7. Intersection Theory I. Once a moduli space is constructed, the next step in "counting" is to convert enumerative questions into intersections. As our first foray into intersection theory, we will discuss Chern classes and the Riemann-Roch theorem. See Fulton's *Intersection Theory* for details.

Throughout, X will be a scheme of finite type over a field k.

The **cycle group** $Z_*(X)$ is the free abelian group on the set of closed subvarieties of X. It is graded by dimension, and a d-cycle α is **rationally equivalent to zero**, written $\alpha \sim 0$, if there exist finitely many subvarieties $W_i \subset X$ of dimension $d+1$ together with rational functions $r_i \in K(W_i)^*$ such that $\alpha = \sum[\mathrm{div}(r_i)]$, where $[\mathrm{div}(r_i)]$ is the divisor (of zeroes minus poles) associated to r_i. The set of d-cycles rationally equivalent to 0 is a subgroup of $Z_d(X)$, denoted $\mathrm{Rat}_d(X)$, and the **Chow group** of X is the graded group:

$$A_*(X) = \bigoplus_{d=0}^{n} Z_d(X)/\mathrm{Rat}_d(X).$$

Examples: (a) If $n = \dim(X)$, then $A_n(X)$ is the free abelian group on the components of X of dimension n.

(b) $A_*(\mathbf{A}^n_k) = A_n(\mathbf{A}^n_k) \cong \mathbf{Z}$.

(c) $A_d(\mathbf{P}^n_k) \cong \mathbf{Z}$, generated by linear subspaces, for all $0 \le d \le n$.

(d) $A_{n-1}(\mathbf{P}^n_k - S) \cong \mathbf{Z}/d\mathbf{Z}$ if S is an irreducible hypersurface of degree d.

(e) $A_0(C)$ is not finitely generated if C is a projective curve of genus > 0.

If $f : X \to Y$ is a **proper** morphism, then there is a functorial graded **proper push-forward** of Chow groups $f_* : A_d(X) \to A_d(Y)$ defined, on the level of closed subvarietes, by:

$$f_*[V] = \begin{cases} 0 \text{ if } \dim(f(V)) < \dim(V) \\ [K(V) : K(f(V))]\,[f(V)] \text{ if } \dim(f(V)) = \dim(V) \end{cases}$$

Example: If X is proper over $\mathrm{Spec}(k)$, then the degree map:

$$\deg : A_0(X) \to \mathbf{Z} = A_0(\mathrm{Spec}(k))$$

is the proper push-forward to $\mathrm{Spec}(k)$.

If $f : X \to Y$ is a **flat** morphism of relative dimension n, then there is a functorial **flat pull-back** $f^* : A_d(Y) \to A_{d+n}(X)$ defined by setting $f^*[V] = [f^{-1}(V)]$ (the cycle associated to the scheme $f^{-1}(V)$). Flat pull-backs also commute with proper push-fowards in the following sense:

If $f : X \to Y$ is proper and $g : Y' \to Y$ is flat of relative dimension n, let:

$$\begin{array}{ccc} X' & \stackrel{g'}{\to} & X \\ f' \downarrow & & f \downarrow \\ Y' & \stackrel{g}{\to} & Y \end{array}$$

Then $g^* f_* = f'_* g'^* : A_d(X) \to A_{d+n}(Y')$.

Suppose $i : Y \hookrightarrow X$ is a closed immersion, and $j : U \hookrightarrow X$ is the open immersion of the complement of Y, Then i is proper, j is flat of relative dimension 0, and the following sequence is exact:

$$A_*(Y) \stackrel{i_*}{\to} A_*(X) \stackrel{j^*}{\to} A_*(U) \to 0$$

Another Example: If $f : E \to X$ is an affine bundle of rank n, then the flat pull-back:

$$f^* : A_d(X) \to A_{d+n}(E)$$

is surjective for all d (including $d < 0$, where $A_d(X) = 0$). If E is a vector bundle, then the f^* are injective, as well.

Suppose L is a **line bundle** on X. Then the **first chern class** $c_1(L)$ should be thought of as an operator:

$$\cap c_1(L) : A_d(X) \to A_{d-1}(X)$$

defined by $[V] \mapsto [\mathrm{div}(s)]$ where s is any (meromorphic) section of $L|_V$ other than the zero section. We let $c_1(L)^i$ be the i-fold iteration of $\cap c_1(L)$.

