looking at the definition of the Koszul differential δ, we have by elementary linear algebra that, for a subspace $\overline{H}$ of a vector space $H^0\left(X;L\right)$,

$$\left(\left(\bigwedge^{r_1+r_2-1}\overline{H}\right)\otimes H^0\left(L\right)\right)\cap\delta\left(\bigwedge^{r_1+r_2}H^0\left(X;L\right)\right)\subseteq\delta\left(\bigwedge^{r_1+r_2}\overline{H}\right).$$

whereas the space $\bigwedge^{r_1+r_2}\overline{H}$ has dimension (r_1+r_2+1). So the intersection of the image of δ

with $\widetilde{H}$ can have at most dimension (r_1+r_2+1) and the proof is complete. □

5. Brill-Noether theory

Let X be a smooth curve of genus g. Define

$$W_d^r=\left\{L\in Pic^dX:h^0\left(L\right)\geq r+1\right\}.$$

Fix a divisor D_0 on X of very high degree d_0. Then for each $L\in Pic^dX$ we have an exact sequence

$$0\to H^0\left(L\right)\to H^0\left(L\left(D_0\right)\right)\to H^0\left(L\left(D_0\right)|_{D_0}\right)\to H^1\left(L\right)\to 0. \tag{13}$$

So

$$\chi\left(L\right)=\left(r_0+1\right)-d_0$$

where $r_0 = \dim |L(D_0)|$. Let $\mathfrak{L}$ denote the Poincaré bundle on $X \times Pic^d$ and let

$$\mathfrak{E} = \pi_* \mathfrak{L}$$

where π is projection to the second factor. Let

$$\mathfrak{F} = \pi_* \left(\mathfrak{L}|_{D_0 \times Pic^d} \right).$$

We have a map of vector bundles

$$\varphi : \mathfrak{E} \to \mathfrak{F}$$

of respective ranks a and b given by restriction and by (13) W_d^r is simply the locus where

$$\dim(\ker \varphi) \geq r+1. \tag{14}$$

In the space of all matrices $Hom\left(\mathbb{C}^a, \mathbb{C}^b\right)$, the locus (14) at a point φ such that

$$\dim(\ker \varphi) = r+1$$

is smooth with tangent space

$$\left\{ \varphi : \varphi\left(\mathbb{C}^{r+1}\right) \subseteq \mathbb{C}^{a-(r+1)} \right\}$$

and so this locus is of codimension

$$\begin{aligned} & ab - (r+1)(a-(r+1)) + (a-(r+1))\,b \\ = \; & (r+1)(b-a+(r+1)). \end{aligned}$$

So in our (possibly non-generic) case

$$\operatorname{co}\dim W_d^r \leq (r+1)(g-d+r)$$

since $a - b = \chi(L) = d+1-g$.

We recall:

Theorem 5.1. *(Petri) If X is generic,*

$$\operatorname{co}\dim W_d^r \leq (r+1)(g-d+r).$$

Furthermore

$$\rho(g,d,r) := g - (r+1)(g-d+r) \geq 0,$$

if and only if

$$W_d^r \neq \emptyset.$$

Now the equivalent inequalities

$$\begin{aligned} g &\geq (r+1)(g-d+r) \\ d &\geq r+g-\frac{g}{r+1} \\ d-2r &\geq g-r-\frac{g}{r+1} \end{aligned}$$

infer that $d-2r$ takes its smallest value when

$$g-r-\frac{g}{r+1}$$

takes its smallest non-negative value for $r \geq 1$. Differentiating this last expression with respect to r we see that the function is increasing for $r < g-1$ and decreasing thereafter. So this minimum occurs for $r = 1$ and so

$$d - 2 = \left\lceil \frac{g}{2} \right\rceil - 1.$$

Let

$$c = Cliff\,(X)$$

for X generic of genus g. Thus, if $g = 2k$,

$$c = k - 1,$$

and if $g = 2k+1$,

$$c = k.$$

But the Green conjecture says (taking into account Remark 4.1) that

$$K_{c-1,2}\,(X, K_X) = 0$$

or equivalently

$$K_{g-c-1,1}\,(X, K_X) = 0.$$

So, for $g = 2k$ or $2k+1$, the Green conjecture says that for a generic curve X,

$$K_{k,1}\,(X, K_X) = 0.$$

A more precise form of the generic Green conjecture would be

$$K_{k,1}\,(X, K_X) \neq 0 \Leftrightarrow Cliff\,(X) < c.$$

In odd genus, Hirschowitz and Ramanan have proved the following result consistent with this more precise form of the conjecture. It provides strong evidence for Green's conjecture (4.3).

Theorem 5.2. *If* $g = 2k+1$ *and*

$$K_{k,1}\,(X, K_X) = 0$$

for X *generic, then*

$$K_{k,1}\,(X, K_X) \neq 0$$

if and only if X *has a* g^1_{k+1}.

Proof. We define the following two algebraic subspaces of the moduli space $\mathfrak{M}_g$:

$$\begin{aligned} D' &= \{X \in \mathfrak{M}_g : K_{k,1}\,(X, K_X) \neq 0\} \\ D &= \left\{X \in \mathfrak{M}_g : \exists g^1_{k+1}\right\}. \end{aligned}$$

Now a count of moduli shows that D should be divisor. Harris-Mumford have shown it is indeed a reduced divisor (with respect to the scheme structure induced by Brill-Noether theory). We first show that D' has naturally the scheme structure of a divisor. For $H = \mathcal{O}_{\mathbb{P}^{g-1}}(1)$ consider the diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
M_H \arrow[r, "\varphi_X"] \arrow[d] & M_{K_X} \arrow[d] \\
H^0(H) \otimes \mathcal{O}_{\mathbb{P}^{g-1}} \arrow[r] \arrow[d] & H^0(K_X) \otimes \mathcal{O}_X \arrow[d] \\
H \arrow[r] \arrow[d] & K_X \arrow[d] \\
0 & 0
\end{tikzcd}$$

where

$$M_{K_X} = M_H|_X$$

and φ_X is the restriction map. We consider the induced maps

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
\bigwedge^{k+1} H^0(K_X) \arrow[d, "\delta"] & = & \bigwedge^{k+1} H^0(K_X) \arrow[d, "\delta"] \\
\left(\bigwedge^{k} H^0(K_X)\right) \otimes H^0(K_X) \arrow[d, "\delta_H"] & = & \left(\bigwedge^{k} H^0(K_X)\right) \otimes H^0(K_X) \arrow[d, "\delta_K"] \\
\left(\bigwedge^{k-1} H^0(K_X)\right) \otimes S^2H^0(K_X) \arrow[rr] \arrow[d, "\delta"] & & \left(\bigwedge^{k-1} H^0(K_X)\right) \otimes H^0\left(K_X^2\right) \arrow[d, "\delta"] \\
\cdots & & \cdots
\end{tikzcd} \tag{15}$$

of Koszul complexes. Since H on $\mathbb{P}^{g-1}$ is Koszul-acyclic, the Snake Lemma applied to the diagram

(16)

$$\begin{tikzcd}
 & H^0\left(\left(\bigwedge^{k-1} M_H\right) \otimes H^2 \otimes \mathcal{I}_X\right) \arrow[d] \\
\left(\bigwedge^{k} H^0(X;K_X) \otimes H^0(K_X)\right) / \delta\left(\bigwedge^{k+1} H^0(X;K_X)\right) \arrow[r, "="] \arrow[d, "="] & H^0\left(\left(\bigwedge^{k-1} M_H\right) \otimes H^2\right) \arrow[d, "\varphi_X"] \\
\left(\bigwedge^{k} H^0(X;K_X) \otimes H^0(K_X)\right) / \delta\left(\bigwedge^{k+1} H^0(X;K_X)\right) \arrow[r, "\alpha"] & H^0\left(\left(\bigwedge^{k-1} M_{K_X}\right) \otimes K_X^2\right)
\end{tikzcd}$$

and the equality

$$\ker(\alpha) = K_{k,1}(X, K_X)$$

gives

$$K_{k,1}(X, K_X) \cong H^0\left(\left(\bigwedge^{k-1} M_H\right) \otimes H^2 \otimes \mathcal{I}_X\right). \tag{17}$$

