Course Notes for Math 738 (Algebraic Geometry)

13. Varieties. Abstract Varieties. The sheaf of regular functions on an affine variety is defined, and abstract varieties are defined as "things" which are locally affine. We investigate some basic properties of varieties (which are analogues in algebraic geometry of differentiable or complex manfolds), define normality, and expand our definitions of closed immersions, dominant maps and finite maps to encompass all varieties. We finish with a discussion of normality and ramification.

Let V be an affine variety over a field k.

Definition: For each nonempty open subset $U \subseteq V$, the ring of regular functions on U is:

$$\mathcal{O}_V(U) := \{\phi \in k(V) \mid U \subseteq \text{ dom}(\phi)\}.$$

In other words, $\mathcal{O}_V(U) \subseteq k(V)$ is the subring consisting of all the rational functions which are "regular" (i.e. defined!) at all the points of U.

Proposition 13.1: Given V as above, the assignments:

(i) $U \mapsto \mathcal{O}_V(U) \subseteq k(V)$ and

(ii) $U \subseteq W \mapsto \rho_{W,U} =$ the inclusion $\mathcal{O}_V(W) \subseteq \mathcal{O}_V(U)$

yield a sheaf (denoted $\mathcal{O}_V$ and called the sheaf of regular functions on V) of commutative rings with 1 on V. Moreover,

(a) If $p \in V$ and $m_p \subset k[V]$ is the corresponding maximal ideal, then the stalk of $\mathcal{O}_V$ at p (denoted $\mathcal{O}_{V,p}$) is the local ring $k[V]_{m_p} \subseteq k(V)$.

(b) If k is algebraically closed, then $\mathcal{O}_V(U_f) = k[V]_f$ for each $f \in k[V]$.

Proof: Fix $\mathcal{O}_V(\emptyset) = 0$. It is immediate that $\mathcal{O}_V$ is a presheaf (see §0.1) and moreover that the additional sheaf requirements (see §0.2) hold.

Notice that the restriction maps $\rho_{W,U}$ are all inclusions and each $\mathcal{O}_V(U)$ is contained in $k(V)$, so it follows that the stalks (see §0.2 for the general definition) can also be realized as subrings of $k(V)$ defined by:

$$\mathcal{O}_{V,p} = \{\phi \in k(V) \mid p \in \text{ dom}(\phi)\}$$

Thus $\phi \in \mathcal{O}_{V,p}$ if and only if $\phi = \frac{g}{f}$ for some $f \notin m_p$, so $\mathcal{O}_{V,p} = k[V]_{m_p}$.

Part (b) is a generalization of Corollary 12.2. Clearly $k[V]_f \subseteq \mathcal{O}_V(U_f)$ since any rational function of the form $\frac{g}{f^n}$ is regular at all points where f fails to vanish(!) On the other hand, if $\phi \in \mathcal{O}_V(U_f)$, consider the ideal of denominators I of ϕ (as in the proof of 12.2). Then $V(I) \subseteq V(f)$ by definition, so by the strong form of the Nulstellensatz, we have $f^n \in I$ for some n, hence $\phi = \frac{g}{f^n}$ for some g.

Remark: Given the affine variety V, then each $\phi \in \mathcal{O}_V(U)$ is in particular a well-defined function $\phi : U \to k$. If we endow $k = \mathbf{A}^1_k$ with the Zariski topology (i.e. the cofinite topology) and U with the induced topology, then it follows from Proposition 11.8 that each $\phi : U \to k$ is **continuous**. That is, $\mathcal{O}_V \subset \mathcal{C}_{V,k}$ is a subsheaf of the sheaf of continuous functions from V to k.

Definition: Suppose X is a topological space and $\mathcal{O}_X$ is a subsheaf of $\mathcal{C}_{X,k}$. Then $(X, \mathcal{O}_X)$ is a **prevariety** over k if there exist:

(i) A covering of X by finitely many open subsets $U_1, ..., U_n$ and

(ii) Homeomorphisms $h_i : U_i \to V_i$ to affine varieties over k satisfying $h_i^*(\mathcal{O}_{V_i}(U)) = \mathcal{O}_X(h_i^{-1}(U))$ for all V_i and all open subsets $U \subset V_i$.

Given a prevariety X, an **affine atlas of** X will refer to any system $\{(U_i, h_i)\}_{i=1}^n$ of open subsets and homeomorphisms satisfying (i) and (ii).

Obviously affine varieties are prevarieties. Here's another example:

Proposition 13.2: Suppose V is an affine variety and $W \subset V$ is a nonempty open subset. Define the sheaf $\mathcal{O}_W$ by:

$$\mathcal{O}_W(U) := \mathcal{O}_V(U) \text{ for all open subsets } U \subseteq W$$

with the same restriction maps. Then $(W, \mathcal{O}_W)$ is a prevariety.

