Genus Zero Gromov-Witten Invariants

(Lectures at the University of Nice in the Fall, 2000)

1. Classes Swept Out by Rational Curves (Ordinary Invariants). Let $X \subset \mathbf{CP}^n$ be a complex projective manifold, and let us assume that the hyperplane class H and the anticanonical class $c_1(TX) = -K_X$ satisfy:

$$-K_X \equiv \alpha H \in \mathrm{H}^2(X, \mathbf{Z})$$

for some $\alpha \in \mathbf{Q}$, which we will call the Fano index of embedding. Not every embedded projective manifold has a Fano index, but if it does, then the sign of α is intrinsic to X. It is a measure of the curvature of X.

In case X is a curve, this leads to the familiar types:

(a) $\mathbf{P}^1$, the only curve with $\alpha > 0$ (positive curvature)

(b) E, elliptic curves, with $\alpha = 0$ (zero curvature)

(c) C, curves of genus ≥ 2, with $\alpha < 0$ (negative curvature)

Some examples in higher dimension include:

(a) Projective space $\mathbf{P}^n$ itself, which has Fano index $n + 1$.

(b) Hypersurfaces of degree l in $\mathbf{P}^n$, which have Fano index $n + 1 - l$.

(c) Abelian varieties, which always have Fano index 0.

A **rational curve** on X will (for now) be the image of a holomorphic map:

$$f : \mathbf{P}^1 \to X$$

These are lines and conics in X and more generally any projection of a rational normal curve that lies in X. Note that $\alpha d = -\deg f^* K_X$ must be an integer for a rational curve of degree d to exist since K_X is an integral class.

Consider the Zariski closure of the locus of points on a rational curve:

$$\mathrm{Rat}(X, d) := \overline{\{p \in X \text{ on a rational curve of degree d}\}} \subseteq X$$

The (expected) dimension of $\mathrm{Rat}(X, d)$ depends only upon the Fano index of the embedding. We will see later that it is:

$$\text{expected } \dim(\mathrm{Rat}(X, d)) = \dim(X) + \alpha d - 2$$

e.g. if $\alpha = 1$, one expects finitely many conics to pass through a general point of X but the lines are only expected to sweep out a divisor in X.

Question: What is the expected cohomology class determined by $\mathrm{Rat}(X, d)$?

Some "classical" examples include:

(a) The smooth cubic surface $S \subset \mathbf{P}^3$. It is well-known that S has 27 distinct lines, and it follows from symmetry that:

$$[\mathrm{Rat}(S, 1)] = \sum [l_i] = 9H \in H^2(S, \mathbf{Z})$$

(each H counts for 3 lines, since it pushes forward to $3l \in H^4(\mathbf{P}^3, \mathbf{Q})$)

(b) A smooth quadric $Q \subset \mathbf{P}^3$. There are two families of lines on the quadric, each covering the quadric with degree one. It follows that the class in cohomology should be the class $2[Q]$. The definition of $[\mathrm{Rat}(X, d)]$ will evidently need to be improved in order to take account of this!

(c) The smooth cubic surface S again. A smooth conic on S is contained in a unique plane, which additionally intersects S in a line. It follows that S is covered by 27 families of conics, and the class in cohomology associated to $\mathrm{Rat}(S, 2)$ should be $27[S]$.

(d) Abelian varieties have no rational curves on them, of any degree! If $f : \mathbf{P}^1 \to X$ is a holomorphic map, then every holomorphic differential form on X must pull back to the zero form. Since the cotangent bundle to an Abelian variety is spanned by the global differential forms, it follows that df is everywhere zero, i.e. that f is the constant map. Note that in this case, the expected $\dim(\mathrm{Rat}(X, d)) = \dim(X) - 2$ is not achieved.

(e) If $X \subset \mathbf{P}^4$ is a smooth quintic hypersurface, then $\alpha = 0$ and the expected $\dim(\mathrm{Rat}(X, d)) = 1$ for all d. That is, $[\mathrm{Rat}(X, d)] = m_d[l]$ where m_d are rational(!) numbers and $[l]$ is the class of a line. Unlike the abelian variety case, there are indeed rational curves of all degrees. But it is an open problem (Clemens conjecture) whether or not the actual set of rational curves of degree d is finite on a general smooth quintic threefold (there are not finitely many lines, for instance, on the Fermat quintic). But even if the set is not finite, it still makes sense to ask for the expected value of m_d. The first big surprise in the subject came when a group of physicists produced a recursive algorithm for computing these numbers based upon the principle of "mirror symmetry." The computation has since been verified and generalized to all complete intersections in $\mathbf{P}^n$ of non-negative Fano index by Givental and others, and is one of the motivations for this set of lectures.

