Genus Zero Gromov-Witten Invariants.

(Lectures at the University of Nice in the Fall, 2000)

4. Generalized Invariants and J Functions.

The ring $H^*(\overline{M}_{0,n}, \mathbf{Z})$ is generated by 1 and the divisor classes $D(S)$. It has distinguished "conormal" divisor classes

$$\psi_i := -\rho_i^* D(\{i, n+1\})$$

i.e. $-\psi_i$ is the Euler class of the embedding $\rho_i : \overline{M}_{0,n} \hookrightarrow \overline{M}_{0,n+1}$. In terms of the divisor classes $D(S)$:

$$\psi_i \sim \sum_{i \in S; j,k \notin S} D(S)$$

for each $j \neq k \in \{1, ..., n\}$ different from i. This follows from:

Proposition: (a) For the forgetful map $f_{n+1} : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$,

$$f_{n+1}^* \psi_i \sim \psi_i - D(\{i, n+1\})$$

(b) For the embeddings $\rho_i : \overline{M}_{0,n} \to \overline{M}_{0,n+1}$:

$$\rho_i^* \psi_i \sim 0 \quad \text{and} \quad \rho_i^* \psi_{n+1} \sim 0 \quad \text{otherwise} \quad \rho_i^* \psi_j \sim \psi_j$$

(c) For the gluing maps $g_S : \overline{M}_{0,S+1} \times \overline{M}_{0,1+S^c} \to \overline{M}_{0,n}$ with $i \in S$,

$$g_S^* \psi_i \sim \pi_1^* \psi_i \quad \text{and} \quad g_S^* D(S) \sim -\pi_1^* \psi_{S+1} - \pi_2^* \psi_1$$

¿From (a), one also gets the **string relation**:

$$\int_{\overline{M}_{0,n+1}} \psi_1^{a_1} \wedge ... \wedge \psi_n^{a_n} = \sum_{i=1}^{n} \int_{\overline{M}_{0,n}} \psi_1^{a_1} \wedge ... \wedge \psi_i^{a_i - 1} \wedge ... \wedge \psi_n^{a_n}$$

and the **dilaton relation**:

$$\int_{\overline{M}_{0,n+1}} \psi_1^{a_1} \wedge ... \wedge \psi_n^{a_n} \wedge \psi_{n+1} = (n-2) \int_{\overline{M}_{0,n}} \psi_1^{a_1} \wedge ... \wedge \psi_n^{a_n}$$

and the string relation already determines the intersection numbers, which are:

$$\int_{\overline{M}_{0,n}} \psi_1^{a_1} \wedge ... \wedge \psi_n^{a_n} = \frac{(n-3)!}{a_1! a_2! ... a_n!}$$

whenever $a_1 + a_2 + ... + a_n = n - 3$.

One could imagine using the ψ_i and the "correspondence":

$$\begin{array}{ccc} \overline{M}_{0,n}(X,\beta) & \stackrel{ev}{\to} & X^n \\ \pi \downarrow & & \\ \overline{M}_{0,n} & & \end{array}$$

to define classes: $ev_*\pi^*\psi_1^{a_1} \wedge ... \wedge \psi_n^{a_n}$ but this is not what we will do.

Instead, we define classes ψ_i directly on $\overline{M}_{0,n}(X,\beta)$ in the same way:

$$\psi_i := -\rho_i^* D(\{i, n+1\}, 0)$$

so $-\psi_i$ is again the Euler class of the embedding ρ_i, and:

Proposition: (a) For the forgetful map f_{n+1},

$$f_{n+1}^*\psi_i \sim \psi_i - D(\{i, n+1\}, 0)$$

(b) For the embeddings ρ_i:

$$\rho_i^*\psi_i \sim 0 \quad \text{and} \quad \rho_i^*\psi_{n+1} \sim 0 \quad \text{otherwise} \quad \rho_i^*\psi_j \sim \psi_j$$

(c) For $g_{S,\alpha} : \overline{M}_{0,S+1}(X,\alpha) \times_X \overline{M}_{0,1+S^c}(X,\beta-\alpha) \to \overline{M}_{0,n}(X,\beta)$

$$g_{S,\alpha}^*\psi_i \sim \pi_1^*\psi_i \quad \text{and} \quad g_{S,\alpha}^* D(S,\alpha) \sim -\pi_1^*\psi_{S+1} - \pi_2^*\psi_1$$

(d) When $n \geq 3$,

$$\pi^*\psi_i \sim \psi_i - \sum_\alpha D(\{i\},\alpha) \quad \text{and thus} \quad \psi_i \sim \sum_{i\in S; j,k \notin S} \sum_\alpha D(S,\alpha)$$

for each $j \neq k \in \{1, ..., n\}$ different from i.

