Genus Zero Gromov-Witten Invariants.

(Lectures at the University of Nice in the Fall, 2000)

5. The Unreasonable Effectiveness of Localization.

The **equivariant cohomology** of a complex manifold X with a holomorphic action of the complex torus $T = (\mathbf{C}^*)^n$ is a graded ring $H_T^*(X, \mathbf{Q})$ together with a graded homomorphism from the cohomology ring of the classifying space:

$$\pi^* : \mathbf{Q}[t_1, ..., t_n] = H^*(BT, \mathbf{Q}) \to H_T^*(X, \mathbf{Q})$$

Two extreme examples are of particular interest:

• If the T-action is **trivial**, then $H_T^*(X, \mathbf{Q}) = H^*(X, \mathbf{Q})[t_1, ..., t_n]$ with the natural structure of a **free** module over $\mathbf{Q}[t_1, ..., t_n]$.

• If the T-action is **free**, then $H_T^*(X, \mathbf{Q}) = H^*(X/T, \mathbf{Q})$ and the maximal ideal $m = \langle t_i \rangle$ is the kernel of π^*.

Equivariant cohomology shares many of the functorial properties of ordinary cohomology. For example, if

$$f : X \to Y$$

is a T-equivariant holomorphic map, then:

• the (graded) pullback f^* is always defined, and

• the pushforward f_* is defined for proper maps f.

There are equivariant **Chern classes**

$$c_d^T(E) \in H_T^{2d}(X, \mathbf{Q})$$

for any holomorphic vector bundle E on X together with a lifting of the T action that is linear on the fibers of E. The tangent bundle TX has a canonical lifting, hence canonical equivariant Chern classes.

The connected components of the locus of fixed points are denoted:

$$i : F \hookrightarrow X$$

They are embedded (holomorphic) submanifolds and i is T-equivariant for the trivial T-action on F. The normal bundle $N_{F/X}$ has a canonical lifting of the (trivial) T action on F, and thus has canonical equivariant Chern classes. In particular, the **equivariant Euler class** of F is the Chern class of degree equal to the codimension of F, and is denoted:

$$\epsilon_T(F) \in H^*(F, \mathbf{Q})[t_1, ..., t_n]$$

The constant (in t) term of $\epsilon_T(F)$ is the ordinary Euler class, which may well be zero. But the following result may be regarded as the most important feature of equivariant cohomology:

Theorem: $\epsilon_T(F)$ is always **invertible** in $H_T^*(F,\mathbf{Q})(t_1,...,t_n)$.

Given a class $c \in H_T^*(X,\mathbf{Q})$, we define the **residue** of c on F to be:

$$\frac{i^*c}{\epsilon_T(F)} \in H_T^*(F,\mathbf{Q})(t_1,...,t_n)$$

and an important corollary of the theorem is the following:

Functoriality of Residues: If $f : X \to Y$ is a T-equivariant proper map, fix a component $j : G \hookrightarrow Y$ of the locus of fixed points of Y. Then for each equivariant cohomology class $c \in H_T^*(X,\mathbf{Q})$,

$$\sum_{F\subset f^{-1}(G)} f|_{F*}\left(\frac{i^*c}{\epsilon_T(F)}\right) = \frac{j^*f_*c}{\epsilon_T(G)} \in H_T^*(G,\mathbf{Q})(t_1,...,t_n)$$

i.e. the pushforward of the residues is the residue of the pushforward

Basic Observation: There are canonical embeddings:

$$i : \overline{M}_{0,n}(X,\beta) \hookrightarrow \overline{G}_{0,n}(X,\beta) := \overline{M}_{0,0}(X \times (\mathbf{P}^1)^n, (\beta, 1^n))$$

that are T-equivariant for the trivial action on $\overline{M}_{0,n}(X,\beta)$ and the standard action on $(\mathbf{P}^1)^n$ with:

$$\epsilon_T(\overline{M}_{0,n}(X,\beta)) = \prod_{i=1}^n t_i(t_i - \psi_i)$$

The embedding is described set-theoretically as follows. Given a stable map $f : C \to X$ with marked points $p_1,...,p_n \in C$, consider the curve C' with no marked points but n additional smooth components $C_1,...,C_n$ meeting C at $p_1,...,p_n$. There is a well-defined class of stable maps $f' : C' \to X \times (\mathbf{P}^1)^n$ such that $f'|_C = (f,0,...,0)$ and $f'|_{C_i} = (f(p_i),0,...,\phi_i,...,0)$ for isomorphisms $\phi_i : C_i \to \mathbf{P}^1$.