Also, since

$$\begin{aligned}
h^1\left(\left(\bigwedge^{k-1} M_{K_X}\right) \otimes K_X^2\right) &= h^0\left(\left(\bigwedge^{k-1} M_{K_X}^{\vee}\right) \otimes K_X^{-1}\right) \\
&= h^0\left(\bigwedge^{g-k} M_{K_X}\right) = 0
\end{aligned}$$

we have

$$h^0\left(\left(\bigwedge^{k-1} M_H\right)\otimes K_X^2\right)=\chi\left(\left(\bigwedge^{k-1} M_{K_X}\right)\otimes K_X^2\right)$$

and, in particular, this last quantity is independent of the smooth curve X . Then one easily computes that the dimensions

$$h^0\left(\left(\bigwedge^{k-1} M_H\right)\otimes H^2\right),\ h^0\left(\left(\bigwedge^{k-1} M_{K_X}\right)\otimes M_{K_X}^2\right)$$

are equal. But we are assuming that

$$K_{k,1}\left(X,K_X\right)=0$$

for generic X so that (17) implies that φ_X is injective for generic X, and hence an isomorphism. Now make these computations for the "universal curve"

$$\begin{tikzcd}
\mathfrak{X} \arrow[rr] \arrow[dr, "\pi"] & & \mathbb{P}\left(\mathfrak{E}_1^{\vee}\right) \arrow[dl, "\rho"] \\
& \mathfrak{M}_g &
\end{tikzcd}$$

over $\mathfrak{M}_g$ where

$$\mathfrak{E}_a=\pi_*\omega^a_{\mathfrak{X}/\mathfrak{M}_g}.$$

We obtain a morphism of bundles

$$\varphi:\mathcal{K}\to\mathcal{H}$$

where

$$\begin{aligned}
\mathcal{H} &= \pi_*\left(\bigwedge^{k-1} M_{K_{\mathfrak{X}/\mathfrak{M}_g}}\otimes\omega^2_{\mathfrak{X}/\mathfrak{M}_g}\right)\\
\mathcal{K} &= \rho_*\left(\bigwedge^{k-1} M_{\mathcal{O}(1)}\otimes\mathcal{O}\left(2\right)\right).
\end{aligned}$$

Thus by the above reasoning D' is the divisor given by the zero-scheme of $\det\left(\varphi\right)$. Now by Theorem 4.4 we know that

$$D\subseteq D'.$$

Since $Pic\left(\mathfrak{M}_g\right)=\mathbb{Z}\left[\lambda\right]$ for $\lambda=c_1\left(\mathfrak{E}\right)$ and $h^0\left(\mathcal{O}_{\mathfrak{M}_g}\right)=1$, in order to show that $D=D'$ set-theoretically, which is the content of the theorem, it will suffice to show that

i) for each $X\in D$

$$\dim K_{k,1}\left(X,K_X\right)\geq k$$

so that D' has multiplicity $\geq k$ there,

ii)

$$c_1\left(\mathcal{K}\right)-c_1\left(\mathcal{H}\right)=k\cdot c_1\left(D\right)\in Pic\mathfrak{M}_g.$$

Now i) is just the content of Lemma 5.3 below. As for ii), the left side of the equation in ii) is computed using the Grothendieck Riemann-Roch theorem. For the right side choose a sufficiently large that

$$\mathbb{P}\left(\mathfrak{E}_a\right)$$

has a section defined through codimension 1 which defines a divisor $\Delta \subseteq \mathfrak{X}$. Let $\mathfrak{L}$ denote the Poincare bundle on

$$\mathfrak{X} \times_{\mathfrak{M}_g} Pic^{2k+1}\left(\mathfrak{X}/\mathfrak{M}_g\right) \to \mathfrak{M}_g$$

with projections

$$\begin{tikzcd}
 & \mathfrak{X} \times_{\mathfrak{M}_g} Pic^{2k+1}\left(\mathfrak{X}/\mathfrak{M}_g\right) \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \\
\mathfrak{X} \arrow[dr, "\pi"'] & & Pic^{2k+1}\left(\mathfrak{X}/\mathfrak{M}_g\right) \arrow[dl, "\rho"] \\
 & \mathfrak{M}_g &
\end{tikzcd} .$$

Then, referring to (13), $c_1(D)$ is computed via the Porteous formula and Grothendieck Riemann-Roch since it is the image under ρ_* of the corank-2 Fitting ideal of the map

$$\tau_*\left(\mathfrak{L} \otimes \sigma^* \omega^N_{\mathfrak{X}/\mathfrak{M}_g}\right) \to \tau_*\left(\mathfrak{L} \otimes \sigma^* \omega^N_{\mathfrak{X}/\mathfrak{M}_g}\Big|_{\Delta}\right).$$

□

Lemma 5.3. *Let X be a generic curve of genus $g = 2k+1$ with a g^1_{k+1}. Then*

$$\dim K_{k,1}(X, K_X) \geq k.$$

Proof. Let $D \in g^1_{k+1}$ and let $L = \mathcal{O}_X(D)$. Notice that

$$M_L = L^{-1}$$

in this case. Let

$$\begin{array}{rcl} \sigma & \in & H^0(L) \\ \tau & \in & H^0(K_X(-D)) \end{array}$$

be such that, as in (12),

$$\tau + \sigma : H^0(L) \oplus H^0(K_X(-D)) \to H^0(K_X)$$

has 1-dimensional kernel generated by $\mathbb{C} \cdot (-\sigma, \tau)$ .As in the argument following (12) we have a map

$$\bigwedge^k (\tau + \sigma) : \left(\bigwedge^k \left(M_L \oplus M_{K_X(-D)}\right)\right) \otimes K_X \to \left(\bigwedge^k M_{K_X}\right) \otimes K_X.$$

which induces a morphism

$$K_X(-D) \otimes \bigwedge^{k-1} M_{K_X(-D)} = M_L \otimes K_X \otimes \bigwedge^{k-1} M_{K_X(-D)} \to \left(\bigwedge^k M_{K_X}\right) \otimes K_X.$$

It follows that we get a composed map

$$\begin{aligned} \varphi_{\sigma,\tau} \quad : \quad & \bigwedge^k H^0(K_X(-D)) \xrightarrow{\delta'} H^0\left(K_X(-D) \otimes \bigwedge^{k-1} M_{K_X(-D)}\right) \\ \to \quad & H^0\left(\left(\bigwedge^k M_{K_X}\right) \otimes K_X\right) \hookrightarrow \left(\bigwedge^k H^0(K_X)\right) \otimes H^0(K_X). \end{aligned}$$

Choose a basis

$$\tau_1, \ldots, \tau_{k+1} = \tau$$

of $H^0\left(K_X\left(-D\right)\right)$ and a basis

$$\sigma_1, \sigma_2 = \sigma$$

of $H^0\left(L\right)$ such that

$$\sigma_1\tau, \sigma_2\tau \notin \sum\nolimits_{j=1}^{k} H^0\left(L\right)\cdot\tau_j$$

and

$$\mu_0 : H^0\left(L\right)\otimes \sum\nolimits_{j=1}^{k-1} H^0\left(L\right)\cdot\tau_j \to H^0\left(K_X\right)$$

is injective.

Then one checks that $\varphi_{\sigma,\tau}\left(\tau_1\wedge\ldots\wedge\tau_k\right)$ is given by the formula

$$\begin{aligned} &\sum\nolimits_i \left(-1\right)^i \sigma\tau_1\wedge\ldots\wedge\widehat{\sigma\tau_i}\wedge\ldots\wedge\sigma\tau_k\wedge\sigma_1\tau\otimes\sigma_2\tau_i \\ -&\sum\nolimits_i \left(-1\right)^i \sigma\tau_1\wedge\ldots\wedge\widehat{\sigma\tau_i}\wedge\ldots\wedge\sigma\tau_k\wedge\sigma_2\tau\otimes\sigma_1\tau_i. \end{aligned}$$

Recalling (15)

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\bigwedge^{k+1} H^0\left(K_X\right) \arrow[d, "\delta"] \arrow[r, equal] & \bigwedge^{k+1} H^0\left(K_X\right) \arrow[d, "\delta"] \\
\left(\bigwedge^{k} H^0\left(K_X\right)\right)\otimes H^0\left(K_X\right) \arrow[d, "\delta_H"] \arrow[r, equal] & \left(\bigwedge^{k} H^0\left(K_X\right)\right)\otimes H^0\left(K_X\right) \arrow[d, "\delta_K"] \\
\left(\bigwedge^{k-1} H^0\left(K_X\right)\right)\otimes S^2H^0\left(K_X\right) \arrow[d, "\delta"] \arrow[r] & \left(\bigwedge^{k-1} H^0\left(K_X\right)\right)\otimes H^0\left(K_X^2\right) \arrow[d, "\delta"] \\
\cdots & \cdots
\end{tikzcd}$$

we apply the differential δ_H and obtain

$$\sum\nolimits_i \left(-1\right)^i \sigma\tau_1\wedge\ldots\wedge\widehat{\sigma\tau_i}\wedge\ldots\wedge\sigma\tau_k\otimes\left(\sigma_1\tau\cdot\sigma_2\tau_i-\sigma_2\tau\cdot\sigma_1\tau_i\right)+T,$$

where, by our choice of bases above, the terms in T are of the form

$$v_j\otimes q_j'$$

where the v_j are linearly independent in

$$\frac{\bigwedge^{k-1} H^0\left(K_X\right)}{\sum_i \mathbb{C}\cdot\left(\sigma\tau_1\wedge\ldots\wedge\widehat{\sigma\tau_i}\wedge\ldots\wedge\sigma\tau_k\right)}.$$