If E is a **vector bundle** of rank $e+1$ on X, let $P(E)$ be the projective bundle, with (proper, flat) projection $p : P(E) \to X$ of relative dimension e and line bundle $\mathcal{O}_{P(E)}(1)$. The i**th Segre class** $s_i(E)$ is the operator

$$\cap s_i(E) : A_d(X) \to A_{d-i}(X); \quad \alpha \mapsto p_*(p^*\alpha \cap c_1(\mathcal{O}_{P(E)}(1))^{e+i})$$

where p^* is the flat pull-back and p_* is the proper push-forward.

The **Chern classes** $c_i(E)$ are defined by:

$$c(E)s(E) = (1 + c_1(E) + c_2(E) + ...)(1 + s_1(E) + s_2(E) + ...) = 1$$

where $c(E)$ and $s(E)$ are the **total** Chern and Segre classes.

Properties of Chern Classes:

(a) They commute. $(\alpha \cap c_i(E)) \cap c_j(F) = (\alpha \cap c_j(F)) \cap c_i(E)$.

(b) (Projection Formula) If $f : X' \to X$ is proper and $\alpha \in A_d(X')$, then:

$$f_*(\alpha \cap c_i(f^*E)) = f_*(\alpha) \cap c_i(E)$$

(c) (Flat Pull-Back) $f : X' \to X$ is flat and $\alpha \in A_k(X)$, then for all i,

$$f^*(\alpha \cap c_i(E)) = f^*(\alpha) \cap c_i(f^*E)$$

(d) (Geometric Interpretation) If $V \subset X$ is nonsingular and the zero scheme Z of a section $s \in \Gamma(V, E|_V)$ has codimension $e + 1$ in V, then

$$[V] \cap c_{e+1}(E) = [Z]$$

(e) (Vanishing) $c_i(E) = 0$ if $i > e + 1$.

(f) (Duals) $c_i(E^*) = (-1)^i c_i(E)$

(f) (Whitney Product Formula) Given $0 \to E' \to E \to E'' \to 0$, then:

$$c(E) = c(E')c(E'')$$

The **Chern roots** $\alpha_1, ..., \alpha_{e+1}$ are formal symbols, used for calculations involving Chern classes. That is, c_i is the ith symmetric polynomial σ_i in the α_j and more generally, any symmetric polynomial in the α_j corresponds to a polynomial in the σ_i, hence to a polynomial in the Chern classes. For example, the i**th Chern character** $ch_i(E)$ is defined by:

$$ch_i(E) = \sum_{j=1}^{e+1} \frac{\alpha_j^i}{i!}$$

The **total Chern character** is $ch(E) = (e + 1) + ch_1(E) + ch_2(E) + ...$

(g) For exact sequences as in (f), $ch(E) = ch(E') + ch(E'')$.

(h) (Tensor Product Formula) $ch(E' \otimes E'') = ch(E')ch(E'')$.

One can derive an expression for $ch(E)$ in terms of $c(E)$ recursively from:

Newton's Formula: Let $p_i = (i!)ch_i(E)$. Then $p_1 = c_1(E)$ and for all $i \geq 2$:

$$p_i = c_1(E)p_{i-1} - c_2(E)p_{i-2} + ... + (-1)^{i-2}c_{i-1}(E)p_1 + (-1)^{i-1}c_i(E)i.$$

Thus, for example,

$$ch_2(E) = \frac{1}{2}(c_1^2 - 2c_2) \text{ and } ch_3(E) = \frac{1}{6}(c_1^3(E) - 3c_1(E)c_2(E) + 3c_3(E))$$

The **total Todd class** of E is defined by:

$$1 + td_1(E) + td_2(E) + ... = td(E) = \prod_{j=1}^{e+1} \frac{\alpha_j}{1 - \exp(-\alpha_j)}$$

and the first few terms are:

$$td(E) = 1 + \frac{1}{2}c_1(E) + \frac{1}{12}(c_1^2(E) + c_2(E)) + \frac{1}{24}(c_1(E)c_2(E)) + ...$$

(i) For exact sequences as in (f), $td(E) = td(E')td(E'')$.

Hirzebruch-Riemann-Roch: If X is a nonsingular projective variety, let TX be its tangent bundle and let $td(X) = td(TX)$. Then:

$$\chi(X, E) = \int_X ch(E).td(X) := \deg([X] \cap ch(E) \cap td(X))$$

Examples: (a) Curves. $td(C) = 1 - \frac{1}{2}K_C$, so:

$$\chi(X, E) = \int_C (e + 1 + c_1(E))(1 - \frac{1}{2}K_C) = \deg(E) + (e + 1)(1 - g)$$

where $\deg(E) = \int_C c_1(E)$ is the degree of E. Thus the Hilbert polynomial:

$$\chi(X, E(d)) = d(e + 1)\int_C c_1(\mathcal{O}_C(1)) + \deg(E) + (e + 1)(1 - g)$$

so the rank and degree of E are determined by the Hilbert polynomial.