Proof: We choose a particular $V = V(f_1, ..., f_l) \subset \mathbf{A}^n_k$. Then W is a finite union: $W = \cup_{i=1}^m U_{g_i}$ for $g_i \in k[V]$ (we will be sloppy and also let g_i refer to the corresponding polynomials in $k[x_1, ..., x_n]$). Consider the affine "hyperbolas" $V_i := V(f_1, ..., f_l, g_i \cdot x_{n+1} - 1) \subset \mathbf{A}^{n+1}_k$ and the rational maps

$$h_i : V - - > V_i \text{ given by } h_i(p_1, ..., p_n) = (p_1, ..., p_n, \frac{1}{g_i(p_1, ..., p_n)}).$$

The h_i are well-defined and continuous on U_{g_i} by Proposition 11.8. The projection maps $\pi_i : V_i \to U_{g_i}$ given by $\pi_i(p_1, ..., p_{n+1}) = (p_1, ..., p_n)$ are inverses to the h_i, which are also continuous. So the h_i give homeomorphisms from U_{g_i} to V_i. Via the isomorphisms $h_i^* : k(V_i) \cong k(V)$, we have:

$$\begin{aligned} h_i^*(\mathcal{O}_{V_i}(U)) &= \{h_i^*(\phi) \in k(V) \mid \phi \in k(V_i) \text{ and } U \subset \text{ dom}(\phi)\} \\ &= \{\psi \in k(V) \mid h_i^{-1}(U) \subset \text{ dom}(\psi)\} = \mathcal{O}_V(h_i^{-1}(U)) \end{aligned}$$

for all V_i and open subsets $U \subset V_i$. Thus W is a prevariety and $\{(U_{g_i}, h_i)\}$ is an affine atlas.

More generally than the proposition, we have:

Open subsets of prevarieties are prevarieties:

If X is a prevariety and $W \subset X$ is an open subset with sheaf $\mathcal{O}_W$ defined as in the Proposition, then $(W, \mathcal{O}_W)$ is a prevariety.

Proof: Let $\{(U_i, h_i : U_i \to V_i)\}_{i=1}^n$ be an affine atlas of X. Given the open subset $W \subset X$, consider the images $W_i := h_i(W \cap U_i) \subset V_i$. Via the proposition, we know that each W_i is a prevariety, say with affine atlas $\{(U_{i,j}, h_{i,j} : U_{i,j} \to V_{i,j})\}_{j=1}^{n_i}$. It is now an exercise to show that W is a prevariety with affine atlas $\{(h_i^{-1}(U_{i,j}), h_{i,j} \circ h_i)\}$.

The above was also true of premanifolds. However, here is a new property:

Irreducible closed subsets of prevarieties are prevarieties:

Given a closed subset $Z \subset X$ of a prevariety, we endow Z with the induced topology. The sheaf $\mathcal{O}_Z$ is a little tricky to define. Here is the definition:

For each open set $W \subset Z$, there are, by definition, open sets $U \subset X$ so that $W = Z \cap U$. A continuous function $f : W \to k$ is defined to be in $\mathcal{O}_Z(W)$ if and only if there is such a $U \subset X$ together with an open (finite) cover $U = \cup_{i=1}^n U_i$ and regular functions $f_i \in \mathcal{O}_X(U_i)$ such that for each i, the f_i and f restrict to the same function from $U_i \cap Z$ to k.

We claim that if Z is irreducible, then $(Z, \mathcal{O}_Z)$ is a prevariety.

Proof: Let's first see how this works when $X = V$ is an affine variety. Then $Z \subset V$ is an affine variety, with $k[Z] \cong k[V]/I(Z)$ since it irreducible and closed in V. As an affine variety, Z has the induced topology (§11). We want to see that its sheaf of regular functions (which I also denoted by $\mathcal{O}_Z$ but will forget for now) coincides with the sheaf $\mathcal{O}_Z$ defined above.

If $\phi \in k(Z)$ and $W \subset \operatorname{dom}(\phi)$ (i.e. ϕ is regular on W), then because the ideal of denominators of ϕ is finitely generated, there exist a finite set of expressions $\phi \sim \frac{f_i}{g_i}$ such that $W = \cup_{i=1}^m (U_{g_i} \cap W)$. Each $f_i, g_i \in k[Z]$ lift to elements $F_i, G_i \in k[V]$, which lets us lift $\frac{f_i}{g_i}$ to $\psi_i := \frac{F_i}{G_i} \in k(V)$ which automatically satisfy $U_{G_i} \subset \operatorname{dom}(\psi_i)$ and $U_{G_i} \cap Z = U_{g_i}$. Choose any open $U \subset V$ such that $U \cap Z = W$, and let $U_i = U_{G_i} \cap U$. It follows that the ψ_i are elements of $\mathcal{O}_V(U_i)$ with the desired properties, so that $\phi \in \mathcal{O}_Z(W)$.

Conversely, given a collection of $\psi_i \in \mathcal{O}_V(U_i)$ determining an element $\phi \in \mathcal{O}_Z(W)$, then by definition $\psi_i \in k(V)$ with $U_i \subset \operatorname{dom}(\phi_i)$. We may assume $U_i \cap Z \neq \emptyset$, and it follows that the ψ_i determine well-defined elements $\phi_i \in k(Z)$ with $U_i \cap Z \subset \operatorname{dom}(\phi_i)$ and which coincide with ϕ as functions from $U_i \cap Z$ to k. But this means each $\phi_i = \phi_j$ as a function from $U_i \cap U_j \cap Z$ to k. Since open subsets are dense in an affine variety, and since rational functions are continuous on their domains, it follows that $\phi_i = \phi_j$ for all i and j. In other words, ϕ comes from a **single** rational function $\phi = \phi_i$ with $W \subset \operatorname{dom}(\phi)$. Thus ϕ is a regular function on W.