So the k linearly independent quadrics

$$q_i = \sigma_1\tau\cdot\sigma_2\tau_i-\sigma_2\tau\cdot\sigma_1\tau_i$$

vanish on X). But the map

$$\sigma\mapsto\delta_H\left(\varphi_{\sigma,\tau}\left(\tau_1\wedge\ldots\wedge\tau_k\right)\right)\in H^0\left(\left(\bigwedge^{k-1} M_H\right)\otimes H^2\otimes I_X\right)\cong K_{k,1}\left(X,K_X\right)$$

is homogeneous of degree $k-1$ and, since the k quadrics are linearly independent, is given by the complete linear system $\mathcal{O}_{\mathbb{P}\left(H^0\left(L\right)\right)}\left(k-1\right)$ and hence its image has rank k. $\square$

6. Curves on $K3$-surfaces

In this section S will denote a smooth $K3$-surface and C will denote a smooth curve on S. Let

$$L = \mathcal{O}_S(C).$$

Assume that L is ample so that, by Kodaira vanishing and the fact that $h^1(\mathcal{O}_S) = 0$,

$$h^1\left(S; L^k\right) = 0$$

for all $k \geq 0$. By adjunction

$$L|_C = K_C$$

so that, for all $k \geq 0$, the restriction-induced map

$$H^0(S; L) \to H^0(C; K_C)$$

is surjective. We will need the Hyperplane-restriction Theorem:

Theorem 6.1. *(Green) Let X be a positive-dimensional projective variety. Let Y be the zero-scheme of a section $\sigma \in H^0(X; L)$. Suppose that*

$$H^0\left(X; L^k\right) \to H^0\left(Y; L^k|_Y\right)$$

is surjective for all $k \geq 0$. Then

$$K_{p,q}(X, L) = K_{p,q}(Y, L|_Y)$$

for all p, q.

Proof. For each $k \geq 0$ we have the exact sequence

$$0 \to H^0\left(X; L^{k-1}\right) \xrightarrow{\sigma} H^0\left(X; L^k\right) \to H^0\left(Y; L^k|_Y\right) \to 0 \tag{18}$$

so that, from the case $k = 1$, we obtain the exact sequence

$$0 \to \bigwedge^{p-1} H^0(Y; L|_Y) \xrightarrow{\wedge\sigma} \bigwedge^{p} H^0(X; L) \to \bigwedge^{p} H^0(Y; L|_Y) \to 0. \tag{19}$$

Now from the definition of the Koszul complexes we deduce from (18) a short exact sequence of complexes

$$0 \to \mathcal{K}_{p,q-1}(X, L) \xrightarrow{\cdot\sigma} \mathcal{K}_{p,q}(X, L) \to \mathcal{K}_{p,q}(X, L; \mathcal{O}_Y) \to 0.$$

We claim that the map

$$K_{p,q-1}(X, L) \xrightarrow{\cdot\sigma} K_{p,q}(X, L)$$

is actually zero. To see this consider the diagram

$$\begin{tikzcd}
 & \left(\bigwedge^p H^0(X; L)\right) \otimes H^0\left(L^{q-1}\right) \arrow[dl, "\wedge\sigma"'] \arrow[d, "1\otimes(\cdot\sigma)"] \\
\left(\bigwedge^{p+1} H^0(X; L)\right) \otimes H^0\left(L^{q-1}\right) \arrow[r, "\delta"] & \left(\bigwedge^p H^0(X; L)\right) \otimes H^0\left(L^{q}\right)
\end{tikzcd}$$

and use the coboundary formula for Koszul cohomology which, up to signs is given by

$$\delta(\alpha \wedge \sigma) = \delta(\alpha) \wedge \sigma + \alpha \cdot \sigma.$$

Thus if $\delta(\alpha) = 0$ then $\alpha \cdot \sigma$ is δ-exact.

It follows from this that we have an exact sequence

$$0 \to K_{p,q}(X,L) \xrightarrow{\beta'} K_{p,q}(X,L;\mathcal{O}_Y) \xrightarrow{\beta} K_{p-1,q}(X,L) \to 0. \tag{20}$$

Again from the definition of the Koszul complexes we deduce from (19) a short exact sequence of complexes

$$0 \to \mathcal{K}_{p-1,q}(Y;L|_Y) \xrightarrow{\wedge\sigma} \mathcal{K}_{p,q}(X,L;\mathcal{O}_Y) \to \mathcal{K}_{p,q}(Y;L|_Y) \to 0$$

induced by (19). In fact this sequence is split; it suffices to choose a splitting

$$H^0(X;L) = \mathbb{C}\cdot\sigma \oplus H^0(Y;L|_Y).$$

Thus we have exact

$$0 \to K_{p-1,q}(Y,L|_Y) \xrightarrow{\alpha} K_{p,q}(X,L;\mathcal{O}_Y) \xrightarrow{\alpha'} K_{p,q}(Y,L|_Y) \to 0. \tag{21}$$

Referring to (20) and (21) one checks directly from the definition that, up to a non-zero coefficient,

$$r_Y \circ \beta \circ \alpha = \textit{identity} \tag{22}$$

where

$$r_Y : K_{p-1,q}(X,L) \to K_{p-1,q}(Y,L|_Y)$$

is restriction. We now argue by induction on p. If r_Y is an isomorphism on $K_{p-1,q}$ then by (22) $\beta\circ\alpha$ is an isomorphism. It follows from this that

$$r_Y = \alpha'\circ\beta' : K_{p,q}(X,L) \to K_{p,q}(Y,L|_Y)$$

is also an isomorphism. □

By this last theorem, if Green's conjecture is true, then for every ample linear system $|L|$ on a $K3$-surface S, the Clifford index of a smooth $C \in |L|$ should be independent of the choice of C. In fact it is a theorem of Green and Lazarsfeld that the Clifford index of a smooth $C \in |L|$ is independent of the choice of C, which is more evidence for the truth of Green's conjecture. We shall now give a proof of a special case of this result of Green and Lazarsfeld.

Theorem 6.2. *Suppose S, C, L are as above and there exists a $D \in Pic^d(C)$ such that $\mathcal{O}_C(D)$ and $K_C(-D)$ are globally generated and*

$$h^0(\mathcal{O}_C(D)) = 2.$$

Then, if $C' \in |L|$ is smooth there is a $D' \in Pic^d(C')$ such that

$$h^0(\mathcal{O}_{C'}(D')), h^0(K_{C'}(-D')) \geq 2$$

and

$$Cliff(D') \leq Cliff(D).$$

Remark 6.1. We do not claim that $h^0(\mathcal{O}_C(D'))$ can be made equal to 2. In fact there is an example of Donagi and Morrison that shows that this is not always the case. Let S is be a double plane branched along a sectic and $L = \mathcal{O}_S(D)$. The general C' is a (smooth) plane sextic and so its Clifford index 2 is achieved by its g^2_6 and not by any g^1_4. However for C which are branched double covers of plane cubics E, we have g^1_4's given by any g^1_2 on E.

Proof. Define the rank-2 bundle F by the exact sequence

$$0 \to F \to H^0(\mathcal{O}_C(D)) \otimes \mathcal{O}_S \to \mathcal{O}_C(D) \to 0. \tag{23}$$

We have $h^1(L^\vee) = 0$ by Kodaira vanishing and so

$$\begin{aligned} h^0(F) &= 0 \\ h^1(F \otimes L^\vee) &= 0. \end{aligned}$$

Let E denote $F^\vee$. Dualizing (23) in the derived category we obtain

$$0 \to H^0(\mathcal{O}_C(D))^\vee \otimes \mathcal{O}_S \to E \to Ext^1_S(\mathcal{O}_C(D), \mathcal{O}_S) \to 0. \tag{24}$$

Since $Ext^1_S(\mathcal{O}_C, \mathcal{O}_S) = \mathcal{O}_S(C)|_C = K_C$ we have that $Ext^1_S(\mathcal{O}_C(D), \mathcal{O}_S) = K_C(-D)$ and, taking global sections in (24) and using that $h^1(\mathcal{O}_S) = 0$, we obtain the exact sequence

$$0 \to H^0(\mathcal{O}_C(D)) \to H^0(E) \to H^0(K_C(-D)) \to 0.$$

So E is generated by global sections and

$$h^0(E) = h^0(\mathcal{O}_C(D)) + h^0(K_C(-D)). \tag{25}$$

Now

$$\begin{aligned} \det F &= -L \\ \det E &= L \end{aligned}$$

and E is rank-2 so that

$$F = E^\vee = E \otimes L^\vee,$$

and so

$$h^0(E \otimes L^\vee) = 0. \tag{26}$$

If $C' \in |L|$ is smooth, the rank-2 bundle

$$E' := E|_{C'}$$

with determinant $K_{C'}$ has

$$h^0(E') \geq h^0(E) = h^0(\mathcal{O}_C(D)) + h^0(K_C(-D))$$

via the exact sequence

$$0 \to E \otimes L^\vee \to E \to E' \to 0$$

and (25) and (26). Also

$$h^0\left((E')^\vee\right) \leq h^0(E^\vee) = 0$$

since

$$h^1(E^\vee \otimes L^\vee) = h^1(F \otimes L^\vee) = 0.$$

We wish to prove that

$$Cliff(C') \leq Cliff(D).$$

We may assume $Cliff(D) < Cliff(\textit{generic curve})$ otherwise there is nothing to prove. By (10)

$$g + 1 - \left(h^0(\mathcal{O}_C(D)) + h^0(K_C(-D))\right) = Cliff(D).$$

If $g = 2k+1$, $Cliff\,(generic\ curve) = k$ and, for $g = 2k$, $Cliff\,(generic\ curve) = k-1$ so in both cases this translates to

$$k + 2 < h^0\left(\mathcal{O}_C\left(D\right)\right) + h^0\left(K_C\left(-D\right)\right). \tag{27}$$

