Genus Zero Gromov-Witten Invariants

(Lectures at the University of Nice in the Fall, 2000)

3. Properties of Kontsevich-Manin Spaces and WDVV.

¿From Deligne-Mumford, we have the following:

Definition: A curve C with marked points $p_1, ..., p_n$ is **prestable** if C is projective and connected, has only simple nodes as singularities, and the marked points are distinct and nonsingular. It is **stable** if, in addition, the automorphism group $\text{Aut}(C; p_i)$ is finite.

The definition of stable maps is entirely analogous.

Definition: A map $f : C \to X$ from a curve with marked points $p_1, ..., p_n$ is **prestable** if the marked curve is prestable. If the group $\text{Aut}(f) \subseteq \text{Aut}(C; p_i)$ of automorphisms fixing the map f is finite, then f is **stable**. It is a (g, n) map if the (arithmetic) genus of C is g.

Examples: (a) A finite prestable map is stable, regardless of the markings or the genus of C.

(b) In general, a prestable map is stable if and only if each of the positive dimensional fibers is a disjoint union of stable marked curves.

Theorem I: For $g, n \geq 0$ and $\beta \in H_2(X, \mathbf{Z})$, the category with objects:

$$\{\text{flat families over } S \text{ of stable } (g,n) \text{ maps } f \text{ with } f_*[C] = \beta\}$$

and the natural morphisms is a proper Deligne-Mumford stack, denoted $\overline{M}_{g,n}(X, \beta)$ and called the Kontsevich-Manin stack of stable (g, n) maps of class β. It has a projective coarse moduli space.

In practice this says that there is a universal stable map

$$\begin{array}{ccc} \mathcal{C} & \stackrel{e}{\to} & X \\ \downarrow & & \\ \overline{M}_{g,n}(X, \beta) & & \end{array}$$

with sections $\rho_i : \overline{M}_{g,n}(X, \beta) \to \mathcal{C}$ and that $\overline{M}_{g,n}(X, \beta)$ resembles a projective variety closely enough that it admits coherent sheaves and has a Chow group (over $\mathbf{Q}$) on which intersection theory applies.

This theorem was proved by Kontsevich and Manin, except for the projectivity of the coarse moduli space, which was proved by Alexeev.

If we replace "stable" with "prestable" then the category defined as above is an Artin stack, which is denoted $\mathfrak{M}_{g,n}(X, \beta)$. This is much more difficult to work with than the Deligne-Mumford stack of stable maps, but can be a useful tool in proofs.

The **total evaluation map** is defined using the universal stable map:

$$ev = (e \circ \rho_1, ..., e \circ \rho_n) : \overline{M}_{g,n}(X, \beta) \to X^n$$

and it is this map that we will use to rigorously define $\mathrm{Rat}_n(X, d)$.

Examples: The spaces $\overline{M}_{0,n}(\mathbf{P}^m, 1)$ for small n are familiar:

- $\overline{M}_{0,0}(\mathbf{P}^m, 1)$ is the Grassmannian $G(2, m+1)$
- $\overline{M}_{0,1}(\mathbf{P}^m, 1)$ is the flag variety $Fl(1, 2, m+1)$ and the projections:

$$\begin{array}{ccc} Fl(1,2,m+1) & \to & \mathbf{P}^m \\ \downarrow & & \\ G(2,m+1) & & \end{array}$$

exhibit the universal stable map over the Grassmannian.

- $\overline{M}_{0,2}(\mathbf{P}^m, 1) \stackrel{ev}{\to} \mathbf{P}^m \times \mathbf{P}^m$ is the blow up of the diagonal.

Gluing maps are maps of the form:

$$\times_{i=1}^{k} \overline{M}_{g_i, n_i}(X, \beta_i) \to \overline{M}_{g,n}(X, \beta)$$

from a fiber product of stable map spaces, corresponding to the graph type of a stable (g, n) map. For example, in genus zero the gluing map:

$$\overline{M}_{0,m+1}(X, \alpha) \times_X \overline{M}_{0,1+n-m}(X, \beta - \alpha) \to \overline{M}_{0,n}(X, \beta)$$

corresponds to stable maps with a node which breaks the curve into two components according to the indicated data. Our convention will be that the last marked point of the first component and the first marked point of the second component correspond to the node. Such maps are indexed by subsets $S \subseteq \{1, ..., n\}$ containing a fixed i and classes α. We will denote the associated divisor classes in the Kontsevich-Manin space by $D(S, \alpha)$ following Fulton and Pandharipande.