Thus we have a diagram:

$$\begin{array}{ccc} \overline{G}_{0,n}(X,\beta) & & \\ \uparrow & & \\ \overline{M}_{0,n}(X,\beta) & \stackrel{ev}{\to} & X^n \end{array}$$

and my philosophy is the following. If we could fill in with:

$$\begin{array}{ccc} \overline{G}_{0,n}(X,\beta) & \stackrel{f}{\to} & Y \\ i\uparrow & & j\uparrow \\ \overline{M}_{0,n}(X,\beta) & \stackrel{ev}{\to} & X^n \end{array}$$

so that f is T-equivariant and X^n is fixed in Y, then we could apply the functoriality of residues to the class 1 to conclude that:

$$J_\beta(t_1, ..., t_n) + \sum_F ev_* \left(\frac{1}{\epsilon_T(F)} \right) = \frac{j^* f_* 1}{\epsilon_T(X^n)}$$

summed over the other fixed loci F in $f^{-1}(X^n)$.

In some cases this is indeed "unreasonably effective". For instance:

Projective Space: If $n = 1$ and $X = \mathbf{P}^m$, then we fill in with:

$$\begin{array}{ccc} \overline{G}_{0,1}(\mathbf{P}^m, d) & \stackrel{f}{\to} & \mathbf{P}^{(m+1)(d+1)-1} \\ i\uparrow & & j\uparrow \\ \overline{M}_{0,1}(\mathbf{P}^m, d) & \stackrel{ev}{\to} & \mathbf{P}^m \end{array}$$

so that f is birational and $f^{-1}(\mathbf{P}^m) = \overline{M}_{0,1}(\mathbf{P}^m, d)$. The projective space $\mathbf{P}^{(m+1)(d+1)-1}$ is the space of $m+1$ homogeneous polynomials of degree d in the variables x, y on $\mathbf{P}^1$. A general element of $\overline{G}_{0,1}(\mathbf{P}^m, d)$ can be interpreted as a morphism $\phi : \mathbf{P}^1 \to \mathbf{P}^m$ of degree d. Since such a morphism is described by $m+1$ homogeneous polynomials of degree d with no common factors, this gives a well-defined birational map:

$$\overline{G}_{0,1}(\mathbf{P}^m, d) - - > \mathbf{P}^{(m+1)(d+1)-1}$$

which extends to the map of the diagram.

The formula then follows from a computation:

$$\epsilon_T(\mathbf{P}^m) = \prod_{k=1}^{d} (H + kt)^{n+1}$$

The space $\overline{G}_{0,1}(X, d)$ in general was studied by gauge theorists well before algebraic geometers got a hold of it. It can be regarded as Gromov's "bubble-tree" compactification of the space $Map(X, d)$ of holomorphic maps $f : \mathbf{P}^1 \to X$ of degree d. There is another compactification, the Uhlenbeck compactification, which is easy to describe set-theoretically:

$$\overline{U}(X, d) = Map(X, d) \cup Map(X, d-1) \times \mathbf{P}^1 \cup ... \cup X \times (\mathbf{P}^1)^d$$

and the diagram for projective space generalizes to:

$$\begin{array}{ccc} \overline{G}_{0,1}(X, d) & \stackrel{f}{\to} & \overline{U}(X, d) \\ i\uparrow & & j\uparrow \\ \overline{M}_{0,1}(X, d) & \stackrel{ev}{\to} & X \end{array}$$

where j is the inclusion of $X = X \times 0^d$.

The fundamental problem with this approach is that in all cases other than $X = \mathbf{P}^m$, the Uhlenbeck compactification is not smooth!