(b) Surfaces. $td(S) = 1 - \frac{1}{2}K_S + \frac{1}{12}(K_S^2 + c_2(TX))$ so

$$\chi(X, \mathcal{O}_S) = \frac{1}{12}\int_S K_S^2 + c_2(TX)$$

For any vector bundle E on S:

$$\chi(X, E) = \frac{1}{2}\int_S (c_1^2(E) - 2c_2(E)) - \frac{1}{2}\int_S c_1(E).K_S + \chi(X, \mathcal{O}_S)$$

and the Hilbert polynomial of E is:

$$\frac{d^2}{2}(e+1)\int_S c_1^2(\mathcal{O}_S(1)) + d\int_S (c_1(\mathcal{O}_S(1)) - \frac{e+1}{2}K_S)\cdot c_1(E) + \chi(X, E)$$

The Hirzebruch-Riemann-Roch theorem generalizes in an important way.

The **Grothendieck ring** $K^0(X)$ is the free abelian group generated by the vector bundles on X modulo the subgroup generated by the relations: $[E] \sim [E'] + [E'']$ whenever $0 \to E' \to E \to E'' \to 0$. This is a commutative ring with 1 via the multiplication $[E]\cdot[F] := [E \otimes F]$ with $[\mathcal{O}_X] = 1$. The pull-back $f^*[E] := [f^*E]$ is well-defined and contravariant.

The **Grothendieck group** $K_0(X)$ is the free abelian group generated by the **coherent sheaves** on X modulo the same relation as in the Grothendieck ring. It is naturally a module over $K^0(X)$ via tensor product. In this case, the pull-back defined as above is only well-defined for flat morphisms f.

Example: (a) If X is nonsingular, then a theorem of Hilbert implies that every coherent sheaf on X has a finite resolution by vector bundles. Thus for such X, the Grothendieck group coincides with the Grothendieck ring! It follows that Hirzebruch-Riemann-Roch is valid for all coherent sheaves.

(b) On $\mathbf{P}_k^n$, every vector bundle may be resolved by line bundles. As a result, $K^0(\mathbf{P}^n) = K_0(\mathbf{P}^n)$ is generated (as a ring) by $[\mathcal{O}(1)]$ and $[\mathcal{O}(-1)]$.

In particular, the Koszul complex:

$$0 \to \wedge^{n+1}V \otimes \mathcal{O}(-n-1) \to \ldots \to \wedge^2 V \otimes \mathcal{O}(-2) \to V \otimes \mathcal{O}(-1) \to \mathcal{O} \to 0$$

is a relation, out of which we can express $[\mathcal{O}(-1)]$ in terms of $[\mathcal{O}(1)]$, and we obtain a surjective ring homomorphism:

$$\mathbf{Z}[x]/(1-x)^{n+1} \to K^0(\mathbf{P}^n);\ x \mapsto [\mathcal{O}(1)]$$

Definition: If X is equidimensional of dimension n, let

$$A^d(X) = A_{n-d}(X) \text{ and } A^d(X)_{\mathbf{Q}} = A^d(X) \otimes_{\mathbf{Z}} \mathbf{Q}$$

and let $1 = [X] \in A^0(X)$.

Then properties (f)-(i) of Chern classes imply that the following maps:

$c_1 : K^0(X) \to A^1(X); [E] \mapsto [X] \cap c_1(E)$,

$c : K^0(X) \to A^*(X); [E] \mapsto [X] \cap c(E)$,

$ch : K^0(X)_{\mathbf{Q}} \to A^*(X)_{\mathbf{Q}}; [E] \mapsto [X] \cap ch(E)$,

$td : K^0(X)_{\mathbf{Q}} \to A^*(X)_{\mathbf{Q}}; [E] \mapsto [X] \cap td(E)$

are well-defined, and compatible with (flat) pull-backs. The maps c_1 and ch preserve addition, while the other two convert addition into "multiplication." Note that the kernel of ch is an ideal in $K^0(X)$ (or $K^0(X)_{\mathbf{Q}}$) so we can think of ch as imposing the structure of a graded commutative ring with 1 on its image. When X is nonsingular, there is a multiplication defined on $A^*(X)$ via intersection theory, with respect to which ch is a ring homomorphism.