We can generalize the question in several ways. First, we define:

$$\mathrm{Rat}_n(X,d) := \overline{\{(p_1,...,p_n) \in X^n \text{ all on a single rational curve of degree } d\}}$$

First Generalization (Ordinary Gromov-Witten Invariants):
What is the class in $\mathrm{H}^*(X^n, \mathbf{Q}) = \mathrm{H}^*(X, \mathbf{Q})^{\otimes n}$ determined by $\mathrm{Rat}_n(X,d)$?

Again, the expected dimension is easy to compute:

$$\text{expected } \dim(\mathrm{Rat}_n(X,d)) = \dim(X) + \alpha d + n - 3$$

Some examples:

(a) "Two points determine a line in $\mathbf{P}^n$" can be interpreted as:

$$[\mathrm{Rat}_2(\mathbf{P}^n, 1)] = [\mathbf{P}^n \times \mathbf{P}^n]$$

(b) If we fix three linear subspaces of $\mathbf{P}^n$ of dimensions a_1, a_2, a_3 satisfying $a_1 + a_2 + a_3 = n - 1$, then there is a unique line meeting the three subspaces. It follows from this that:

$$[\mathrm{Rat}_3(\mathbf{P}^n, 1)] = \sum_{a_1+a_2+a_3=n-1} H^{a_1} \otimes H^{a_2} \otimes H^{a_3}$$

(c) The expected dimension of $\mathrm{Rat}_{3d-1}(\mathbf{P}^2, d)$ is $2(3d-1)$, so the class is $n_d[{\mathbf{P}^2}^{3d-1}]$ where n_d is the number of rational plane curves of degree d passing through $3d-1$ general points of $\mathbf{P}^2$. The result in this case is a wonderful and surprising recursive formula for the n_d, which is a good indication of the power of the "modern" point of view on Gromov-Witten invariants. Here is the result. Consider the formal power series:

$$\Phi(t_1, t_2) = \sum_{d=1}^{\infty} n_d \frac{t_2^{3d-1}}{(3d-1)!} e^{dt_1}$$

As a consequence of big quantum cohomology Φ satisfies the equation:

$$\frac{\partial \Phi}{\partial t_2^3} = \left(\frac{\partial \Phi}{\partial t_1^2 \partial t_2}\right)^2 - \left(\frac{\partial \Phi}{\partial t_1^3}\right)\left(\frac{\partial \Phi}{\partial t_1 \partial t_2^2}\right)$$

It is a simple exercise to use this to conclude the recursive formula:

$$n_d = \sum_{k+l=d} n_k n_l k^2 l \left[l \binom{3d-4}{3k-2} - k \binom{3d-4}{3k-1} \right]$$

for $d > 1$ (this formula was later given a "classical" proof by Kontsevich)

2. A Brief Overview of the Research. I'll separate the research on Gromov-Witten invariants into two categories: foundations and structure.

Foundations (How to rigorously define Gromov-Witten invariants) A rigorous definition was given by symplectic and algebraic geometers at about the same time, but I will only talk about the algebraic geometry side. The strategy, which goes back to Grothendieck, is to construct a proper **moduli space** $M_n(X,d)$ together with an "evaluation" morphism:

$$ev : M_n(X,d) \to X^n$$

such that the class we want is defined by $[\mathrm{Rat}_n(X,d)] := ev_*[M_n(X,d)]$ the push-forward of the fundamental class.

The spaces $\overline{M}_{g,n}(X,d)$ of stable maps were constructed by Kontsevich and Manin for this purpose, building on earlier work of Deligne-Mumford and Gromov (actually, $\overline{M}_{g,n}(X,d)$ is a Deligne-Mumford stack, a scheme equivalent of an orbifold). These moduli spaces exist for any X and curves of any genus mapping to X. When X is a homogeneous space, pushing forward the fundamental class of $\overline{M}_{0,n}(X,d)$ is a satisfactory definition for $[\mathrm{Rat}_n(X,d)]$. In other cases, the definition is not satisfactory, but Behrend and Fantechi have shown how to use the intersection theory of Fulton and MacPherson to find a canonical "virtual" class in the moduli space whose push-forward is a satisfactory definition of $[\mathrm{Rat}_n(X,d)]$.