This brings up an important point. The spaces $\overline{G}_{0,n}(X,\beta)$ are not smooth either, but a theorem of Graber and Pandharipande states that they are "virtually smooth" in the sense that the functoriality of residues holds with "c" replaced by "$c \cap [\overline{G}_{0,n}(X,\beta)]^{\mathrm{vir}}$" in an equivariant Chow group. The Uhlenbeck compactification has no such virtual smoothness property.

Nevertheless, a variation of this approach works for Fano toric varieties.

Product Formula: There is a diagram:

$$\begin{array}{ccccc} \overline{G}_{0,1}(X \times Y, (\alpha,\beta)) & \stackrel{f_G}{\to} & \overline{G}_{0,1}(X,\alpha) \times \overline{G}_{0,1}(Y,\beta) & & \\ i_{(X,Y)} \uparrow & & (i_X, i_Y) \uparrow & & \\ \overline{M}_{0,1}(X \times Y, (\alpha,\beta)) & \stackrel{f_M}{\to} & \overline{M}_{0,1}(X,\alpha) \times \overline{M}_{0,1}(Y,\beta) & \stackrel{ev}{\to} & X \times Y \end{array}$$

where f_G and f_M are pairs of forgetful maps. One expects for dimension reasons that f_G (but not f_M) is birational, and a theorem of Behrend states that in fact, the virtual class pushes forward to the product of virtual classes, from which it follows that:

$$J_{(\alpha,\beta)}(t) = ev_* \left(\frac{[\overline{G}_{0,1}(X \times Y, (\alpha,\beta))]^{\mathrm{vir}}}{t(t-\psi)} \right)$$

$$= ev_* \left(\frac{[\overline{G}_{0,1}(X,\alpha)]^{\mathrm{vir}} \times [\overline{G}_{0,1}(Y,\beta)]^{\mathrm{vir}}}{\epsilon_T(\overline{M}_{0,1}(X,\alpha)) \otimes \epsilon_T(\overline{M}_{0,1}(Y,\beta))} \right) = J_\alpha(t) \otimes J_\beta(t)$$

Hypersurfaces: Let $i : X \hookrightarrow \mathbf{P}^m$ be a hypersurface of degree l. Consider the diagram:

$$\begin{array}{ccc} \overline{G}_{0,1}(X,d) & \stackrel{f_G}{\to} & \overline{G}_{0,1}(\mathbf{P}^m,d) \\ \uparrow & & \uparrow \\ \overline{M}_{0,1}(X,d) & \stackrel{f_M}{\to} & \overline{M}_{0,1}(\mathbf{P}^m,d) \end{array}$$

One of the basic properties of the virtual class is:

$$f_{M*}([\overline{M}_{0,1}(X,d)]^{\mathrm{vir}}) = c_{dl+1}E$$

where E is the rank $dl+1$ bundle $\pi_* e^* \mathcal{O}_{\mathbf{P}^m}(l)$, and the same is true of the pushforward under f_G of the virtual class, which we will denote c.

It follows that:

$$i_* J_d^X(t) = ev_* \left(\frac{c}{t(t-\psi)} \right) \in H^*(\mathbf{P}^m, \mathbf{Q})[t^{-1}]$$

and one can extract the J-function itself from this pushforward.

So we need a method for computing such things. By the functoriality of residues and the diagram for projective space, we have:

$$ev_*\left(\frac{c}{t(t-\psi)}\right) = \frac{j^*f_*c}{\prod_{k=1}^{d}(H+kt)^{m+1}}$$

and we can thus reduce the problem to one of computing j^*f_*c.

There are several ways to do this. Using the additional action of the "big" torus $(\mathbf{C}^*)^m$ on $\mathbf{P}^m$ and considering residues for the total action of $\mathbf{C}^* \times (\mathbf{C}^*)^m$, Givental found his original proof.

The proof of Lian-Liu-Yau is similar, and in their series of papers Mirror Principle I-IV, they develop a machine for computing push-forwards of residues of any class c satisfying "nice" properties. This also applies, for example, to classes of the form $c_{dl-1}F$ where F is the bundle $R^1\pi_*e^*\mathcal{O}_{\mathbf{P}^m}(-l)$. These latter computations are useful for "local" mirror symmetry.