To complete the proof we claim that it will be sufficient to establish that there exists a $D' \in C'^{(k+1)}$ for which

$$h^0\left(E'\left(-D'\right)\right) \neq 0. \tag{28}$$

First let's assume (28) and finish the proof. Using (28) we have an injection of sheaves

$$\mathcal{O}_{C'}\left(D'\right) \to E'$$

from which we construct a saturation

$$\mathcal{O}_{C'}\left(D'\right) \subseteq \mathcal{O}_{C'}\left(D''\right)$$

such that, referring to (24), the induced exact sequence

$$0 \to \mathcal{O}_{C'}\left(D''\right) \to E' \to K_{C'}\left(-D''\right) \to 0$$

is a sequence of vector bundles. Since

$$h^0\left(\mathcal{O}_{C'}\left(D''\right)\right) + h^0\left(K_{C'}\left(-D''\right)\right) \geq h^0\left(E'\right) \geq h^0\left(\mathcal{O}_C\left(D\right)\right) + h^0\left(K_C\left(-D\right)\right) \tag{29}$$

the theorem will follow as long as we check that

$$h^0\left(\mathcal{O}_{C'}\left(D''\right)\right), h^0\left(K_{C'}\left(-D''\right)\right) \geq 2.$$

Now $K_{C'}\left(-D''\right)$ is generated by sections because E' is. Also $K_{C'}\left(-D''\right)$ is not trivial since $h^0\left(\left(E'\right)^{\vee}\right) = 0$. So $h^0\left(K_{C'}\left(-D''\right)\right) \geq 2$. To check $h^0\left(\mathcal{O}_{C'}\left(D''\right)\right) \geq 2$, suppose $h^0\left(\mathcal{O}_{C'}\left(D''\right)\right) = 1$. By Riemann-Roch

$$-h^0\left(K_{C'}\left(-D''\right)\right) = \deg D'' - 1 + 1 - g.$$

But

$$\deg D'' \geq \deg D' = k + 1.$$

So

$$h^0\left(\mathcal{O}_{C'}\left(D''\right)\right) + h^0\left(K_{C'}\left(-D''\right)\right) = g - \deg D'' + 1 \leq k + 1$$

which contradicts the inequality coming from (27) and (29).

Finally we need to check (28). Consider the bundle $\mathfrak{E}$ on $C'^{(k+1)}$ with fiber $H^0\left(E'\left(Z\right)\right)$ at $Z \in C'^{(k+1)}$. Now

$$h^0\left(E'\right) > k + 2$$

and it is only $2\left(k+1\right)$ equations to find a section in the image

$$\varphi : H^0\left(E'\right) \to H^0\left(\mathfrak{E}\right)$$

which vanishes at a given Z so that the set of Z where a section vanishes must have codimension less than

$$2\left(k+1\right) - \left(k+2\right) = k.$$

A Chern class argument on $C'^{(k+1)}$ then yield the existence of the desired Z. □

We are now ready to prove an important result of Lazarsfeld:

Theorem 6.3. *Let S be a $K3$-surface having a line bundle L generating $Pic\,(S)$. Let $C \in |L|$ be smooth. Then C is generic for Brill-Noether theory in the sense that*

$$\rho\,(g,d,r) < 0 \Rightarrow W_d^r = \emptyset.$$

So in particular

$$Cliff\,(C) = Cliff\,(generic\ curve)\,.$$

Proof. Suppose that $\rho = \rho\,(g,d,r) < 0$ and $D \in W_d^r$. I may assume that $|D|$ and $|K_C\,(-D)|$ are basepoint-free since eliminating basepoints lowers ρ. As before form

$$0 \to F \to H^0\,(\mathcal{O}_C\,(D)) \otimes \mathcal{O}_S \to \mathcal{O}_C\,(D) \to 0.$$

Then as before $E = F^\vee$ has determinant L and is globally generated, and $h^0\,(E^\vee) = 0$. One then computes from (24) or by Riemann-Roch for vector bundles on surfaces that

$$\chi\,(Hom\,(E,E)) = 2 - 2\rho. \tag{30}$$

So $\rho < 0$ implies that

$$\chi\,(Hom\,(E,E)) = 2h^0\,(Hom\,(E,E)) - h^1\,(Hom\,(E,E)) \geq 4.$$

Thus

$$h^0\,(Hom\,(E,E)) \geq 2.$$

So for $\varphi \in Hom\,(E,E)$ which is not a multiple of the identity map ι, and for any point of S,

$$\varphi - \lambda \cdot \iota$$

is non-zero and has a kernel for some value λ_0 of λ. So

$$\det\,(\varphi - \lambda_0 \cdot \iota)$$

is a section of $\mathcal{O}_S$ with a zero at some point, so is identically zero. Let G denote the image of $(\varphi - \lambda_0 \cdot \iota)$ and G' denote its cokernel. Then G and G' are rank-1 sheaves and

$$\det G + \det G' = \det E = L.$$

But both G and G' are quotients of E and therefore generated by global sections. Also

$$\det G, \det G' > 0 \tag{31}$$

since otherwise we would have that $h^0\,(E^\vee) \neq 0$. But (31) contradicts the minimality of $\det E = L$ in $Pic\,(S)$. ☐

7. GREEN'S CONJECTURE IN EVEN GENUS

We now consider a $K3$-surface S for which the ample line bundle L generates $PicS$ and for smooth $C \in |L|$ we have $g = genus\,(C) = 2k$. Since we have seen in the last section that C is generic for Brill-Noether, Green's conjecture for the generic curve predicts that

$$K_{k,1}\,(C, K_C) = 0$$

and if we prove this last equality, we will have established the generic Green conjecture for even genus. But for this it suffices by Theorem 6.1 to prove the following:

Theorem 7.1.

$$K_{k,1}(S,L)=0.$$

Before giving the proof of Theorem 7.1, which will occupy the remainder of these notes, we state and prove:

Corollary 7.2. *Let X be a generic curve of genus $2k-1$. Then*

$$K_{k,1}(X,K_X)=0.$$

Remark 7.1. The generic Green conjecture in odd genus $2k-1$ is the statement

$$K_{k-1,1}(X,K_X)=0.$$

Proof. For S,L,C as above, let $C'\in|L|$ be nodal and let X be the normalization of C'. Thus

$$\pi:X\to C'=\frac{X}{\{p=q\}}$$

for $p,q\in X$. By Theorem 6.1 applied to the singular curve C',

$$K_{k,1}(C',K_{C'})=0.$$

Then

$$\pi^*K_{C'}=K_X(p+q)$$

so that

$$\begin{array}{rcl}H^0(C';K_{C'}) & = & H^0(X;K_X(p+q))\\ H^0\left(C';K_{C'}^2\right) & \subseteq & H^0\left(X;K_X^2(2p+2q)\right).\end{array}$$

So from the definition of Koszul cohomology

$$K_{k,1}(X,K_X(p+q))=0$$

We finish by checking the injectivity of the natural map

$$K_{k,1}(X,K_X)\to K_{k,1}(X,K_X(p+q)).$$

To see this notice that, if

$$\Lambda:\left(\bigwedge^k H^0(X;K_X)\right)\otimes H^0(X;K_X)\to\bigwedge^{k+1}H^0(X;K_X) \tag{32}$$

is the natural wedge-product map, the injection

$$\delta:\bigwedge^{k+1}H^0(X;K_X)\to\left(\bigwedge^k H^0(X;K_X)\right)\otimes H^0(X;K_X) \tag{33}$$

satisfies (up to sign)

$$\Lambda\circ\delta=(k+1)\,(\mathit{identity})$$

We have the same formula for the analogues of (32) and (33) for $H^0(X;K_X(p+q))$ into which the spaces of (32) and (33) inject. We denote images under these injections by a " $'$ ". Suppose, for

$$\alpha\in\left(\bigwedge^k H^0(X;K_X)\right)\otimes H^0(X;K_X)$$

with $\delta\alpha = 0$, the image α' of α in $\left(\bigwedge^k H^0(X;K_X(p+q))\right) \otimes H^0(X;K_X(p+q))$ satisfies

$$\alpha' = \delta\beta'$$

for $\beta' \in \bigwedge^k H^0(X;K_X(p+q))$. Then (again up to sign)

$$\begin{aligned} \delta\beta' &= \frac{1}{k+1}(\delta\circ\Lambda\circ\delta)(\beta') \\ &= \frac{1}{k+1}(\delta\circ\Lambda)(\alpha') \\ &= \frac{1}{k+1}(\delta(\Lambda\alpha))' \end{aligned}$$

and so

$$\alpha = \frac{1}{k+1}(\delta(\Lambda\alpha)).$$

□

For the proof of Theorem 7.1 we start with the following geometric interpretation of $K_{k,1}(X,L)$ for any smooth quasi-projective variety X and line bundle L. Denote by

$$H^{k+1}_{curv.}$$

the subscheme of the Hilbert scheme of Artinian subschemes of X of length $k+1$ given by the curvilinear subschemes, i.e. those lying in smooth curves on X. Let

$$I_{k+1} \subseteq X \times H^{k+1}_{curv.}$$

denote the corresponding incidence scheme. We have a natural truncation morphism

$$\begin{aligned} \tau \quad &: \quad I_{k+1} \to X \times H^k_{curv.} \\ (x,z) \quad &\mapsto \quad (x, z-x) \end{aligned}$$