Thus $\mathcal{O}_Z$ really is the sheaf of regular functions on the affine variety Z.

In general, let $\{(U_i, h_i : U_i \to V_i)\}$ be an affine atlas for a prevariety X. If $Z \subset X$ is closed and irreducible, let $Z_i = h_i(Z \cap U_i) \subset V_i$. It is an easy exercise to see that each Z_i is closed and irreducible in V_i, hence an affine variety, and it follows as another exercise that Z is a prevariety with affine atlas: $\{U_i \cap Z, h_i|_{U_i \cap Z} : U_i \cap Z \to Z_i\}$.

Remark: It would be a lot simpler if we could define:

$$\mathcal{O}_Z(W) := \{f : W \to k \mid \exists\, g \in \mathcal{O}_X(U) \text{ such that } U \cap Z = W \text{ and } f|_W = g|_W\}$$

to avoid taking open covers of U. Unfortunately, the presheaf defined in this way may not satisfy property (ii) of sheaves, even when X is affine!

Next, we'd like to define a variety over k as a "Hausdorff" prevariety. Similarly, we'd like a notion of compactness. Unfortunately, because of the peculiar nature of the Zariski topology, we cannot use the topological definitions, as in the case of manifolds. Instead, we will use the category-theoretic replacements of "separatedness" and "properness". To do this, we need to construct the category of prevarieties and to show that products exist in this category (see §0.3).

Definition: If $(X, \mathcal{O}_Y)$ and $(Y, \mathcal{O}_Z)$ are prevarieties over a field k, then a mapping $\Phi : X \to Y$ is called a **regular map** if:

(i) Φ is continuous, and

(ii) $\Phi^*(\mathcal{O}_Y(U)) \subseteq \mathcal{O}_X(\Phi^{-1}(U))$ for all open $U \subseteq Y$.

Example: (a) Let V, W be affine varieties and $\Phi : V - - > W$ a rational map such that $V =$ dom(Φ). If $\phi \in k(W)$, if $U \subset$ dom(ϕ), and if $\Phi^{-1}(U) \neq \emptyset$, then $\Phi^*(\phi) \in k(V)$ is well-defined and $\Phi^{-1}(U) \subset$ dom$(\Phi^*(\phi))$, so Φ is a regular map of prevarieties. Conversely, if $\Phi : V \to W$ is regular as a map of prevarieties, let $x_1, ..., x_n \in k[W] \subset \mathcal{O}_W(W)$ be the coordinate functions for $W \subset \mathbf{A}^n_k$. Then $\Phi = (\Phi^*(x_1), ..., \Phi^*(x_n))$ and $\Phi^*(x_i) \in \mathcal{O}_V(V) \subset k(V)$. Thus Φ is a rational map which is everywhere defined. In case k is algebraically closed, this condition on a rational map is equivalent to being regular (exercise) so in that case, the two notions of a regular map coincide.

(b) If $W \subset X$ is an open subset or $Z \subset X$ is an irreducible closed subset, then the inclusion maps are regular maps of prevarieties, when W and Z are thought of as prevarieties as above. Indeed, the inclusions are continuous because the topology is induced. And the definitions of $\mathcal{O}_W$ and $\mathcal{O}_Z$ are set up so that the inclusions satisfy (ii) of the definition.

Definition: (a) The isomorphism classes of prevarieties over k with regular maps are a category, called the **regular category** of prevarieties over k.

(b) A regular map $\Phi : X \to Y$ of prevarieties is an **open** (resp. **closed**) **immersion** if $\Phi(X) \subset Y$ is open (resp. closed and irreducible), and Φ is an isomorphism to its image. It is **dominant** if the image is dense in Y.

(c) A prevariety is called **affine** if it is isomorphic to an affine variety and **quasi-affine** if it admits an open immersion to an affine variety.

Examples: (a) The open subsets $U_g \subseteq V$ of an affine variety are affine since they are isomorphic (by Proposition 13.2) to affine "hyperbolas."

(b) Conversely if k is algebraically closed, $W \subseteq V$ is a nonempty open subset, and if $k[V]$ is a UFD, then the quasi-affine W is affine if **and only if** $W = U_g$ for some $g \in k[V]$. (Exercise!) For example, $\mathbf{A}^n_k - \{(0, ..., 0)\}$ is not affine if $n \geq 2$ and k is algebraically closed.

(c) If X is a prevariety and $\{U_i, h_i : U_i \to V_i\}$ is an affine atlas, then from the definition of an atlas, each $h_i^{-1} : V_i \to X$ is an open immersion.

Here is a useful lemma.

Lemma 13.3: Suppose X is a prevariety and $X = \cup X_i$ is an open cover.

(a) If Y is a prevariety and $\Phi : X \to Y$ is a mapping with the property that each $\Phi_i := \Phi|_{X_i} : X_i \to Y$ is regular as a map of prevarieties, then Φ is also regular as a map of prevarieties.

(b) If $\{U_i, h_i : U_i \to V_i\}$ is an affine atlas for X, then each composition $h_j \circ h_i^{-1}$ is the restriction of a birational isomorphism $\Phi_{i,j} : V_i - - > V_j$.