For my approach, recall that $f : \overline{G}_{0,1}(\mathbf{P}^m, d) \to \mathbf{P}^{(m+1)(d+1)-1}$ is birational. If c were the pullback of an equivariant cohomology class c' on $\mathbf{P}^{(m+1)(d+1)-1}$, then we'd have $j^*f_*c = j^*c'$ and the computation would be easy. This doesn't happen, but c does decompose as a sum of pullbacks for a sequence of maps from boundary strata starting with the map f. The point is that there is a map:

$$\Phi : E = \pi_*e^*\mathcal{O}_{\mathbf{P}^m}(l) \to f^*\mathcal{O}(l)^{\oplus dl+1}$$

of vector bundles on $\overline{G}_{0,1}(\mathbf{P}^m, d)$. Given $\phi : \mathbf{P}^1 \to \mathbf{P}^m$ of degree d, then $\phi^*\mathcal{O}_{\mathbf{P}^m}(l) \cong \mathcal{O}_{\mathbf{P}^1}(dl)$ and the fiber of E over ϕ is thus $\mathrm{H}^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(dl))$. This globalizes to the map Φ, which is an isomorphism over the points corresponding to maps ϕ. The precise relationship between the top equivariant Chern classes:

$$c^T_{dl+1}(E) \text{ and } \prod_{k=0}^{dl}(lH+kt)$$

is determined by means of MacPherson's graph construction and the "self-similar" nature of the boundary leads to the proofs.

WDVV-Like Relations: Assume that X is embedded in $\mathbf{P}^m$, for simplicity and for $n \geq 1$, consider the diagram:

$$\begin{array}{ccccccc}
\mathbf{P}^{(m+1)(d+1)-1} & \overset{f_{n+1}}{\leftarrow} & \overline{G}_{0,n+1}(X,d) & \overset{f_G}{\to} & \overline{G}_{0,n}(X,d) \times \mathbf{P}^3 & & \\
 & & \uparrow & & \uparrow & & \\
 & & F & \overset{f_M}{\to} & \overline{M}_{0,n}(X,d) & \overset{ev}{\to} & X^n
\end{array}$$

where f is made up of two forgetful maps to:

$$\overline{G}_{0,n}(X,d) \quad \text{and} \quad \overline{M}_{0,0}(\mathbf{P}^1 \times \mathbf{P}^1, (1,1)) \cong \mathbf{P}^3$$

The fixed loci can be described explicitly, yielding:

$$ev_* f_{M*} \sum_F \left(\frac{1}{\epsilon_T(F)} \right) = \sum_{1 \in S} \sum_e P \left(\widehat{J}_{d-e}(t_S, t) \otimes J_e(-t, t_{S^c}) \right)$$

and more generally, for positive integers b, we have:

$$ev_* f_{M*} \sum_F \left(\frac{i^* f_{n+1}^* H_T^b}{\epsilon_T(F)} \right) = \sum_{1 \in S} \sum_e P_{b,e} \left(\widehat{J}_{d-e}(t_S, t) \otimes J_e(-t, t_{S^c}) \right)$$

with the conventions on Poincaré duality from the previous section.

On the other hand, by functoriality of residues, we have:

$$ev_* f_{M*} \sum_F \left(\frac{i^* f_{n+1}^* H_T^b}{\epsilon_T(F)} \right) = \frac{j^* f_{G*} f_{n+1}^* H_T^b}{\prod t_i(t_i - \psi_i)} \cdot \frac{1}{t_1 t(t_1 + t)}$$

(the extra factor is the equivariant Euler class of a fixed point in $\mathbf{P}^3$)

Here's the punchline.

The numerator: $j^* f_{G*} f_{n+1}^* H_T^b$ is an equivariant cohomology class, so without calculating it, we know that it is **polynomial** in t. Hence the relation of the previous section. In a couple of cases, it can be computed. When $b = 0$, it is 1 since f_G is birational. When $b = 1$, it is $e_1^* H + dt_1$. This requires more work. (It may follow from the divisor relation) And for $b > 1$, it seems to be hard to calculate. But as we've noted, the calculation is not necessary for reconstruction!