For example, if $k=0$, I_1 is obtained by blowing up the diagonal in $X \times X$ and τ is the natural map to $X \times X$. In general I_{k+1} is a large (complement codimension is > 1) Zariski-open set in the scheme obtained by blowing up

$$I_k \subseteq X \times H^k_{curv.}$$

and τ is the natural map to $X \times H^k_{curv.}$. So $H^{k+1}_{curv.}$ is smooth and reduced. From the diagram

$$\begin{tikzcd} & I_{k+1} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X & & H^{k+1}_{curv.} \end{tikzcd}$$

define the rank-$(k+1)$ vector bundle

$$\mathfrak{E}_{k+1} = p_*q^*L.$$

Let

$$L_{k+1} := \det \mathfrak{E}_{k+1} = \bigwedge^{k+1} \mathfrak{E}_{k+1}.$$

Lemma 7.3.

$$K_{k,1}(X,L) \cong \frac{H^0\left(I_{k+1}; p^* L_{k+1}\right)}{p^* H^0\left(H^{k+1}_{curv.}; L_{k+1}\right)}. \tag{34}$$

Proof. To prove (34) we claim that the natural map

$$H^0(L) \otimes \mathcal{O}_{H^{k+1}_{curv.}} \to \mathfrak{E}_{k+1}$$

induced by $p_* q^*$ induces an isomorphism

$$\bigwedge^{k+1} H^0(X; L) \overset{\cong}{\longrightarrow} H^0\left(H^{k+1}_{curv.}; L_{k+1}\right). \tag{35}$$

To prove (35) let $U \subseteq H^{k+1}_{curv.}$ be the open subset where at most 2 of the $k+1$ points coincide. Then U can be constructed from

$$\tilde{U} \subseteq X^{k+1}$$

where at most 2 points coincide by taking the smooth locus Δ where 2 points actually do coincide and forming the quotient

$$U = \frac{Bl_{\Delta}\left(\tilde{U}\right)}{S_{k+1}}$$

of the blow-up under the natural action of the symmetric group S_{k+1}. For the diagram

$$\begin{tikzcd}
& Bl_{\Delta}\left(\tilde{U}\right) \arrow[dl, "u"] \arrow[dr, "r"] & \\
\tilde{U} & & U
\end{tikzcd}$$

we have that

$$r^* L_{k+1} = u^*\left(L \boxtimes \ldots \boxtimes L\right)\left(-\tilde{E}\right) \tag{36}$$

where $\tilde{E}$ is the exceptional divisor of u. Now since U and $\tilde{U}$ are large,

$$\begin{aligned}
H^0\left(H^{k+1}_{curv.}; L_{k+1}\right) &= H^0\left(\tilde{U}; u^*\left(L \boxtimes \ldots \boxtimes L\right)\left(-\tilde{E}\right)\right)^{S_{k+1}} \\
&\subseteq \left(H^0(X; L)^{\otimes k+1}\right)^{S_{k+1}}.
\end{aligned}$$

But by direct computation in local coordinates, the action of $\sigma \in S_{k+1}$, induced by the isomorphism (36), is by the rule

$$\left(l_1 \otimes \ldots \otimes l_{k+1}\right) \mapsto sgn(\sigma)\left(l_{\sigma(1)} \otimes \ldots \otimes l_{\sigma(k+1)}\right)$$

so that

$$\left(H^0(X; L)^{\otimes k+1}\right)^{S_{k+1}} = \bigwedge^{k+1} H^0(X; L)$$

and

$$\left(H^0(X; L)^{\otimes k+1}\right)^{S_{k+1}} \subseteq H^0\left(\tilde{U}; u^*\left(L \boxtimes \ldots \boxtimes L\right)\left(-\tilde{E}\right)\right)^{S_{k+1}}.$$

To finish our proof of (34) we return to our diagram

$$\begin{tikzcd}
\tilde{E}_{k+1} \arrow[r, phantom, "\subseteq"] \arrow[d] & I_{k+1} \arrow[d, "\tau"] \arrow[dr, "p"] & \\
I_k \arrow[r, phantom, "\subseteq"] & X \times H^k_{curv.} & H^{k+1}_{curv.}
\end{tikzcd} \tag{37}$$

where $\tilde{E}_{k+1}$ is the exceptional locus of the blow-up τ. Again one checks by direct computation that

$$p^* L_{k+1} = \tau^* \left(L \boxtimes L_k\right) \left(-\tilde{E}_{k+1}\right).$$

It follows from the projection formula that

$$H^0 \left(I_{k+1}; p^* L_{k+1}\right) = \ker \left(H^0 \left(X \times H^k_{curv.}; L \boxtimes L_k\right) \to H^0 \left(I_k; L \boxtimes L_k|_{I_k}\right)\right).$$

By (35)

$$\begin{aligned} H^0 \left(X \times H^k_{curv.}; L \boxtimes L_k\right) &= H^0 \left(X; L\right) \otimes H^0 \left(H^k_{curv.}; L_k\right) \\ &= H^0 \left(X; L\right) \otimes \bigwedge^k H^0 \left(X; L\right) \end{aligned}$$

and

$$H^0 \left(H^{k+1}_{curv.}; L_{k+1}\right) = \bigwedge^{k+1} H^0 \left(X; L\right).$$

Using the inclusion

$$H^0 \left(H^{k+1}_{curv.}; L_{k+1}\right) \subseteq H^0 \left(I_{k+1}; p^* L_{k+1}\right)$$

we finish the proof of (34) by checking that we have a natural injection

$$H^0 \left(I_k; L \boxtimes L_k|_{I_k}\right) \hookrightarrow H^0 \left(X; L^2\right) \otimes \bigwedge^{k-1} H^0 \left(X; L\right).$$

For this last point, consider the diagram obtained by replacing k with $k-1$ in (37) and restrict to $\tilde{E}_k$. Then

$$p^* L_k = \tau^* \left(L \boxtimes L_{k-1}\right) \left(-\tilde{E}_k\right) \subseteq \tau^* \left(L \boxtimes L_{k-1}\right)$$

so that

$$\begin{aligned} L \boxtimes L_k &= q^* L \otimes p^* L_k \\ &\subseteq q^* L \otimes \tau^* \left(L \boxtimes L_{k-1}\right) = \tau^* \left(L^2 \boxtimes L_{k-1}\right) \end{aligned}$$

and

$$H^0 \left(I_k; L \boxtimes L_k|_{I_k}\right) \subseteq H^0 \left(I_k; \tau^* \left(L^2 \boxtimes L_{k-1}\right)\right).$$

But now using (35) again

$$\begin{aligned} H^0 \left(I_k; \tau^* \left(L^2 \boxtimes L_{k-1}\right)\right) &= H^0 \left(X \times H^{k-1}_{curv.}; L^2 \boxtimes L_{k-1}\right) \\ &= H^0 \left(X; L^2\right) \otimes \bigwedge^{k-1} H^0 \left(X; L\right). \end{aligned}$$

The proof then follows by checking that the diagram

$$\begin{tikzcd} H^0 \left(H^{k+1}_{curv.}; L_{k+1}\right) \arrow[r, phantom, "\cong"] \arrow[d, "p^*"] & \bigwedge^{k+1} H^0 \left(X; L\right) \arrow[d, "\delta"] \\ H^0 \left(I_{k+1}; p^* L_{k+1}\right) \arrow[r, hook] & H^0 \left(X; L\right) \otimes \bigwedge^k H^0 \left(X; L\right) \end{tikzcd}$$

is commutative and by a similar check for the diagram

$$\begin{tikzcd} H^0 \left(X \times H^k_{curv.}; L \boxtimes L_k\right) \arrow[r, phantom, "\cong"] \arrow[d] & H^0 \left(X; L\right) \otimes \bigwedge^k H^0 \left(X; L\right) \arrow[d, "\delta"] \\ H^0 \left(I_k; L \boxtimes L_k|_{I_k}\right) \arrow[r, hook] & H^0 \left(X; L^2\right) \otimes \bigwedge^{k-1} H^0 \left(X; L\right) \end{tikzcd}.$$

□

The strategy of the proof of Theorem 7.1 is then as follows. With reference to the map

$$p : I_{k+1} \to H^{k+1}_{curv.}$$

and (34) the proof reduces to showing the surjectivity of

$$p^* H^0 \left(H^{k+1}_{curv.}; L_{k+1} \right) \hookrightarrow H^0 \left(I_{k+1}; p^* L_{k+1} \right).$$

We shall use the following:

Criterion: For the $(k+1)$-to-1 branched cover $I_{k+1}/H^{k+1}_{curv.}$ assume we have a diagram

$$\begin{tikzcd}
\tilde{Z} \arrow[r] \arrow[d, "\varphi"] \arrow[dr, phantom, "\square"] & I_{k+1} \arrow[d, "p"] \\
Z \arrow[r, "\gamma"] & H^{k+1}_{curv.}
\end{tikzcd}$$

such that γ is generically injective and:

Property Ci): We have the desired isomorphism

$$\varphi^* H^0 \left(Z; L_{k+1}|_Z \right) = H^0 \left(\tilde{Z}; \varphi^* \left(L_{k+1}|_Z \right) \right),$$

on the restriction over Z.