Remark: When the $D(S,\alpha)$ are not, actually, divisors, they need to be replaced with their virtual counterparts. One way to do this would be to use the embedding $X \subset \mathbf{P}^m$ and pull back $D(S,\alpha)$ from the smooth stack $\overline{M}_{0,n}(\mathbf{P}^m, d)$ via the embedding:

$$\overline{M}_{0,n}(X,\beta) \hookrightarrow \overline{M}_{0,n}(\mathbf{P}^m, d)$$

This is only necessary in cases (c) and (d).

We will be interested in the following generalized classes:

$$[\mathrm{Rat}^{a_1,...,a_n}(X,\beta)] := ev_*\left(\psi_1^{a_1} \wedge ... \wedge \psi_n^{a_n} \cap [\overline{M}_{0,n}(X,\beta)]^{\mathrm{vir}}\right)$$

with which one defines **generalized Gromov-Witten invariants**:

$$\langle \psi^{a_1}(\gamma_1), ..., \psi^{a_n}(\gamma_n)\rangle_\beta := \int_{X^n} [\mathrm{Rat}^{a_1,...,a_n}(X,\beta)] \wedge \gamma_1 \otimes ... \otimes \gamma_n$$

The basic relations of the previous section may be modified to include the ψ classes as follows. Their proofs follow from the projection formula and the proposition above.

Basic Relations II: (a) The string equation:

$$\langle \psi^{a_1}(\gamma_1), ..., \psi^{a_n}(\gamma_n), 1\rangle_\beta = \sum_{i=1}^{n} \langle \psi^{a_1}(\gamma_1), ..., \psi^{a_i-1}(\gamma_i), ..., \psi^{a_n}(\gamma_n)\rangle_\beta$$

(b) The divisor equation: For all divisor classes D on X:

$$\langle \psi^{a_1}(\gamma_1), ..., \psi^{a_n}(\gamma_n), D\rangle_\beta = \deg(D \cap \beta)\langle \psi^{a_1}(\gamma_1), ..., \psi^{a_n}(\gamma_n)\rangle_\beta$$

$$+ \sum_{i=1}^{n} \langle \psi^{a_1}(\gamma_1), ..., \psi^{a_i-1}(\gamma_i \wedge D), ..., \psi^{a_n}(\gamma_n)\rangle_\beta$$

(c) The dilaton equation:

$$\langle \psi^{a_1}(\gamma_1), ..., \psi^{a_n}(\gamma_n), \psi\rangle_\beta = (n-2)\langle \psi^{a_1}(\gamma_1), ..., \psi^{a_n}(\gamma_n)\rangle_\beta$$

Using these, one can also prove:

Dubrovin's Reconstruction Theorem: The proposition and basic relations determine $[\mathrm{Rat}^{a_1,...,a_n}(X,\beta)]$ in terms of the $[\mathrm{Rat}_n(X,\beta)]$.

So why consider generalized classes at all?

There are fancy motivations coming from physics, but my point of view is more mundane. The most powerful tool yet discovered for computing the basic classes $[\mathrm{Rat}_n(X,\beta)]$ also computes the generalized classes $[\mathrm{Rat}^{a_1,...,a_n}(X,\beta)]$, in such a way that it is unreasonable to distinguish the basic classes. The tool is the localization theorem, and it is used to compute J-functions, which I will now describe.

J-Functions: Let t be a parameter, and define:

$$J_\beta(t) := ev_*\left(\frac{[\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}}}{t(t-\psi_1)}\right) \in \mathrm{H}^*(X,\mathbf{Q})[t^{-1}]$$

That is,

$$J_\beta^X(t) = \sum_{a=0} t^{-a-2}[\mathrm{Rat}^a(X,\beta)]$$

More generally, define:

$$J_\beta^X(t_1,...,t_n) := ev_*\left(\frac{[\overline{M}_{0,n}(X,\beta)]^{\mathrm{vir}}}{\prod t_i(t_i-\psi_i)}\right) \in \mathrm{H}^*(X,\mathbf{Q})^{\otimes n}[t_1^{-1},...,t_n^{-1}]$$

so that $[\mathrm{Rat}^{a_1,...,a_n}(X,\beta)]$ is the coefficient of $\prod t_i^{-a_i-2}$.