Proof: (a) If $U \subset Y$ is open, then each $\Phi_i^{-1}(U) \subset X_i$ is open, because the Φ_i are continuous, hence $\Phi^{-1}(U) = \cup \Phi_i^{-1}(U)$ is open, so Φ is continuous. If $\phi \in \mathcal{O}_Y(U)$, then each $\Phi_i^*(\phi) \in \mathcal{O}_{X_i}(\Phi_i^{-1}(U)) = \mathcal{O}_X(\Phi^{-1}(U) \cap X_i)$ because the Φ_i are regular. But by the second sheaf property for $\mathcal{O}_X$, this implies $\Phi^*(\phi) \in \mathcal{O}_X(\Phi^{-1}(U))$. Thus Φ is regular.

(b) Consider the open subset $h_i(U_i \cap U_j) \subset V_i$. By the the definition of an affine atlas, the map $h_j \circ h_i^{-1} : h_i(U_i \cap U_j) \to V_j$ pulls back rational functions defined on V_j to rational functions on V_i defined on $h_i(U_i \cap U_j)$. If we choose a particular $V_j \subset \mathbf{A}_k^n$, and let $x_1, ..., x_n \in k[V_i] \subset \mathcal{O}_{V_i}(V_i)$ be the coordinate functions, then the rational map $\Phi_{i,j} := ((h_j \circ h_i^{-1})^*(x_1), ..., (h_j \circ h_i^{-1})^*(x_n))$ is defined and agrees with $h_j \circ h_i^{-1}$ on $h_i(U_i \cap U_j)$. It is a birational isomorphism, because it has a rational inverse, namely $\Phi_{j,i}$ (!)

Proposition 13.4: If V and W are affine varieties over k, then their product $V \times W$ in the category of affine varieties is also a product of prevarieties.

Proof: Suppose X is a prevariety and $p : X \to V$ and $q : X \to W$ are regular maps of prevarieties. We need to prove that $(p, q) : X \to V \times W$ is also regular as a map of prevarieties.

Suppose $\{U_i, h_i : U_i \to Y_i\}$ is an affine atlas for X. It follows from the definition of an atlas that each $p \circ h_i^{-1} : Y_i \to V$ and $q \circ h_i^{-1} : Y_i \to W$ is a regular map of prevarieties, hence rational as a map of affine varieties (see the earlier example). But since products in the regular category of affine varieties agree with products in the rational category (HW §11) it follows that the maps $(p \circ h_i^{-1}, q \circ h_i^{-1}) : Y_i \to V \times W$ are also rational maps, hence regular as maps of prevarieties. It follows that the restrictions $(p, q)|_{U_i} : U_i \to V \times W$ are regular maps of prevarieties. But now by Lemma 13.3(a), we have that (p, q) is itself regular, as desired.

Proposition 13.5: Products of prevarieties over k always exist. Moreover, the product agrees with the Cartesian product.

Proof: Suppose we are given prevarieties X and Y over k with affine atlases $\{S_i, g_i : S_i \to V_i\}$ and $\{T_j, h_j : T_j \to W_j\}$ respectively. I want to prove that there exists a topology and sheaf on the Cartesian product so that $\{S_i \times T_j, (g_i, h_j) : S_i \times T_j \to V_i \times W_j\}$ is an affine atlas for $X \times Y$. (Here $V_i \times W_j$ is the ordinary product of affine varieties and $(g_i, h_j)(x, y) := (g_i(x), h_j(y))$.) By definition, then, $(X \times Y, \mathcal{O}_{X \times Y})$ would be a prevariety, and moreover,

(i) The projection map $\pi_X : X \times Y \to X$ restricts to the projections $\pi_{i,j} : S_i \times T_j \to S_i \subset X$. Each of these would be regular maps of prevarieties (they are really the projections $V_i \times W_j \to V_i$ in disguise), and by Lemma 13.3(a), it would follow that π_X (and similarly π_Y) is regular.

(ii) If Z is a third prevariety with regular maps $p : Z \to X$ and $q : Z \to Y$, let $U_{i,j} := p^{-1}(S_i) \cap q^{-1}(T_j) \subset Z$. The "Cartesian" lift $(p, q) : Z \to X \times Y$ restricts to $(p, q) : U_{i,j} \to S_i \times T_j \subset X \times Y$ which would all be regular maps by Proposition 13.4 (again, think of $S_i \times T_j$ as the affine $V_i \times W_j$ in disguise). But then by Lemma 13.3(a) it would follow that (p, q) is regular.

In other words, it would follow that $(X \times Y, \mathcal{O}_{X \times Y})$ is the product!

Construction of the topology and sheaf: Let $\Phi_{i,k} : V_i - - > V_k$ and $\Psi_{j,l} : W_j - - > W_l$ be the birational isomorphisms from Lemma 13.3(b). Since products agree in the regular and rational categories of affine varieties, there is a birational isomorphism $(\Phi_{i,k}, \Psi_{j,l}) : V_i \times W_j - - > V_k \times W_l$ whose domain includes the open subset $\mathrm{dom}(\Phi_{i,k}) \times \mathrm{dom}(\Psi_{j,l})$ and which is the map $(\Phi_{i,k}, \Psi_{j,l})(x, y) = (\Phi_{i,k}(x), \Psi_{j,l}(y))$ on that open set. We now construct the desired topology and sheaf by "gluing". Namely, consider the data:

- The affine varieties $V_i \times W_j$
- Open subsets $U_{i,j,k,l} := (g_i, h_j)(S_i \cap S_k, T_j \cap T_l)$
- Birational isomorphisms $\Phi_{i,j,k,l} := (\Phi_{i,j}, \Psi_{k,l}) : V_i \times W_j - - > V_k \times W_l$.