Property Cii): The map

$$H^0 \left(I_{k+1}; p^* L_{k+1} \right) \to H^0 \left(\tilde{Z}; \varphi^* \left(L_{k+1}|_Z \right) \right)$$

is injective.

Lemma 7.4. *If the above Criterion is satisfied, then*

$$K_{k,1}(S, L) = 0.$$

Proof. Let

$$\sigma \in H^0 \left(I_{k+1}; p^* L_{k+1} \right).$$

Then by Property Ci)

$$\sigma|_{\tilde{Z}} = \varphi^* \tau$$

for some $\tau \in H^0 \left(Z; L_{k+1}|_Z \right)$. Then

$$\begin{aligned} \tau &= \frac{1}{k+1} \mathrm{trace}_{\tilde{Z}/Z}(\sigma) \\ &= \frac{1}{k+1} \left. \mathrm{trace}_{I_{k+1}/H^{k+1}_{curv.}}(\sigma) \right|_Z . \end{aligned}$$

So

$$\sigma - \frac{1}{k+1} p^* \mathrm{trace}_{I_{k+1}/H^{k+1}_{curv.}}(\sigma)$$

vanishes on $\tilde{Z}$ and so on I_{k+1} by Property Cii) above. □

So the rest of the proof is reduced to constructing Z satisfying Property Ci) and Property Cii) above. We proceed as follows. For smooth $C \in |L|$ of genus $2k$ we have

$$\begin{aligned} \rho(g, k+1, 1) &= g - 2(g - (k+1) + 1) \\ &= 2k - 2k = 0. \end{aligned}$$

Thus C has a g^1_{k+1} which must be basepoint-free since there is no g^1_k by Theorem 6.3 of Lazarsfeld. So as in (23)-(26) we have the rank-2 bundle

$$E = F^{\vee} = F \otimes L$$

with $\det E = L$ and the sequence

$$0 \to H^0\left(\mathcal{O}_C(D)\right)^{\vee} \otimes \mathcal{O}_S \xrightarrow{\alpha} E \xrightarrow{\beta} K_C(-D) \to 0. \tag{38}$$

So

$$h^0(E) = 2 + h^0\left(K_C(-D)\right) = k + 2.$$

α induces an inclusion $h^0(\alpha) : H^0\left(\mathcal{O}_C(D)\right)^{\vee} \to H^0(E)$. If $\tau \in H^0\left(\mathcal{O}_C(D)\right)^{\vee}$, the zero-scheme of $h^0(\alpha)(\tau)$ in (38) lies in C and the restriction of (38) to C gives

$$\ker\left(\beta|_C\right) = K_C \otimes \left(K_C(-D)\right)^{\vee} = \mathcal{O}_C(D).$$

Thus we have a surjection

$$\alpha|_C : H^0\left(\mathcal{O}_C(D)\right)^{\vee} \otimes \mathcal{O}_C \to \mathcal{O}_C(D)$$

and taking global sections we get an isomorphism

$$H^0\left(\mathcal{O}_C(D)\right)^{\vee} \cong H^0\left(\mathcal{O}_C(D)\right),$$

which we verify to be generated by a generator of $\bigwedge^2 H^0\left(\mathcal{O}_C(D)\right)$. It follows that

$$zero\left(h^0(\alpha)(\tau)\right) = zero(\tau).$$

We next check that the isomorphism class of the bundle E is independent of the choice of C and of the choice of the g^1_{k+1} on C. To prove this, we firs check that E is stable. For this it suffices to recall that

$$Pic(S) = \mathbb{Z} \cdot L$$

and that

$$\det E = L$$

and

$$h^0\left(Hom(L, E)\right) = 0.$$

Next, by (30),

$$\chi\left(Hom(E, E)\right) = 2 - 2\rho(2k, k+1, 1) = 2.$$

Suppose now that we have another curve C' and $D' \in g^1_{k+1}(C')$ giving rise to a bundle E'. Then

$$\begin{aligned} \chi\left(Hom(E, E')\right) &= 2 \\ &= h^0\left(Hom(E, E')\right) - h^1\left(Hom(E, E')\right) + h^0\left(Hom(E', E)\right). \end{aligned}$$

So either $h^0\left(Hom(E, E')\right)$ or $h^0\left(Hom(E', E)\right)$ is non-zero. Then stability implies that E and E' are isomorphic. So also by stability

$$\begin{aligned} h^0\left(Hom(E, E)\right) &= 1 \\ h^1\left(Hom(E, E)\right) &= 0 \end{aligned}$$

so E is (infinitesimally) rigid.

So we conclude that for all $\left(C, D\in g^1_{k+1}\right)$, D is recovered as the zero-scheme of a section of E. Conversely given a non-zero section

$$\alpha\in H^0\left(E\right),$$

by stability α cannot vanish along a divisor in S so

$$z\left(\alpha\right)\in Hilb^{k+1}.$$

Comparing

$$0\to\mathcal{O}_S\xrightarrow{\cdot\alpha}E\to Q\to 0$$

with

$$0\to\mathcal{O}_S\xrightarrow{\cdot\alpha}E\xrightarrow{\wedge\alpha}L \tag{39}$$

we see that

$$Q=I_{z(\alpha)}\cdot L.$$

The exact sequence (39) induces the exact sequence

$$0\to\mathbb{C}\xrightarrow{\cdot\alpha}H^0\left(E\right)\to H^0\left(I_{z(\alpha)}\cdot L\right)\to 0$$

so that

$$\begin{aligned} h^0\left(I_{z(\alpha)}\cdot L\right) &= h^0\left(E\right)-1\\ &= k+1 \end{aligned} \tag{40}$$

and, if $z\left(\alpha\right)\subseteq C$,

$$h^0\left(I_{z(\alpha)}\cdot L\big|_C\right)=k.$$

Thus as long as C is smooth the divisor D of degree $k+1$ imposes k conditions on $L|_C=K_C$ so that by Riemann-Roch

$$\begin{aligned} h^0\left(K_C\left(-D\right)\right) &= 2k-k=k\\ h^1\left(K_C\left(-D\right)\right) &= 2. \end{aligned}$$

Thus

$$D\in g^1_{k+1}.$$

So we can identify the space of $\left(C, g^1_{k+1}\right)$ with C smooth with a large open subset of $Gr\left(\mathbb{P}^1,\mathbb{P}\left(H^0\left(E\right)\right)\right)$.

The map which to $\sigma\in H^0(E)$ associates its zero set gives an immersion

$$\mathbb{P}\left(H^0(E)\right)\hookrightarrow Hilb^{k+1}(S).$$

The subvariety $\mathbb{P}\left(H^0(E)\right)\subset Hilb^{k+1}(S)$, or large open set $\mathbb{P}$ parametrizing curvilinear schemes, could be an interesting candidate for our subscheme Z. In fact letting

$$r:\tilde{\mathbb{P}}:=p^{-1}\left(\mathbb{P}\right)\to\mathbb{P}$$

be the restriction of p, one can show that the map

$$r^*:H^0(\mathbb{P};L_{k+1}|_{\mathbb{P}})\to H^0(\tilde{\mathbb{P}};p^*L_{k+1}|_{\tilde{\mathbb{P}}})$$

is surjective, which was the desired Property Ci) above. However, the restriction map

$$H^0(I_{k+1};p^*L_{k+1})\to H^0(\tilde{\mathbb{P}};p^*L_{k+1}|_{\tilde{\mathbb{P}}})$$

is certainly not injective. Indeed, we have the following:

Lemma 7.5. *The restriction map*

$$H^0(Hilb^{k+1}(S), L_{k+1}) \to H^0(\mathbb{P}; L_{k+1}|_{\mathbb{P}})$$

is equal to 0.