Back to our $\mathbf{P}^n$ example: The "chern character" ring homomorphism:

$$ch : \mathbf{Q}[x]/(1-x)^{n+1} \to \mathbf{Q}[h]/h^{n+1}$$

defined by:

$$ch(x) = \exp(h) = 1 + h + \frac{h^2}{2!} + ... \pmod{h^{n+1}}$$

is an **isomorphism**. The point is that:

$$1, \exp(h), \exp(2h), ..., \exp(nh)$$

are all linearly independent! In this case, we have the ring structure:

$$\mathbf{Q}[h]/h^{n+1} \quad \text{on} \quad A^*(\mathbf{P}^n)_{\mathbf{Q}}$$

and in particular, $\mathbf{Z}[x]/(1-x)^{n+1} \cong K^0(\mathbf{P}^n)$, because there can be no kernel.

Next, we introduce the proper push-forward on $K_0(X)$.

The push-forward of a coherent sheaf under a proper morphism remains coherent, but this operation is only left exact, so to obtain a homomorphism of Grothendieck groups, we will need to consider the right-derived functors. Specifically, if $f : X \to Y$ is a proper morphism and $\mathcal{F}$ is a coherent sheaf on X, then the sheaves $R^i f_*\mathcal{F}$ associated to the presheaves:

$$U \mapsto \mathrm{H}^i(f^{-1}(U), \mathcal{F})$$

are coherent, and are the right derived functors for the functor: $\mathcal{F} \rightsquigarrow f_*\mathcal{F}$. Thus, if we are given a short exact sequence: $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, of coherent sheaves on X, then we obtain a (terminating) long exact sequence:

$$... \to R^{i-1}f_*\mathcal{F}'' \to R^i f_*\mathcal{F}' \to R^i f_*\mathcal{F} \to R^i f_*\mathcal{F}'' \to R^{i+1}f_*\mathcal{F}' \to ...$$

$R^i f_*\mathcal{F}$ restricts to $\mathrm{H}^i(X_A, \mathcal{F}_A)$ over open affines $\mathrm{Spec}(A) \subset Y$. It follows that many of the results obtained for these cohomology groups are also valid for the higher direct image sheaves. For instance:

Serre's Theorem B: If f is projective, then $R^i f_*\mathcal{F}(d) = 0$ for all $d >> 0$.

Flat Base Change: If f is any morphism and $g : Y' \to Y$ is flat, consider the fiber square:

$$\begin{array}{ccc} X' & \stackrel{\widetilde{g}}{\to} & X \\ \widetilde{f}\downarrow & & f\downarrow \\ Y' & \stackrel{g}{\to} & Y \end{array}$$

Then:

$$g^* R^i f_*(\mathcal{F}) = R^i \widetilde{f}_*(\widetilde{g}^*\mathcal{F})$$

for all coherent sheaves $\mathcal{F}$ on X.

Thus if f is a proper morphism we obtain a well-defined proper push-forward on Grothendieck groups by taking the "Euler characteristic":

$$f_* : K_0(X) \to K_0(Y); \quad f_*[\mathcal{F}] := \sum(-1)^i[R^i f_*\mathcal{F}]$$

If f is proper in the fiber diagram above, then:

$$\widetilde{g}_*\widetilde{f}^* = f^* g_* : K_0(X) \to K_0(Y')$$

so that flat pull-backs commute with proper push-forwards.

The proper push-forward is functorial (this requires spectral sequences) and there is a projection formula. If $f : X \to Y$ is a proper morphism, $\mathcal{F}$ is a coherent sheaf on X, and E is a vector bundle on Y, then:

$$R^i f_*(f^*E \otimes \mathcal{F}) = E \otimes R^i f_*(\mathcal{F})$$

from which we obtain the projection formula:

$$f_*(f^*[E] \cdot [\mathcal{F}]) = [E] \cdot f_*[\mathcal{F}]$$

Grothendieck-Riemann-Roch: If $f : X \to Y$ is a projective morphism of nonsingular varieties over k and $\mathcal{F}$ is a coherent sheaf on X, then:

$$f_* \left(ch(\mathcal{F}) \cap td(X) \right) = ch(f_*[\mathcal{F}]) \cap td(Y)$$

In other words, $ch(\mathcal{F}) \cap td(X)$ commutes with proper push-forward.