Initial Conditions: In case $n = 1$ or 2 and $\beta = 0$, the definition above does not suffice, because the moduli space of stable maps does not exist. In those cases, we define:

$$J_0(t_1) = 1 \quad \text{and} \quad J_0(t_1, t_2) = \frac{\delta}{t_1 t_2 (t_1 + t_2)}$$

These initial conditions fit in with the pattern:

$$J_0(t_1, ..., t_n) = \delta \left(\prod_{i=1}^{n} t_i^{-2} \right) (t_1^{-1} + ... + t_n^{-1})^{n-3}$$

which follows for $n \geq 3$ from the formula for intersection numbers on $\overline{M}_{0,n}$. The initial condition for $J_0(t_1, t_2)$ in particular, is a bit strange. It is the only J-function that is not a Laurent polynomial in $t_1, ..., t_n$ with coefficients in $\mathrm{H}^*(X^n, \mathbf{Q})$.

Computing the one-variable J-functions has become known in the literature (unfortunately) as "proving the mirror conjecture" because of the important role of these J-functions in the phenomoneon of mirror symmetry in case X is a Calabi-Yau threefold. The following results are due to Givental, with later proofs by Lian-Liu-Yau, Kim and myself.

The Mirror Theorem for Fano Complete Intersections in $\mathbf{P}^n$

(a)

$$J_d^{\mathbf{P}^n}(t) = \frac{1}{\prod_{k=1}^{d} (H + kt)^{n+1}}$$

If $X \subset \mathbf{P}^n$ is a complete intersection of type $(l_1, ..., l_r)$, define:

$$I_d^X(t) = \frac{\prod_{i=1}^{r} \prod_{k=1}^{dl_i} (l_i H + kt)}{\prod_{k=1}^{d} (H + kt)^{n+1}}$$

(b) Whenever $l_1 + ... + l_r < n$, then $J_d^X(t) = I_d^X(t)$

(c) Whenever $l_1 + ... + l_r = n$, then:

$$J_d^X(t) = \sum_{e=0}^{d} \frac{(-\prod_{i=1}^{r} l_i!)^e I_{d-e}^X(t)}{e! t^e}$$

Define formal power series:

$$I^X(q) := \sum_{d=0}^{\infty} I_d^X(t) q^d \quad \text{and} \quad J^X(q) := \sum_{d=0}^{\infty} J_d^X(t) q^d$$

Mirror Theorem for Calabi-Yau Complete Intersections in $\mathbf{P}^n$ Whenever $l_1 + ... + l_r = n + 1$, there are $f(q), g(q) \in \mathbf{Q}[[q]]$ such that:

$$J^X(q) = e^{\frac{H}{t} f(q) + g(q)} I^X(qe^{f(q)})$$

and this determines $f(q)$ and $g(q)$ in terms of $I^X(q)$.

There are similar explicit results for complete intersections in toric varieties, due to Givental and Lian-Liu-Yau. The J-functions of flag varieties have also been studied by Givental and Kim.

There is a product formula for the classes $[\mathrm{Rat}_3(X, \beta)]$ which is quite intricate, in contrast with the following very simple:

Product Formula: Given classes $\alpha \in \mathrm{H}^2(X, \mathbf{Z})$ and $\beta \in \mathrm{H}^2(Y, \mathbf{Z})$, then:

$$J^{X \times Y}_{(\alpha,\beta)}(t) = J^X_\alpha(t) \otimes_{\mathbf{Q}[t^{-1}]} J^Y_\beta(t)$$

i.e. the J-function of a product is the tensor product of the J-functions.

It is natural to ask for a form of WDVV for J-functions. I offer the following theorem (proved jointly with Holger Kley) which, if not an exact analogue, does have similar "reconstructive" properties.

First, generalize the notion of ampleness by letting H denote a set of linearly independent nef divisors $H_1, ..., H_m$ such that all positive linear combinations are ample. In that case define

$$\deg(\beta) = (\deg(H_1 \cap \beta), ..., \deg(H_n \cap \beta))$$

and for $d = (d_1, ..., d_m)$, let

$$J_d(t_1, ..., t_n) := \sum_{\deg(\beta) = d = (d_1, ..., d_m)} J^X_\beta(t_1, ..., t_n)$$

For each $I = (i_1, ..., i_m)$ with all $i_k \geq 0$, define classes

$$H^I = \prod H_k^{i_k} \text{ and } (H - dt)^I = \prod (H_k - d_k t_k)^{i_k} \in \mathrm{H}^*(X, \mathbf{Q})[t]$$

and define a bilinear pairing:

$$P_{I,d} : V \otimes_{\mathbf{Q}[t^{-1}]} V \to \mathbf{Q}[t^{-1}, t]; \quad P_{I,d}(\gamma_1 \otimes \gamma_2) = \int_X \gamma_1 \wedge \gamma_2 \wedge (H - dt)^I$$

generalized to $P_{I,d} : V^{\otimes S+1} \otimes V^{\otimes 1+S^c} \to V^{\otimes n}$ as for WDVV.