These constructions are somewhat technical, and I will not say much about the them in these talks, besides stating the appropriate theorems. I will focus instead on the properties of the moduli spaces of stable maps and their "virtual" classes that have proven useful to the theory.

Structure (Properties of Gromov-Witten invariants) The collection of cohomology classes $[\mathrm{Rat}_n(X,d)]$ has a rich structure which has yet to be understood completely. Explaining the structure that we do understand will be the main point of these lectures. For now, I'll give some examples:

In the *reduction to the diagonal* a product $\gamma_1 \wedge \gamma_2$ of cohomology classes on X is expressed as the pull-back: $\gamma_1 \wedge \gamma_2 = \Delta^*(\pi_1^*\gamma_1 \wedge \pi_2^*\gamma_2)$ where $\Delta : X \to X \times X$ is the diagonal embedding. One can also reduce to the triple diagonal, and express the product as:

$$\gamma_1 \wedge \gamma_2 = \pi_{3*}(\delta \wedge \pi_1^*\gamma_1 \wedge \pi_2^*\gamma_2)$$

where $\delta \in H^*(X^3, \mathbf{Q})$ is the class of the diagonal $\Delta : X \to X \times X \times X$.

Small quantum cohomology is the product on cohomology which we obtain by adding all the three-point classes to δ. That is, define:

$$\delta_{sm} := \delta + \sum_{d=1}^{\infty} [\mathrm{Rat}_3(X,d)]$$

and define the small quantum product of cohomology classes by:

$$\gamma_1 \wedge_{sm} \gamma_2 := \pi_{3*}(\delta_{sm} \wedge \pi_1^* \gamma_1 \wedge \pi_2^* \gamma_2)$$

This is well-defined when the Fano index is positive or negative, but not when the Fano index is zero. To cover that case, and for other reasons, we modify the definition to depend upon a parameter q:

$$\delta_{sm}(q) := \delta + \sum_{d=1}^{\infty} [\mathrm{Rat}_3(X,d)] q^d$$

and we think of the corresponding small quantum product either as a product of cohomology classes depending upon q or as a twisted product structure on $\mathrm{H}^*(X, \mathbf{Q})[[q]]$ (or $\mathrm{H}^*(X, \mathbf{Q})[q]$ in the positive or negative case).

There is also a **big quantum product** defined using a potential function:

$$\Phi(\gamma) = \frac{1}{6} \int_{X^3} \gamma \wedge \gamma \wedge \gamma + \sum_{d=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{X^n} [\mathrm{Rat}_n(X,d)] \wedge \gamma^{\otimes n}$$

specializing to the small product. The WDVV equations satisfied by the third derivatives of this potential function encode the associativity of quantum cohomology and also lead to Dubrovin's Frobenius manifolds.

There is more structure to be seen using a further generalization of the invariants defined by means of "cotangent classes" on the spaces $\overline{M}_{g,n}(X,d)$. These are divisor classes $\psi_1, ..., \psi_n \in H^2(\overline{M}_{g,n}(X,d), \mathbf{Q})$ obtained from the internal structure of the moduli space of stable maps with respect to which we define:

$$[\mathrm{Rat}^{a_1,\dots,a_n}(X,d)] := ev_*(\psi_1^{a_1} \wedge ... \wedge \psi_n^{a_n}) \in H^*(X^n, \mathbf{Q})$$

A second generalization of the original question is then to determine all of these classes. The first result is due to Dubrovin:

Reconstruction I: The generalized invariants in genus zero are determined by the ordinary invariants.

So the generalized invariants (in genus zero) only represent a new way to organize the same data. On the other hand, there are some distinct advantages to considering generalized invariants. For example in the case $n = 1$,generating functions (known as J-functions) made up of the generalized invariants play an important role in mirror symmetry. These J-functions are frequently computable. Some results include:

- Explicit formulas when X is a toric variety.
- Explicit formulas for Fano and Calabi-Yau complete intersections in terms of the invariants of the ambient variety.
- Explicit formulas for products in terms of the invariants of the factors.

And a final theorem (of -,Kley) states that the J-functions sometimes suffice.

Reconstruction II If $H^*(X, \mathbf{Q})$ is generated by divisor classes, then all invariants are determined by the $n = 1$ generalized invariants.

This should probably be thought of as an analogue for generalized invariants of an earlier theorem of Kontsevich, stating that the small quantum cohomology sometimes suffices:

Reconstruction III If $H^*(X, \mathbf{Q})$ is generated by divisor classes, then all invariants are determined by the $n = 3$ ordinary invariants.