Let $Z := (\coprod V_i \times W_j)/ \sim$, where $p \sim q$ if $p \in U_{i,j,k,l}$ and $\Phi_{i,j,k,l}(p) = q$. There is a natural bijection $Z \leftrightarrow X \times Y$, and the desired properties are obtained from the Gluing Construction (HW # 8).

Definition: A **variety** over k is a separated prevariety over k. Regular maps of varieties are the same as for prevarieties.

Examples: (a) Affine varieties are varieties by 13.4 and HW §11.

(b) Quasi-affine varieties are varieties. If V is affine and $W \subset V$ is open, then $W \times W \subset V \times V$ is open (why?), and by the universal property, this must be the product. But the diagonal $\Delta_W \subset W \times W$ is the intersection $\Delta_V \cap (W \times W)$, which is closed since V is separated.

(c) More generally, if $\Phi : X \to Y$ is an open (resp. closed) immersion and Y is a variety, then the natural map $X \times X \to Y \times Y$ is an open (resp. closed) immersion and X is also a variety.

(d) Consider the "affine line with doubled origin" formed by applying the gluing construction (see HW # 8) to the following data:

- $V_1 = V_2 = \mathbf{A}^1_k$
- $U_{1,2} = U_{2,1} = \mathbf{A}^1_k - \{0\}$,
- $\Phi_{1,2} = \Phi_{2,1} =$ the identity map.

The resulting prevariety X is not a variety. If $\{U_i, h_i : U_i \to \mathbf{A}^1_k\}$ is the associated affine atlas, then the two open immersions $\Phi_i := h_i^{-1} : \mathbf{A}^1_k \to X$ have the property that $\{\Phi_1 = \Phi_2\} = \mathbf{A}^1_k - \{0\}$ is not closed in $\mathbf{A}^1_k$.

Here is an important result from commutative algebra:

Finiteness of Integral Closure: Suppose A is an integral domain which is finitely generated as an algebra over a field k. If $K \subset L$ is a finite field extension of the field of fractions of A, then the integral closure of A in L is a finitely generated A-module. In particular, it is finitely generated as an algebra over k.

(See Zariski-Samuel, Commutative Algebra I for a proof.)

Definition: A prevariety X is **normal at** $p \in X$ if $\mathcal{O}_{X,p}$ is integrally closed. X is **normal** if it is normal at all points $p \in X$.

Proposition 13.6: Every prevariety X is normal at some point, and the set $N(X) := \{p \in X \mid X \text{ is normal at } p\}$ is open in X.

Proof: Let $\{U_i, h_i : U_i \to V_i\}$ be an affine atlas for X. Since $N(X) = \cup N(U_i)$, it suffices to show that each $N(U_i) \subset U_i$ is open and nonempty, i.e. that each $N(V_i) \subset V_i$ is open and nonempty. So we can assume $X = V$ is an affine variety. Let $\overline{k[V]}$ be the integral closure of $k[V]$ in $k(V)$.

By the finiteness theorem, $\overline{k[V]}$ is finitely generated as a $k[V]$ module, say by $\phi_1, ..., \phi_m \in k(V)$. If we write each $\phi_i = \frac{a_i}{b_i}$ and let $f = \prod_{i=1}^m b_i$, then $\overline{k[V]} \subset k[V]_f$, and it follows that $k[V]_f$ is integrally closed, and each $k[V]_{m_q}$ is integrally closed whenever $f \notin m_q$. (See Exercise # 5.) By 13.1(a), $U_f \subset N(V)$, which is therefore not empty. If $p \in N(V)$, we can choose the b_i above so that $b_i \notin m_p$ since $k[V]_{m_p} \subset k(V)$ is integrally closed, so if f is chosen as above, then $f \notin m_p$ and $p \in U_f \subset N(V)$ which is then open.

Definition: Suppose X is a prevariety over k. Let

$$k(X) := \{(U, \phi) \mid U \subset X \text{ is open and nonempty, and } \phi \in \mathcal{O}_X(U)\}/\sim$$

where $(U, \phi) \sim (V, \psi)$ if $\phi = \psi$ as functions from $U \cap V$ to k. We call $k(X)$ the **field of rational functions on** X.

Observations: (a) $k(X)$ is a field. If X is affine, then this agrees with the earlier definition of the field of rational functions.

(b) If $\Phi : X \to Y$ is a dominant regular map of prevarieties, then Φ^* determines a well-defined injection of fields $\Phi^* : k(Y) \hookrightarrow k(X)$. If Φ is an open immersion, then Φ^* is an isomorphism of fields.

(c) There is a natural inclusion $\mathcal{O}_{X,p} \subset k(X)$ whenever $p \in X$. I claim that $k(X)$ is the field of fractions of $\mathcal{O}_{X,p}$. To see this, observe that there is an open immersion $\Phi : V \to X$ of an affine variety with $p \in \Phi(V)$. Then $\Phi^* : \mathcal{O}_{X,p} \cong k[V]_{m_p}$ and $\Phi^* : k(X) \cong k(V)$. The same may not be true of $\mathcal{O}_X(U) \subset k(X)$ for an open $U \subset X$. This integral domain can be quite small.