Proof. Indeed, the fiber of L_{k+1} at $z \in Hilb^{k+1}(S)$ is equal to $\bigwedge^{k+1} H^0(L|_z)$, the space $H^0(Hilb^{k+1}(S); L_{k+1})$ is isomorphic to $\bigwedge^{k+1} H^0(S; L)$ by (35), and the evaluation map

$$H^0(Hilb^{k+1}(S); L_{k+1}) \to L_{k+1}|_z \tag{41}$$

is the $(k+1)$-st exterior power of the evaluation map

$$H^0(S; L) \to H^0(L|_z).$$

But by (40) we have that, for $z \in \mathbb{P}(H^0(E))$, the evaluation map (41) is not surjective, hence its $(k+1)$-st exterior power is equal to 0. ☐

The variety Z that we shall construct, and for which we shall verify the Properties Ci) and Cii), is built from $\mathbb{P}$ as follows. Roughly speaking, it will be made of the cycles of the form

$$z = z(\sigma) - x + y,$$

where $z(\sigma) \in \mathbb{P}$, x is any point in the support of $z(\sigma)$ and y is any point of S. To make this more rigorous, we do the following. Consider the variety $\tilde{\mathbb{P}}$ defined above. Then it admits a morphism

$$\psi = pr_2 \circ \tau : \tilde{\mathbb{P}} \to H^k_{curv.}(S)$$

$$(x, z) \mapsto z - x$$

which is the restriction to $\tilde{\mathbb{P}} \subset I_{k+1}$ of the morphism τ in (37). Now recall that I_{k+1} identifies to a large open set of the blow-up of $S \times H^k_{curv.}(S)$ along the incidence set I_k. It follows that by taking a large open set in the blow-up of $S \times \tilde{\mathbb{P}}$ along

$$I := \tau^{-1}(I_k) \subseteq S \times \tilde{\mathbb{P}},$$

one gets a variety Z together with a morphism

$$j : Z \to I_{k+1}.$$

A typical point of I is a triple

$$(x', x, x + w)_{(x+w) = z(\sigma) \in \mathbb{P}, x' \in w}$$

A typical point $z \in Z$ is a triple

$$(y, x, z(\sigma))_{x \in z(\sigma)}$$

such that

$$x \in z(\sigma)$$

and

$$j(y, x, z(\sigma)) = (y, y + (z(\sigma) - x)).$$

The composition $(p \circ j)$ can be shown to be of degree 1 on its image, which is precisely the set described above. We now describe the variety

$$\tilde{Z} := Z \times_{Hilb^{k+1}(S)} I_{k+1},$$

which is birationally equivalent to $p^{-1}\left((p \circ j)(Z)\right)$ but has a more suitable scheme structure. A general point of $\tilde{Z}$ has the form

$$\Big((y, x, z(\sigma))_{x \in z(\sigma)}, (x', y + (z(\sigma) - x))\Big)_{x' \in (y + (z(\sigma) - x))}.$$

$\tilde{Z}$ has two components, one obvious component being Z which is given by setting

$$x' = x.$$

The second component Z' is described as follows. For the ramified cover $I_k \to H^k_{curv.}(S)$, form

$$\tilde{W} = \tilde{\mathbb{P}} \times_{H^k_{curv.}(S)} I_k$$

so that a typical point of $\tilde{W}$ is of the form

$$(x', x, z(\sigma))_{x \in z(\sigma), x' \in (z(\sigma) - x)}.$$

Let

$$\eta : \tilde{W} \to \tilde{\mathbb{P}}$$

be the induced cover (of degree k) and let Z'' be the blow-up of $S \times \tilde{W}$ along $(ident., \eta)^{-1}(I)$. Then, for a large open set Z' of Z'', there is a natural morphism

$$\begin{array}{rcl} j' & : & Z' \to I_{k+1}. \\ (y, x', x, z(\sigma) - x)_{x \in z(\sigma), x' \in (z(\sigma) - x)} & \mapsto & (x', y + (z(\sigma) - x))_{x' \in (z(\sigma) - x)} \end{array}$$

Here the blow-up along

$$\{(x'', x', x, z(\sigma) - x)\}_{x \in z(\sigma), x'', x' \in (z(\sigma) - x)}$$

is necessary to make the union $(y + (z(\sigma) - x))$ well-defined in $Hilb^{k+1}(S)$.

One important point is now the fact that in $\tilde{Z}$, the two components Z and Z' will be glued along a divisor D they both naturally contain. Namely, inside Z, D will simply be the exceptional divisor of the blow-up map

$$\tau : Z \to S \times \tilde{\mathbb{P}}.$$

along

$$I = \{(x', x, z(\sigma))\}_{x \in z(\sigma) \in \mathbb{P}, x' \in z(\sigma) - x}.$$

Inside Z', D will be one component of the exceptional divisor of the blow-up map

$$\tau' : Z' \to S \times \tilde{W}.$$

Namely, the blow-ups above are along the incidence subschemes I and $\tilde{I} = (ident., \eta)^{-1}(I)$ respectively. But now $\tilde{I}$ has a component which is isomorphic to I, and, abusing notation, we denote that component also as D. The entire picture so far can be put in the following diagrams:

$$\begin{tikzcd}
\tilde{Z} \arrow[d] & = & Z \times_{Hilb^{k+1}(S)} I_{k+1} & = & \underset{(y,y,x,z(\sigma))}{Z} \cup \underset{(y,x',x,z(\sigma))}{Z'} \arrow[d, "\tilde{j}=j\cup j'"] \\
\underset{(y,x,z(\sigma))}{Z} \arrow[rrrr, "j"] & & & & \underset{(y,y+z(\sigma)-x)}{I_{k+1}}
\end{tikzcd}$$

where

$$\begin{tikzcd}
D\cup D' \arrow[r] \arrow[d, phantom, "\cap"] & \tilde{I} = \underset{(x',x',x,z(\sigma))}{I} \cup \underset{(x',x'',x,z(\sigma))}{I'} \arrow[d, phantom, "\cap"] \arrow[drr] & & & \\
Z' \arrow[r] & \underset{(y,x',x,z(\sigma))}{S\times\tilde{W}} \arrow[d] \arrow[drr, "(ident.{,}\eta)"] & \square & \underset{(x'',x,z(\sigma))}{I} \arrow[d, phantom, "\cap"] & D \arrow[l] \arrow[d, phantom, "\cap"] \\
 & \underset{(x',x,z(\sigma))}{\tilde{W}} = \tilde{\mathbb{P}} \times_{H^k_{curv.}(S)} I_k \arrow[drr, "\eta"] & & \underset{(y,x,z(\sigma))}{S\times\tilde{\mathbb{P}}} \arrow[d] \arrow[ddll, "(ident.{,}\psi)"'] & Z . \arrow[l, "\tau"'] \arrow[d, "j"] \\
\tilde{E}_{k+1} \arrow[r, phantom, "\subseteq"] \arrow[d] & \underset{(y,y+z(\sigma)-x)}{I_{k+1}} \arrow[d, "\tau"] & & \underset{(x,z(\sigma))}{\tilde{\mathbb{P}}} \arrow[r, phantom, "\subseteq"] \arrow[d, "r"] & \underset{(y,y+z(\sigma)-x)}{I_{k+1}} \arrow[d, "p"] \\
I_k \arrow[r, phantom, "\subseteq"] & \underset{(y,z(\sigma)-x)}{S\times H^k_{curv.}} & & \underset{z(\sigma)}{\mathbb{P}} \arrow[r, phantom, "\subseteq"] & \underset{y+z(\sigma)-x}{H^{k+1}_{curv.}}
\end{tikzcd}$$

We want now to check the Properties Ci) and Cii) for Z and $\tilde{Z}$. Property Cii) says that the pull-back map

$$H^0\left(I_{k+1};p^*L_{k+1}\right) \xrightarrow{\tilde{j}^*} H^0\left(\tilde{Z};\left(p\circ\tilde{j}\right)^*L_{k+1}\right)$$

is injective. Here, abusing notation slightly, we write $\tilde{j}$ is equal to j on the component Z of $\tilde{Z}$ and as j' on the component Z'. Now it obviously suffices to show that the pull-back map

$$H^0(I_{k+1};p^*L_{k+1}) \xrightarrow{j^*} H^0(Z;(p\circ j)^*L_{k+1}) \tag{42}$$

is injective.

But recall that I_{k+1} identifies to a large open set in the blow-up of $S\times H^k_{curv.}$ along the incidence divisor I_k, and that Z is similarly the blow-up of $S\times W$ along $I=(Id,\psi)^{-1}(I_k)$. Furthermore if τ denote the blow-up morphism we have the formula

$$p^*L_{k+1} = \tau^*\left(L\boxtimes L_k\right)\left(-\tilde{E}_{k+1}\right), \tag{43}$$

where $\tilde{E}_{k+1}$ is the exceptional divisor of τ so that

$$(p\circ j)^*L_{k+1} = b^*\left(\left(L\boxtimes\psi^*L_k\right)\left(-\left(\tilde{E}_{k+1}\cap\tilde{\mathbb{P}}\right)\right)\right).$$

But then the injectivity of the map (42) will follow by proving the following:

Proposition 7.6. *The pull-back map*

$$\psi^* : H^0(H^k_{curv.}; L_k) \to H^0(\tilde{\mathbb{P}}; \psi^* L_k)$$

is injective.