Examples: (a) Hirzebruch-Riemann-Roch is the special case $Y = \text{Spec}(k)$ since in that case, $f_*[\mathcal{F}] = \chi(X, \mathcal{F})$ and $td(Y) = 1$.

(b) If f is a closed immersion, let $N_{X/Y}$ be the normal bundle. Then:

$$f_* \left(ch(\mathcal{F}) \cap td^{-1}(N_{X/Y}) \right) = ch(f_*\mathcal{F})$$

tells how to calculate Chern classes of $f_*\mathcal{F}$ in terms of Chern classes of $\mathcal{F}$.

The Chow Ring of the Grassmannian: On an equidimensional X with an exact sequence:

$$(*) \quad 0 \to K \to \mathcal{O}_X^n \to Q \to 0$$

and K, Q vector bundles of ranks m and $n - m$, we obtain some non-trivial relations among the Chern classes of K. Namely $c(K)c(Q) = c(\mathcal{O}_X^n) = 1$, hence $c_i(Q) = s_i(K)$ for all i and in particular, $s_i(K) = 0$ for all $i > n - m$. Using this, we obtain a map:

$$e : \mathbf{Z}[x_1, ..., x_m]/I \to A^*(X); \quad p(x_i) \mapsto [X] \cap p(c_i(K))$$

with I the ideal generated by the polynomials $s_{m-n+1}, ..., s_n$ defined by:

$$(1 + s_1 t + s_2 t^2 + ...)(1 + x_1 t + ... + x_e t^e) = 1$$

($s_{n+1}, s_{n+2}, ...$ are automatically in this ideal!)

This is a graded homomorphism if we let the degree of x_i be i.

Theorem: If $X = G(m,n)$ and $(*)$ is the universal short exact sequence, then e is an isomorphism of graded abelian groups.

(This gives $A^*(G(m,n))$ the structure of a graded commutative **ring**.)

Example: When $m = 1$, this gives:

$$\mathbf{Z}[x]/(-x)^n \cong A^*(G(1,n)) = A^*(\mathbf{P}_k^{n-1})$$

The minus sign is explained by the fact that $[\mathbf{P}_k^{n-1}] \cap c_1(K) = -H$.

The theorem may be proved in several steps.

Step 1: Properties of the graded ring $R(m,n) := \mathbf{Z}[x_1, ..., x_m]/\langle s_{n-m+1}, ..., s_n\rangle$.

The **Schur polynomial** of a partition $m \geq a_1 \geq ... \geq a_{n-m} \geq 0$ is:

$$\Delta_{a_1,\ldots,a_{n-m}} := \det \begin{pmatrix} x_{a_1} & x_{a_1+1} & \cdots & x_{a_1+n-m-1} \\ x_{a_2-1} & x_{a_2} & \cdots & x_{a_2+n-m-2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{a_{n-m}-(n-m)+1} & x_{a_{n-m}-(n-m)+2} & \cdots & x_{a_{n-m}} \end{pmatrix}$$

with the convention that $x_i = 0$ if $i < 0$ or $i > m$. Note that $\Delta_{a_1,\ldots,a_{n-m}}$ is homogeneous of degree $a_1 + ... + a_{n-m}$.

These Schur polynomials have the following remarkable properties:

(i) The $\Delta_{a_1,\ldots,a_{n-m}}$ freely generate $R(m,n)$.

Thus $R(m,n)$ is a free of rank $\binom{n}{m}$ and $R(m,n)_{m(n-m)} = \mathbf{Z}x_m^{n-m}$.

(ii) $\Delta_{a_1,\ldots,a_{n-m}}\Delta_{m-a_{n-m},\ldots,m-a_1} = x_m^{n-m}$ in $R(m,n)$ for all partitions.

Thus $R(m,n)$ satisfies "Poincaré duality". That is, the multiplications:

$$R(m,n)_d \times R(m,n)_{m(n-m)-d} \to R(m,n)_{m(n-m)} \cong \mathbf{Z}$$

are perfect pairings, inducing isomorphisms

$$R(m,n)_{m(n-m)-d} \cong \operatorname{Hom}(R(m,n)_d, \mathbf{Z}).$$

This is considerably more information than simply that $R(m,n)_d$ and $R(m,n)_{m(n-m)-d}$ have the same rank!

These facts may be proved directly. As it would take us too far afield, I will not spend class time on this. I encourage the interested student to go find a proof or better yet, to construct one.