(d) $\mathcal{O}_X(U) = \cap_{p \in U} \mathcal{O}_{X,p} \subset k(X)$. This is simply the following statement: "a function is defined on U iff it is defined at all the points of U"(!)

Proposition 13.1(b) will play a prominent role in the rest of this section. For this reason, we will now (and quite frequently in the future) assume k is algebraically closed. Varieties over general fields were studied in the early part of this century by similar techniques, but the theory of schemes, due to Grothendieck (where the analogue of 13.1(b) does hold) provides a much better framework for studying such varieties. For this reason, we will not worry about the loss of generality inherent in assuming k is algebraically closed, referring the interested reader to Hartshorne's Algebraic Geometry for an exhaustive introduction to the scheme-theoretic approach.

Assume k is algebraically closed.

Proposition 13.7: If $\Phi : V \hookrightarrow X$ is an open immersion from an affine variety to a variety over k, then $\Phi(V) \subset N(X)$ if and only if $k[V]$ is integrally closed. In particular, an affine variety V is normal iff $k[V]$ is integrally closed.

Proof: By Proposition 13.1(a), the Nullstellensatz and HW §12 # 2(d),

$$k[V] = \cap_{p \in V} k[V]_{m_p} \text{ and each } k[V]_{m_p} = \mathcal{O}_{V,p} \cong \mathcal{O}_{X,\Phi(p)}$$

so $k[V] = \overline{k[V]}$ iff $\mathcal{O}_{X,q}$ is integrally closed for all $q \in \Phi(V)$ (HW §12 # 3(d)).

Definition: A regular map $\Phi : X \to Y$ of varieties over any field is **finite** if there is an affine atlas $\{U_i, h_i : U_i \to V_i\}$ of Y with the following properties:

(a) Each $\Phi^{-1}(U_i)$ is affine, say by an isomorphism $g_i : \Phi^{-1}(U_i) \to W_i$.

(b) Each regular map $\Phi_i : h_i \circ \Phi \circ g_i^{-1} : W_i \to V_i$ is a finite regular map of affine varieties.

Proposition 13.8: If $\Phi : X \to Y$ is a finite map of varieties over k and $V \subset Y$ is open and affine, with the property that $W := \Phi^{-1}(V) \subset X$ is also affine, then $\mathcal{O}_X(W)$ is integrally dependent upon $\Phi^*(\mathcal{O}_Y(V))$. Thus the new definition agrees with the old one if X and Y are affine varieties.

Proof: I'm going to be sloppy and write $k[W] = \mathcal{O}_X(W), k[V] = \mathcal{O}_Y(V)$, assuming W and V are literally affine varieties. Similarly, instead of the affine atlas of the definition, let's write $V_i \subset Y$ and $W_i \subset X$ instead of the open immersions. The only effect of this sloppiness is to reduce notation(!)

First, we assume $V \subset V_i$ for some i. In that case, cover $V = \cup U_{f_j}$ with basic open sets $U_{f_j} \subset V_i$, and **further** cover $U_{f_j} = \cup U_{h_{j,l}}$ with basic open sets $U_{h_{j,l}} \subset V$. Since by assumption $k[W_i] = \mathcal{O}_X(W_i)$ is integrally dependent upon $k[V_i] = \mathcal{O}_Y(V_i)$, it follows that each $k[W_i]_{\Phi^*(f_j)} = \mathcal{O}_X(\Phi^{-1}(U_{f_j}))$ (13.1(b)) is also integrally dependent upon $k[V_i]_{f_j} = \mathcal{O}_Y(U_{f_j})$. Let $\phi \in k[W] = \mathcal{O}_X(W)$. Then each image of ϕ in $\mathcal{O}_X(\Phi^{-1}(U_{f_j}))$ is integrally dependent upon $\mathcal{O}_Y(U_{f_j})$, hence each **further** image of ϕ in $k[W]_{\Phi^* h_{j,l}}$ is integrally dependent upon $k[V]_{h_{j,l}}$. Recall that the $U_{h_{j,l}}$ cover V so that the $h_{j,l}$ generate $k[V]$ as an ideal (Nullstellensatz). It is an exercise now to see that ϕ is integral over $k[V] = \mathcal{O}_Y(V)$.

In general, cover V by affines U_{g_m} each of whose image lies in some V_i. Then by the above argument, each $k[W]_{\Phi^* g_m}$ is integrally dependent upon $k[V]_{g_m}$ and as before, the fact that the g_m generate $k[V]$ as an ideal implies that $k[W]$ is integrally dependent upon $k[V]$.

Observations: (a) A finite map of varieties over k is closed and the preimage of every point is a finite set. This follows from Proposition 12.4 and the fact that $Z \subset X$ is closed iff $Z \cap U_i$ is closed in U_i for each U_i in an open cover.

(b) The degree of a finite dominant map $\Phi : X \to Y$ is well-defined. It is the degree of the field extension $\Phi^* k(Y) \subset k(X)$, which is finite by Proposition 12.6 because it is the degree of $\Phi^* k(V) \subset k(W)$ for any affine V with affine preimage $W := \Phi^{-1}(V)$ by Proposition 13.8.