A second theorem dates back in some form to Mumford-Knudsen.

Theorem II. A family of prestable (g, n) maps $\mathcal{C} \to X$ over a base scheme S always admits a unique *stabilization*:

$$\begin{array}{ccc} \mathcal{C} & & \\ \downarrow & \searrow & \\ \mathcal{C}^{st} & \to & X \\ \downarrow & & \\ S & & \end{array}$$

where $\mathcal{C}^{st}$ is a family of stable (g, n) maps.

As a direct consequence of the theorem, there is a morphism of stacks:

$$\mathfrak{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}(X,\beta)$$

As another consequence, morphisms $\phi : X \to Y$ push forward to:

$$\phi_* : \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}(Y,\phi_*\beta)$$

and in particular from the structure map, we get:

$$\pi : \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n} = \overline{M}_{g,n}(pt,0)$$

As a third consequence, there are forgetful maps:

$$f_i : \overline{M}_{g,n+1}(X,\beta) \to \overline{M}_{g,n}(X,\beta)$$

forgetting the ith marked point of the curve C.

Example: With the forgetful map and evaluation map:

$$\begin{array}{ccc} \overline{M}_{g,n+1}(X,\beta) & \stackrel{e\circ\rho_{n+1}}{\to} & X \\ f_{n+1}\downarrow & & \\ \overline{M}_{g,n}(X,\beta) & & \end{array}$$

the stable map space $\overline{M}_{g,n+1}(X,\beta)$ is the universal stable map over $\overline{M}_{g,n}(X,\beta)$ and $D(\{i,n+1\},0)$ is the image of the section ρ_i. This implies that the forgetful maps are flat, in particular.

The stack of stable maps is an open substack of a relative Hilbert scheme of morphisms over the smooth Artin stack $\mathfrak{M}_{g,n}$:

$$\begin{array}{ccc} \overline{M}_{g,n}(X,\beta) & \hookrightarrow & \mathrm{Mor}_\beta(\mathcal{C},X) \\ & \searrow & \downarrow \\ & & \mathfrak{M}_{g,n} \end{array}$$

¿From this point of view, the expected dimension:

$$\chi(f^*TX) + 3g - 3 + n = (\dim(X) - 3)(1-g) + n - \deg(K_X \cap \beta)$$

is just the expected dimension of the relative Hilbert scheme, and more generally, the relative deformation theory is particularly simple.

This is the point of view taken by Behrend in his proof of:

Theorem III. For any g, n, β there is a class of the expected dimension:

$$[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}} \in CH_*(\overline{M}_{g,n}(X,\beta))$$

in the Chow group of the Kontsevich-Manin stack, defined canonically by intersection theory, which also satisfies the following properties:

(o) It is invariant under the action of the symmetric group S_n.

(i) If X is a homogeneous space, then $[\overline{M}_{0,n}(X,\beta)]^{\rm vir} = [\overline{M}_{0,n}(X,\beta)]$. If X is the zero locus of a general section of a spanned vector bundle on a homogeneous space, the virtual class is the "expected" Chern class.

(ii) Under forgetful maps, $f_i^*[\overline{M}_{g,n}(X,\beta)]^{\rm vir} = [\overline{M}_{g,n+1}(X,\beta)]^{\rm vir}$.

(iii) Under gluing maps, virtual classes "pull back" to fiber products of virtual classes. (This requires Gysin maps for a precise statement.)

As suggested in the introduction, we may now define:

$$[\mathrm{Rat}_n(X,d)] = \sum_{\deg(\beta)=d} [\mathrm{Rat}_n(X,\beta)] := \sum_{\deg(\beta)=d} ev_*[\overline{M}_{0,n}(X,\beta)]^{\rm vir}$$

as (totally symmetric) elements of $\mathrm{H}^*(X,\mathbf{Q})^{\otimes n}$.