Step 2:

$$\int_{G(m,n)} c_m^{n-m}(K) = (-1)^{m(n-m)}$$

The sections of the dual bundle K^* vanish along the (reduced) sub-Grassmannians $G(m, n-1) \subset G(m, n)$. A general section of $\oplus_{i=1}^{n-m} K^*$ will then vanish at a single reduced point, corresponding to a $G(m, m) \subset G(m, n)$. Thus $\int c_{m(n-m)}(\oplus K^*) = 1$ by the geometric realization of Chern classes and Step 2 follows from the Whitney formula and the formula for duals.

Corollary (of Steps 1 and 2): e is injective.

Proof: Suppose $e(P) = 0$ for some $0 \neq P \in R(m, n)$. We may as well assume $0 \neq P$ is homogeneous, say of degree d. Using the perfect pairing of Step 1, we know that there is a $Q \in R(m, n)_{m(n-m)-d}$ with the property that $0 \neq PQ \in R(m, n)_{m(n-m)}$, i.e. $PQ = rx_m^{n-m}$ for some nonzero $r \in \mathbf{Z}$. But then $e(PQ) \neq 0$, by Step 2, since $\int e(PQ) = \pm r$. But the kernel of e is clearly an ideal, so this contradicts $e(P) = 0$.

Step 3: A cellular decomposition of the Grassmannian.

Define a partial ordering on the sequences: $0 < i_1 < ... < i_m \leq n$:

$$I \preceq J \text{ if } i_k \leq j_k \text{ for all } k = 1, ..., m,$$

Recall the Plücker embedding $G(m, n) \subset \mathbf{P}_k^{\binom{n}{m}-1}$ with coordinates x_J. We define closed subsets $V_J := G(m, n) \cap V(\{x_I\}_{I \prec J})$ and (relatively) open subsets $Y_J = V_J - V(x_J)$.

The Y_J are clearly disjoint and cover $G(m, n)$. They are the isomorphic images of $m \times n$ matrices with 1's in the (k, j_k) positions and zeroes to the left. Thus, $Y_J \cong \mathbf{A}^{m(n-m)-\sum_{k=1}^{m}(j_k-k)}$. Moreover,

$$(\dagger) \quad \overline{Y_J} - Y_J = \cup_{I \succ J} Y_I$$

as the reader is invited to check.

Corollary (of all the Steps): e is surjective.

Proof: First of all, $A_*(G(m,n))$ is a quotient of a free group of rank $\binom{n}{m}$. This is seen by induction and the exact sequence on Chow groups, starting with the point $Y_{(n-m+1,\ldots,n)} \in G(m,n)$.

Assume we are given a closed union $Z \subset G(m,n)$ of finitely many Y_J. From (†) above and downward induction, there is a $Y = Y_I$ with the property that $Y \cup Z$ is also closed. Thus we then obtain an exact sequence:

$$A_*(Z) \to A_*(Z \cup Y) \to A_*(Y) \to 0$$

which is a split epimorphism with inverse $[Y] \mapsto [\overline{Y}]$, since the Y's are affine spaces. Proceeding in this way we fill the Grassmannian and untimately obtain a surjection:

$$\rho : \bigoplus_J \mathbf{Z}[\overline{Y_J}] \to A_*(G(m,n)).$$

By the previous corollary, we know that $A_*(G(m,n))$ contains a free abelian group of rank $\binom{n}{m}$, so it follows that ρ is an isomorphism.

We are not quite done yet(!). Namely, we have an injective map:

$$e : R(m,n) \to A^*(G(m,n))$$

and both sides are free abelian groups of the same rank. It follows that e embeds $R(m,n)$ as a lattice. Choose $\alpha \in A_d(G(m,n))$. Via the pairing

$$t_\alpha(P) := \deg(\alpha \cap P(c_i(K)))$$

we determine from α an element $t_\alpha \in \mathrm{Hom}(R(m,n)_d, \mathbf{Z})$.

By the perfect pairing of Step 1 and also using Step 2, there is an element $Q \in R(m,n)_{m(n-m)-d}$ such that $\int e(PQ) = t_\alpha(P)$ for every $P \in R(m,n)_d$. If we consider $\beta := e(Q)$ it follows that $t_\beta(P) = t_\alpha(P)$ for all $P \in R(m,n)_d$. Some multiple $r\alpha$ must be in the image, say $r\beta = e(S)$, since $R(m,n)$ embeds as a lattice in $A^*(G(m,n))$. But then $S - rQ$ pairs to zero with every element $P \in R(m,n)_d$. By Poincaré duality, it follows that $S = rQ$, and that $\alpha = \beta = e(Q)$. Thus e is surjective.