The sufficiency of this Proposition follows from the consequent injective composition

$$\begin{array}{c} H^0\left(S \times H^k_{curv.}; (L \boxtimes L_k)(-I_k)\right) \\ \downarrow \\ H^0\left(S \times \tilde{\mathbb{P}}; (L \boxtimes \psi^* L_k)\left(-\left(\tilde{E}_{k+1} \cap \tilde{\mathbb{P}}\right)\right)\right) \\ \downarrow \\ H^0(Z; (p \circ j)^* L_{k+1}) \end{array} .$$

Finally we also want to reduce Ci), that is, the surjectivity of the pull-back map

$$p^* : H^0\left(Z; (p \circ j)^* L_{k+1}\right) \to H^0\left(\tilde{Z}; (p \circ \tilde{j})^* L_{k+1}\right). \tag{44}$$

to the proof of a surjectivity result which is simple to state. Let us denote by

$$\phi : Z' \to Z$$

the restriction of the map

$$\tilde{Z} \to Z$$

to Z'. Recall that in $\tilde{Z}$, the components Z and Z' are glued along a divisor $D \subset Z$. It follows immediately that the surjectivity of the map (44) will be a consequence of the injectivity of the restriction map

$$H^0\left(Z; (p \circ j)^* L_{k+1}\right) \to H^0\left(D; (p \circ j)^* L_{k+1}\big|_D\right) \tag{45}$$

and of the surjectivity of the pull-back map

$$\phi^* : H^0\left(Z; (p \circ j)^* L_{k+1}\right) \to H^0\left(Z'; (p \circ j')^* L_{k+1}\right). \tag{46}$$

Next recall that Z' is the blow-up of $S \times \tilde{W}$ along $\tilde{I} = (ident., \eta)^{-1}(I)$, while Z is the blow-up of $S \times W$ along I, and the map ϕ is induced by $(ident., \eta)$. Using formula (43) as before, we see now that the surjectivity of (46) will be a consequence of

Proposition 7.7. *The pull-back map*

$$\eta^* : H^0\left(\tilde{\mathbb{P}}; \psi^* L_k\right) \to H^0\left(\tilde{W}, \eta^*(\psi^* L_k)\right)$$

is surjective.

In fact, we show a stronger statement. To state it, we note first the following

Lemma 7.8. *On $\tilde{\mathbb{P}}$, the line bundles $\psi^* L_k$ and $r^* \mathcal{O}_{\mathbb{P}}(k)$ coincide.*

In fact it is easy to check that $\psi^* L_k$ is the pull-back to $\tilde{\mathbb{P}}$ of some line bundle on $\mathbb{P}$. Indeed, the map

$$\chi_k : H^k_{curv.} \to Gr\left(k, H^0(L)^\vee\right)$$

induced by the linear system $\mid L_k \mid$ associates to w the subspace $H^0(I_w \cdot L)$. If $w = \psi(x, z) \in \tilde{\mathbb{P}}$, with $z \in \mathbb{P}$ and x in the support of z, we have $w = z - x$. But we know that z does not impose the maximal number of conditions to L, hence

$$H^0(I_w \cdot L) = H^0(I_z \cdot L).$$

It follows that $\chi_k \circ \psi$ factors through r. An analogous argument and some computation then leads to a complete proof of the lemma above.

The proposition above is then a consequence of the following theorem

Theorem 7.9. *The pull-back map*

$$(r \circ \eta)^* : H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)) \to H^0\left(\tilde{W}; (r \circ \eta)^* L_k\right)$$

is surjective.

So to conclude the proof of Green's conjecture for the generic curve of genus $2k$, we have only to prove Proposition 7.6 and Theorem 7.9. We do not want to give the full details here, but explain two important points, which lead to these proofs.

The first point concerns Theorem 7.9. We observe that we can describe our variety $\tilde{W}$ as follows. Consider the blow-up

$$\widetilde{S \times S}$$

of $S \times S$ along the diagonal. It parametrizes couples (y, y') of points of S together with a schematic structure z of length 2 on $y \cup y'$. There is a vector bundle E_2 on $\widetilde{S \times S}$ which has for fiber $H^0(E|_z)$ at (y, y', z). Now there is a universal section

$$\tilde{\sigma} \in H^0\left(\widetilde{S \times S} \times \mathbb{P}; E_2 \boxtimes \mathcal{O}_{\mathbb{P}}(1)\right)$$

which takes the value $E|_z$ at the point (x, y, z, σ). Recalling that a point of $\tilde{W}$ is a point $\sigma \in \mathbb{P}$ together with a point x in the support $z(\sigma)$, and a point x' in the support of $(z(\sigma) - x)$, we see immediately that we can identify $\tilde{W}$, with the zero-set of $\tilde{\sigma}$.

To prove Theorem7.9, it suffices to show that

$$H^1\left(\widetilde{S \times S} \times \mathbb{P}; I_{\tilde{W}} \boxtimes \mathcal{O}_{\mathbb{P}}(k)\right) = 0. \tag{47}$$

To this effect, we write the Koszul resolution

$$0 \to \bigwedge^4 E_2^\vee(-4) \to \ldots \to E_2^\vee(-1) \to I_{\tilde{W}} \to 0$$

of $I_{\tilde{W}}$ which is given by $\tilde{\sigma}$.

Computing now the cohomology groups

$$H^i\left(\widetilde{S \times S} \times \mathbb{P}; \bigwedge^i (E_2^\vee \boxtimes \mathcal{O}_{\mathbb{P}}(k-i))\right)$$

we find that there are 0 for odd i, and a spectral sequence argument shows that (47) will hold if the two maps induced by $\tilde{\sigma}$

(48)
$$H^2\left(\widetilde{S \times S} \times \mathbb{P}; \bigwedge^2 (E_2^\vee \boxtimes \mathcal{O}_{\mathbb{P}}(k-2))\right) \to H^2\left(\widetilde{S \times S} \times \mathbb{P}; E_2^\vee \boxtimes \mathcal{O}_{\mathbb{P}}(k-1)\right),$$

and

(49)
$$H^4\left(\widetilde{S \times S} \times \mathbb{P}; \bigwedge^4 (E_2^\vee \boxtimes \mathcal{O}_{\mathbb{P}}(k-4))\right) \to H^4\left(\widetilde{S \times S} \times \mathbb{P}; \bigwedge^3 (E_2^\vee \boxtimes \mathcal{O}_{\mathbb{P}}(k-3))\right)$$

are injective. Let

$$\det : \bigwedge^2 H^0(E) \to H^0(\det E) = H^0(L)$$

be the determinant map. Computing the cohomology groups above and dualizing, Then the injectivity of (48) is equivalent to the surjectivity of the composed map

$$\begin{aligned} &H^0(E) \otimes S^{k-1}H^0(E) \to H^0(E) \otimes H^0(E) \otimes S^{k-2}H^0(E) \\ &\to \bigwedge^2 H^0(E) \otimes S^{k-2}H^0(E) \overset{\det \otimes ident.}{\longrightarrow} H^0(L) \otimes S^{k-2}H^0(E). \end{aligned} \tag{50}$$

Furthermore the injectivity of (49) is equivalent to the surjectivity of the similarly defined map

$$Q_{E,L} \otimes S^{k-3}H^0(E) \oplus Q_{L,E} \otimes S^{k-3}H^0(E) \to Q_{L,L} \otimes S^{k-4}H^0(E), \tag{51}$$

where

$$\begin{aligned} Q_{E,L} &: = \ker\left(H^0(E) \otimes H^0(L) \to H^0(E \otimes L)\right), \\ Q_{L,E} &: = \ker\left(H^0(L) \otimes H^0(E) \to H^0(E \otimes L)\right), \\ Q_{L,L} &: = \ker\left(H^0(L) \otimes H^0(L) \to H^0\left(L^2\right)\right). \end{aligned}$$

And finally we note a second ingredient which plays an essential role both in the proof of Proposition 7.6 and in the surjectivity of the (50) and (51) above. Let σ, σ' be two linearly independent sections of E. Then $\det(\sigma, \sigma') \neq 0$. Indeed there would be otherwise a rank-1 subsheaf of E with two sections. But this contradicts the fact that $Pic\, S$ is generated by L and that $H^0(E \otimes (-L)) = 0$.

It follows from this that we have a well-defined morphism

$$d : Grass\left(2, H^0(E)\right) \to \mathbb{P}\left(H^0(L)\right). \tag{52}$$

But note that both spaces have dimension $2k$, so that this morphism is in fact finite to one. Another way to say this is to say that the dual map

$$d^* : H^0(L)^\vee \to \bigwedge^2 H^0(E)^\vee$$

gives a regular sequence on $Grass\left(2, H^0(E)\right)$, that is a sublinear system of rank $(2k+1)$ and without basepoint of the Plücker linear system.

It turns out that the proof of Proposition 7.6 and a large part of the proof of the surjectivity of the two maps (50) and (51) use only this fact, Koszul resolution associated to the regular sequence above, and vanishing theorems for the tautological bundles on the Grassmannian. In other words, we shift from the geometry of S to that of the Grassmannian, via the morphism (52).

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