We can use these foundations to rigorously prove the:

WDVV Relations: Let $V = \mathrm{H}^*(X,\mathbf{Q})$ and $P : V\otimes V \to \mathbf{Q}$ be the perfect pairing of Poincaré duality. More generally, let:

$$P : V^{\otimes m+1}\otimes V^{\otimes 1+n-m} = V^{\otimes m}\otimes V\otimes V\otimes V^{\otimes n-m} \to V^{\otimes n}$$

be the pairing of Poincaré duality applied to the neighboring factors. Even more generally, for $S\subset\{1,...,n\}$ let $V^{S+1} = V^{\otimes|S|}\otimes V$ but also remembering the set S, and likewise for $V^{\otimes 1+S^c}$. Then let

$$P : V^{\otimes S+1}\otimes V^{\otimes 1+S^c} = V^{\otimes S}\otimes V\otimes V\otimes V^{\otimes S^c} \to V^{\otimes n}$$

be defined as above, but permuting the factors of $V^{\otimes n}$ according to S.

(a) (Small WDVV) For all β, the following is totally symmetric:

$$\sum_\alpha P\left([\mathrm{Rat}_3(X,\alpha)]\otimes[\mathrm{Rat}_3(X,\beta-\alpha)]\right)\in V^{\otimes 4}$$

(b) (Big WDVV) For all β and all $n\ge 4$, fix $i,j,k,l\in\{1,...,n\}$. Then the following is symmetric under the S_4 action on $\{i,j,k,l\}$:

$$\sum_S\sum_\alpha P\left([\mathrm{Rat}_{S+1}(X,\alpha)]\otimes[\mathrm{Rat}_{1+S^c}(X,\beta-\alpha)]\right)\in V^{\otimes n}$$

where S ranges over subsets of $\{1,...,n\}$ with $i,j\in S$ but $k,l\notin S$.

Proof: (a) is a special case of (b). Note that symmetry under the transpositions $(i\ j)$ and $(k\ l)$ is immediate. Symmetry under the transposition $(j\ k)$ we prove using the "correspondence"

$$\begin{array}{ccc} \overline{M}_{0,n}(X,\beta) & \stackrel{ev}{\to} & X^n \\ \downarrow & & \\ \overline{M}_{0,n} & & \\ \downarrow & & \\ \overline{M}_{0,4} & & \end{array}$$

where the map $\overline{M}_{0,n} \to \overline{M}_{0,4} = \mathbf{P}^1$ remembers only p_i, p_j, p_k, p_l.

¿From property (iii) of virtual classes, it follows that:

$$ev_*([D(S,\alpha)]^{\text{vir}}) = P\left([\text{Rat}_{S+1}(X,\alpha)] \otimes [\text{Rat}_{1+S^c}(X,\beta-\alpha)]\right)$$

On the other hand, consider the points $D(\{i,j\}), D(\{i,k\}) \in \overline{M}_{0,4}$. Since they are points of $\mathbf{P}^1$, their fibers determine rationally equivalent divisors in $\overline{M}_{0,n}(X,\beta)$. More precisely, by properties (ii),(iii) of the virtual class, it follows that the "virtual" divisors determined by the points satisfy:

$$\sum_{i,j\in S;k,l\notin S} \sum_{\alpha} [D(S,\alpha)]^{\text{vir}} \sim \sum_{i,k\in S;j,l\notin S} \sum_{\alpha} [D(S,\alpha)]^{\text{vir}} \in CH_*(\overline{M}_{0,n}(X,\beta))$$

and the symmetry under $(j\ k)$ is obtained by pushing foward.

On could imagine getting other relations from the Chow groups of the other $\overline{M}_{0,n}$, but a theorem of Sean Keel says that all such relations are a consequence of $D(\{i,j\}) \sim D(\{i,k\}) \in CH_*(\mathbf{P}^1)$.

Examples: The only nonzero class $[\text{Rat}_n(X,0)]$ is:

$$[\text{Rat}_3(X,0)] = \delta$$

the class of the diagonal, and one may check that

$$P([\text{Rat}_3(X,0)] \otimes [\text{Rat}_3(X,0)]) = \delta$$

the class of the diagonal in X^4 which is, of course, symmetric.

For $n = 3$ on $\mathbf{P}^m$ and $d = 1$, (a) gives:

$$\sum H^{a_1} \otimes H^{a_2} \otimes H^{b_1} \otimes H^{b_2} + H^{b_1} \otimes H^{b_2} \otimes H^{a_1} \otimes H^{a_2}$$

summed over integers a_1, a_2, b_1, b_2 such that $a_1+a_2+b_1+b_2 = 2m-1$ and $b_1 + b_2 \leq m - 1$. This doesn't look symmetric under the permutation (2 3) until one realizes that $b_1+b_2 > m-1$ if and only if $a_1+a_2 \leq m-1$, so the sum above is just:

$$\sum_{a_1+a_2+a_3+a_4=2m-1} H^{a_1} \otimes H^{a_2} \otimes H^{a_3} \otimes H^{a_4}$$

which is evidently symmetric.