Definition: A dominant, finite map $\Phi : X \to Y$ of prevarieties is:

(a) **ramified at** $p \in \Phi^{-1}(q) \subset X$ if $\Phi^*(m_q)$ does not generate m_p, where m_q and m_p are the maximal ideals in $\mathcal{O}_{Y,q}$ and $\mathcal{O}_{X,p}$, respectively.

(b) **ramified over** $q \in Y$ if it is ramified at **some** $p \in \Phi^{-1}(q)$.

Exercise: This agrees with the definition of ramification in §12.

Finally we answer a question that was begged by the examples of §12.

Proposition 13.9: Let $\Phi : X \to Y$ be a finite and dominant map of degree d of prevarieties over k and $q \in Y$. Let e be the number of points in $\Phi^{-1}(q)$.

(a) If Φ is not ramified over q, then $e \geq d$.

(b) If $q \in N(Y)$, then $e \leq d$, and $e = d$ iff Φ is not ramified over q.

We will need two more tools from commutative algebra for this proof:

Nakayama's Lemma: If A is a local ring with maximal ideal m and $F : M \to N$ is a homomorphism of finitely generated A-modules with the property that the induced map $\overline{F} : M/mM \to N/mN$ of vector spaces over $k = A/mA$ is surjective, then F is also surjective.

Coefficients of the Minimal Polynomial: Suppose A is an integral domain which is integrally closed and $K \subset L$ is a finite extension of the field of fractions of A. If $b \in L$ is integrally dependent upon A, then the coefficients of the minimal polynomial of b are in A (as opposed to K).

Proof: By definition of finiteness, there is an affine neighborhhod V of $q \in Y$ with the property that $\Phi^{-1}(V) \subset X$ is also affine. Moreover, if $q \in N(Y)$, we can find a basic open set $U_f \subset V$ which is a neighborhood of q contained in $N(X)$ such that $k[U_f]$ is integrally closed, by 13.6 and 13.7. Note that $\Phi^{-1}(U_f) = U_{\Phi^* f} \subset \Phi^{-1}(V)$ is also affine. Thus in (a) we can assume X and Y are affine, and in (b) we can assume that $k[Y] = \overline{k[Y]}$.

(a) It follows from from 12.3 and 12.4 that if Φ is not ramified over q, then $k[X]/I \cong k^e$, where $I = m_q k[X]$. Choose $a_1, ..., a_e \in k[X]$ whose images generate $k[X]/I$. I would like to say that the a_i generate $k[X]$ as a $k[Y]$-module, because the proof of 12.6 would imply that the a_i generate $k(X)$ as a field extension of $k(Y)$. Unfortunately, this may not be true. However, if we let $M = k[Y] - m_q$ and localize everything at M, then Nakayama's lemma tells us that the homomorphism of $k[Y]_M$-modules:

$$\oplus_{i=1}^{e} k[Y]_M \to k[X]_M;\ (b_1, ..., b_e) \mapsto \sum \frac{a_i}{1} \Phi^*(b_i)$$

is surjective, because $k[X]_M / m_q k[X]_M$ is isomorphic to $k[X]/m_q k[X]$ and because the images of the a_i generate this vector space. In other words, the a_i generate $k[X]_M$ as a module over $k[Y]_M$, and the same argument as in 12.6 shows that they generate the field extension, so $d \leq e$.

(b) Suppose first that Φ is not ramified over q. We can find $a \in k[X]$ which takes different values at each $p \in \Phi^{-1}(q)$. Since $k[Y]$ is integrally closed, it follows from the coefficient lemma that the minimal polynomial satisfied by a for the field extension $k(Y) \subset k(X)$ has the form:

$$x^n + \Phi^*(b_{f-1}) x^{n-1} + ... + \Phi^*(b_0) = 0$$

where each $b_i \in k[Y]$ and, of course, $n \leq d$. If we evaluate each b_i at the point $q \in Y$, then we get a polynomial equation with coefficients in k which is satisfied by each of the e distinct values $a(p)$ for $p \in \Phi^{-1}(q)$. But this means that $e \leq n \leq d$, and using (a) we get equalities all around.

Finally, suppose Φ is ramified at $p \in \Phi^{-1}(q)$. By definition, this means that $\Phi^*(m_q)$ does not generate m_p. We can therefore find an $a \in k[X]$ with the property that $a(p) = 0$, the image of a in $\mathcal{O}_{X,p}$ is not in the ideal generated by $\Phi^*(m_q)$, and the values $a(p)$ are distinct and nonzero for the other $p \in \Phi^{-1}(q)$. It follows that the minimal polynomial for a as above satisfies $b_0(q) = 0$ **and** $b_1(q) = 0$, hence the root at p counts for two zeroes, and we get $e < d$.

Eighth Homework Assignment

1. The topology on a prevariety X is called the Zariski topology. Prove:

(a) X is irreducible, so open subsets are dense.

(b) Every open cover of an open subset of X has a finite subcover.

(c) Every closed subset of X is a union of finitely many irreducibles.

2. If k is algebraically closed, show that the regular category of affine varieties over k is a full subcategory of the regular category of prevarieties over k.

3. Suppose V is an affine variety over an algebraically closed field and $k[V]$ is a UFD. If $W \subset V$ is open and affine as a prevariety, show that $W = U_g$ for some $g \in k[V]$. Conclude that $\mathbf{A}^n_k - \{(0, ..., 0)\}$ is not affine if $n \geq 2$.