Theorem (Relations for J-functions): Let $V = H^*(X, \mathbf{Q})[t^{-1}]$ and identify each $V^{\otimes n}$ with $H^*(X^n, \mathbf{Q})[t_1^{-1}, ..., t_n^{-1}]$.

Then for each n and I, d,

$$(*) \quad t(t + t_1) \sum_{1 \in S} \sum_e P_{I,e} \left(\widehat{J}_{d-e}(t_S, t) \otimes J_e(-t, t_{S^c}) \right) \in V^{\otimes n}[t]$$

where $t_S = (t_{s_1}, ..., t_{s_k})$ is indexed by the elements of S, and:

$$\widehat{J}_{d-e}(t_S, t) = \begin{cases} J_d(t_S, t) - J_d(t_1, .., t_n) \text{ if } e = 0 \text{ and } |S^c| = 1 \\ \\ J_{d-e}(t_S, t) \text{ otherwise} \end{cases}$$

Remark: The point is that $(*)$ a priori belongs to $V^{\otimes n}[t^{-1}, t]$ so the fact that it is polynomial in t gives us relations on the coefficients of the negative powers of t, that is, relations in $V^{\otimes n}$.

Special Terms: These are the terms in $(*)$ involving one of the J functions $J_0(t)$ or $J_0(t, t_j)$ that required a special definition. These terms are:

$$P\left(J_d(t_1, ..., t_n, t) \otimes J_0(-t)\right), \quad P_{I,d}\left(J_0(t_1, t) \otimes J_d(-t, t_2, ..., t_n)\right)$$

and, for each $j = 2, ..., n$,

$$P\left(\widehat{J}_d(t_1, ..., \hat{t}_j, ..., t_n, t) \otimes J_0(-t, t_j)\right)$$

When integrated against the class $\gamma_1 \otimes ... \otimes \gamma_n$, this becomes:

$$\left\langle \frac{\gamma_1}{t_1(t_1 - \psi)}, ..., \frac{\gamma_n}{t_n(t_n - \psi)}, \frac{H^I}{t(t - \psi)} \right\rangle_d,$$

$$\frac{1}{tt_1(t + t_1)} \left\langle \frac{\gamma_1 \wedge (H - dt)^I}{-t(-t - \psi)}, ..., \frac{\gamma_n}{t_n(t_n - \psi)} \right\rangle_d,$$

and, for each $j = 2, ..., n$:

$$\left\langle \frac{\gamma_1}{t_1(t_1 - \psi)}, ..., H^I \left(\frac{(t(t - \psi))^{-1} - (t_j(t_j - \psi))^{-1}}{tt_j(t_j - t)} \right), ..., \frac{\gamma_n}{t_n(t_n - \psi)} \right\rangle_d$$

Of all the terms, only the first special term is of degree d and involves $n + 1$ cohomology classes. Moreover, this term belongs to $t^{-2}\mathbf{Q}[t_1^{-1}, ..., t_n^{-1}, t^{-1}]$ so it is inductively determined by the relation $(*)$ and the other terms, and we obtain as a corollary:

Reconstruction of J-functions: If $\mathrm{H}^*(X, \mathbf{Q})$ is generated by classes of divisors, then all the J-functions are determined by the $J_\beta(t)$.

Example: If $X \subset \mathbf{P}^r$ is a hypersurface of degree l, let H be the hyperplane class, let $n = 1$ and $I = b$ and integrate the coefficient of t_1^{-1} against H^a. Then only the first two special terms occur, and relation $(*)$ becomes:

$$\left\langle H^a, \frac{H^b}{t - \psi} \right\rangle_d + \left\langle \frac{H^a(H - dt)^b}{-t(-t - \psi)} \right\rangle_d +$$

$$\frac{1}{l}\sum_{e=1}^{d-1}\sum_{c=0}^{r-1}\left\langle H^a, \frac{H^c}{t-\psi}\right\rangle_{d-e}\left\langle \frac{H^{r-1-c}(H-et)^b}{-t(-t-\psi)}\right\rangle_e \in \mathbf{Q}[t]$$

One can turn this formula into a very effective algorithm for computing the classes $[\mathrm{Rat}_2(X,d)]$ of a Fano hypersurface, which were previously unkown (for $d>1$).

Exercise! In the case $I=0$ we can prove more, namely:

$$(*) = t(t+t_1)P\left(J_0(t,t_1)\otimes J_d(t_1,t_2,...,t_n)\right)$$

Use this to deduce the string and dilaton relations.

In case $I=1_k$, then we can also prove that:

$$(*) = t(t+1)P_{I,-d}\left(J_0(t,t_1)\otimes J_d(t_1,t_2,...,t_n)\right)$$

Use this to deduce the divisor relation.

We do not know a simple expression for $(*)$ in general, and we do not know how to directly deduce WDVV from the theorem.