Some other relations among the classes $[\text{Rat}_n(X,\beta)]$ are also easily obtained from the foundations. They are best expressed in terms of the n-point **Gromov-Witten invariants**:

$$\langle \gamma_1, ..., \gamma_n \rangle_\beta^X := \int_{X^n} [\text{Rat}_n(X,\beta)] \wedge \gamma_1 \otimes ... \otimes \gamma_n$$

and have obvious analogues in higher genus, too. Note that by Poincaré duality, the complete set of n-point Gromov-Witten invariants determines the class $[\mathrm{Rat}_n(X,\beta)]$.

Basic Relations: (a) $\langle \gamma_1, ..., \gamma_n \rangle_0^X = 0$ unless $n = 3$, in which case:

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle_0^X = \int_X \gamma_1 \wedge \gamma_2 \wedge \gamma_3$$

(b) $\langle \gamma_1, ..., \gamma_n, 1 \rangle_\beta^X = 0$ unless $\beta = 0$ and $n = 2$

(c) If D is the class of a divisor, then:

$$\langle \gamma_1, ..., \gamma_n, D \rangle_\beta^X = \deg(D \cap \beta) \langle \gamma_1, ..., \gamma_n \rangle_\beta^X$$

unless $\beta = 0$ and $n = 2$.

Proof: The degree zero moduli spaces with evaluation maps are:

$$ev : \overline{M}_{0,n}(X,0) = \overline{M}_{0,n} \times X \to X \xrightarrow{\Delta} X^n$$

Relation (a) follows immediately from this.

Relations (b) and (c) follow from the diagram:

$$\begin{array}{ccc} \overline{M}_{0,n+1}(X,\beta) & \xrightarrow{ev} & X^{n+1} \\ f_{n+1} \downarrow & & \pi \downarrow \\ \overline{M}_{0,n}(X,\beta) & \xrightarrow{ev} & X^n \end{array}$$

and the projection formula. For (b), this gives:

$$\langle \gamma_1, ..., \gamma_n, 1 \rangle_\beta^X = \int_{X^n} ev_* f_{n+1*} [\overline{M}_{0,n+1}(X,\beta)]^{\mathrm{vir}} \wedge \gamma_1 \otimes ... \otimes \gamma_n$$

which vanishes since $[\overline{M}_{0,n+1}(X,\beta)]^{\mathrm{vir}} = f_{n+1}^* [\overline{M}_{0,n}(X,\beta)]^{\mathrm{vir}}$. For (c),

$$\langle \gamma_1, ..., \gamma_n, D \rangle_\beta^X = \int_{X^n} ev_* f_{n+1*} (e^* D \cap [\overline{M}_{0,n+1}(X,\beta)]^{\mathrm{vir}}) \wedge \gamma_1 \otimes ... \otimes \gamma_n$$

and $f_{n+1*}(e^* D \cap [\overline{M}_{0,n+1}(X,\beta)]^{\mathrm{vir}}) = \deg(D \cap \beta) [\overline{M}_{0,n}(X,\beta)]^{\mathrm{vir}}$.

It is an immediate consequence of basic relation (c) that the classes $[\mathrm{Rat}_n(X,\beta)]$ are determined by the classes $[\mathrm{Rat}_{n+1}(X,\beta)]$. When we throw in the WDVV relations, there is a sort of converse:

Kontsevich Reconstruction Theorem:

If $\mathrm{H}^*(X, \mathbf{Q})$ is generated by divisor classes, then the WDVV relations and the basic relations determine all the $[\mathrm{Rat}_n(X,\beta)]$ in terms of the $[\mathrm{Rat}_3(X,\beta)]$.

Proof: Choose a basis $D_1, ..., D_k$ for $\mathrm{H}^2(X, \mathbf{Q})$ consisting of divisor classes, and a basis $D_J = \prod D_i^{j_i}$ for $\mathrm{H}^*(X, \mathbf{Q})$.

For $n \geq 3$ and $|K| \geq 2$, we will show how to inductively solve for Gromov-Witten invariants of the form:

$$\langle \gamma_1, ..., \gamma_n, D_K \rangle_\beta$$

in terms of invariants of the same form involving either:

(i) smaller β and at most $n+1$ cohomology classes or

(ii) the same β and number of classes but smaller $|K|$.