4. (a) Prove that if $Z \subset X$ is a closed irreducible subset of a prevariety, then the object $\mathcal{O}_Z$ constructed in the notes really is a sheaf.

Let $Z = V(xy - zw) \subset \mathbf{A}^4_k$, and let $\phi \sim \frac{x}{w} \in k(Z)$.

(b) Prove that $k[Z]$ is not a UFD for any infinite field k.

(c) Find $\mathrm{dom}(\phi)$, and let $Y = Z - \mathrm{dom}(\phi)$. Suppose k is algebraically closed, and there is a polynomial $f \in k[x, y, z, w]$ such that $V(f) \cap Z = Y$. Conclude that $y^n - fg \in \langle xy - zw \rangle$ for some $g \in k[x, y, z, w]$. Conclude that there is a **homogeneous** $F \in k[x, y, z, w]_d$ such that $V(F) \cap Z = Y$. Derive a contradiction by considering the images of everything in $V(xy - zw) \subset \mathbf{P}^3_k$. Conclude finally that the "simpler" construction of $\mathcal{O}_Z$ suggested in the notes does not produce a sheaf for this choice of $Z \subset V = \mathbf{A}^4_k$.

5. If A is an integral domain and $M \subset A$ is a multiplicative subset, prove that $\overline{(A_M)} = (\overline{A})_M$, where the integral closures are taken in the field of fractions of A. Show how this completes the proof of Proposition 13.6.

6. If $A \subset B$ is an inclusion of integral domains, if $f_1, ..., f_m \in A$ generate A as an A-module, and if $b \in B$ has the property that the image of b in each B_{f_i} is integral over A_{f_i}, conclude that b is integral over A. This is the key algebraic lemma we needed to complete Proposition 13.8.

7. Look up a proof of Nakayama's Lemma and the result on coefficients of the minimal polynomial in a book on commutative algebra, then close the book and reproduce the proof in your own words. Or better yet, come up with proofs on your own.

8. The Gluing Construction: Suppose we are given the following data:

- A finite collection of birational affine varieties $V_1, ..., V_m$.
- Nonempty open subsets $U_{i,j} \subseteq V_i$ for each $i, j = 1, ..., m$ (with $U_{i,i} = V_i$).
- Birational isomorphisms $\Phi_{i,j} : V_i - - > V_j$ with the following properties:

(o) $U_{i,j} \subseteq \operatorname{dom}(\Phi_{i,j})$ and $\Phi_{i,j}(U_{i,j}) = U_{j,i}$ for all i, j

(i) $\Phi_{i,i} : V_i \to V_i$ is the identity map for all i.

(ii) $\Phi_{i,j}^{-1} = \Phi_{j,i} : U_{j,i} \to U_{i,j}$ for all i, j and

(iii) $\Phi_{i,j}(U_{i,j} \cap U_{i,k}) = U_{j,i} \cap U_{j,k}$ for all i, j, k and:

$$\Phi_{j,k} \circ \Phi_{i,j} = \Phi_{i,k} \text{ as functions from } U_{i,j} \cap U_{i,k} \text{ to } V_k.$$

Let $X = \coprod_{i=1}^m V_i / \sim$ where $p \sim q \Leftrightarrow p \in U_{i,j}, q \in U_{j,i}$ and $\Phi_{i,j}(p) = q$.

(a) Prove that $\sim$ is an equivalence relation, so the definition of X makes sense, and prove that each $f_i : V_i \to \coprod V_i \to X$ is injective. Let $U_i = f_i(V_i)$.

Put the topology on X generated by the $f_i(U)$, where U range over all open subsets of V_i. We call this the Zariski topology on X.

(b) Prove that $f_i : V_i \to U_i$ are homeomorphisms, and let $h_i = f_i^{-1}$.

Definition: The sheaf $\mathcal{O}_X$ is defined as follows:

(i) $\mathcal{O}_X(U) = \{f : U \to k \mid f|_{U \cap U_i} \circ f_i \in \mathcal{O}_{V_i}(h_i(U \cap U_i)) \text{ for each } i\}$.

(ii) ρ is the restriction of functions.

(c) Prove that X is a prevariety and $\{U_i, h_i\}$ is an affine atlas.

(d) Show that (o)-(iii) are satisfied for the data in Proposition 13.5, and finish the proof of that proposition.

(e) Given a prevariety X with affine atlas $\{U_i, h_i : U_i \to V_i\}$, prove that the data $V_i, U_{i,j} := h_i(U_i \cap U_j), \Phi_{i,j} := h_j \circ h_i^{-1}$ (as in 13.3(b)) satisfy (o)-(iii), producing a prevariety isomorphic to Y.

(f) If X is a prevariety over an algebraically closed field, with affine atlas $\{U_i, h_i : U_i \to V_i\}$, let W_i be the affine variety associated to the integral closure of $k[V_i]$ in $k(V_i)$, and let $\Phi_i : W_i \to V_i$ be the resulting finite, dominant regular map. Show that the $\Phi_{i,j}$ induce birational isomorhisms from W_i to W_j and show that the data $W_i, \Phi_i^{-1}(h_i(U_i \cap U_j)), \Phi_{i,j}$ produce a prevariety Y which is normal together with a finite, dominant regular map $\pi : Y \to X$. This is called the **normalization** of X.