This will prove the theorem by induction and basic relation (c).

Let (g^{IJ}) be the inverse of the intersection matrix defined by:

$$g_{IJ} := \int_X D_I \wedge D_J$$

For each $S \subset \{1, ..., n+2\}$, let $s_1, ..., s_m$ denote the elements of S and $s_1^c, ..., s_{n+2-m}^c$ the elements of the complement. Then:

$$\int_{X^n} P\left([\mathrm{Rat}_{S+1}(X, \alpha)] \otimes [\mathrm{Rat}_{1+S^c}(X, \beta - \alpha)]\right) \wedge \gamma_1 \otimes ... \otimes \gamma_{n+2}$$

$$= \sum_{I,J} \langle \gamma_{s_1}, ..., \gamma_{s_m}, D_I \rangle_\alpha g^{IJ} \langle D_J, \gamma_{s_1^c}, ..., \gamma_{s_{n+2-m}^c} \rangle_{\beta-\alpha}$$

and in particular, in case $S = \{1, ..., n\}$ and $\alpha = \beta$, this is:

$$\langle \gamma_1, ..., \gamma_n, \gamma_{n+1} \wedge \gamma_{n+2} \rangle_\beta$$

Now let $\{i, j, k, l\} = \{1, 2, n+1, n+2\}$ in the big WDVV relation (for $n+2$ total classes) and write $D_K = D_L \wedge D_M$ with $|L|, |M| < |K|$. When we integrate against the class $\gamma_1 \otimes ... \otimes \gamma_n \otimes D_L \otimes D_M$, we get on the one hand:

$$\sum_{1,2 \in S, n+1, n+2 \in S^c} \sum_{\alpha, I, J} \langle \gamma_1, \gamma_2, ..., \gamma_{s_m}, D_I \rangle_\alpha g^{IJ} \langle D_J, \gamma_{s_1^c}, ..., D_L, D_M \rangle_{\beta-\alpha}$$

and on the other hand, applying the transposition $(2\ n+1)$ we get:

$$\sum_{1,2 \in S, n+1, n+2 \in S^c} \sum_{\alpha, I, J} \langle \gamma_1, D_L, ..., \gamma_{s_m}, D_I \rangle_\alpha g^{IJ} \langle D_J, \gamma_{s_1^c}, ..., \gamma_2, D_M \rangle_{\beta-\alpha}$$

The only terms in either double sum which involve $n+1$ classes and the same β are the terms with $\alpha = \beta$ and $S = \{1, ..., n\}$ or else $\alpha = 0$ and $S = \{1, 2\}$. These yield the two pairs:

$$\langle \gamma_1, \gamma_2, ..., \gamma_n, D_K \rangle_\beta + \langle \gamma_1 \wedge \gamma_2, ..., D_L, D_M \rangle_\beta$$

in the first double sum, and, in the second:

$$\langle \gamma_1, D_L, ..., \gamma_n, \gamma_2 \wedge D_K \rangle_\beta + \langle \gamma_1 \wedge D_L, \gamma_3, ..., \gamma_n, \gamma_2, D_M \rangle_\beta$$

But the first of these terms is the one we wish to solve for and the others involve smaller D_I classes, so the equality of the double sums gives the desired induction.

Example: As in the introduction, let n_d be the number of rational plane curves of degree d through $3d-1$ general points of $\mathbf{P}^2$. In terms of Gromov-Witten invariants,

$$n_d = \langle H^2, ..., H^2 \rangle_d$$

If we apply the proof of Kontsevich reconstruction with $n = 3d-2$ and $D_L = H = D_M$, and simplify the double summands by means of the basic relations, we get the relation:

$$n_d + \sum_{e=1}^{d-1} e n_e (d-e)^3 n_{d-e} \binom{3d-4}{3e-3} = \sum_{e=1}^{d-1} e^2 n_e (d-e)^2 n_{d-e} \binom{3d-4}{3e-2}$$

This is precisely the recursive formula of the introduction!

Remark: In case X is Fano in the theorem, one can further determine the $[\mathrm{Rat}_3(X,\beta)]$ in terms of the $[\mathrm{Rat}_2(X,\beta)]$. This follows, as noted by Siebert and Tian, from the associativity of small quantum cohomology, which in turn follows from the small WDVV equations. Evidently one cannot do better than this since when $X = \mathbf{P}^n$, for instance, the one-point invariants all vanish!