Course Notes for Math 737 (Algebraic Geometry)

5. Introduction. Embedding the Torus. Let $L = \mathbf{Z}\sigma + \mathbf{Z}\tau$ be a lattice subgroup of the complex numbers (i.e. σ and τ are complex numbers which are linearly independent as vectors in $\mathbf{R}^2$). Our goal in this section: To find lots of holomorphic maps from the torus $\mathbf{C}/L$ to complex projective spaces, and to find some algebraic geometry in this "transcendental" problem.

Some Preliminaries: If $z \in \mathbf{C} - \{0\}$, then multiplication by z takes L to the lattice $zL = \mathbf{Z}\sigma z + \mathbf{Z}\tau z$, and there is no way to holomorphically distinguish between the tori $\mathbf{C}/L$ and $\mathbf{C}/zL$. So we may as well assume (choosing $z = \frac{1}{\sigma}$) that $\sigma = 1$. Moreover, by an appropriate choice for the basis of the lattice, we can and will assume that $\mathrm{Im}(\tau) > 0$ and that $|\tau| \geq \frac{1}{2}$ (exercise).

The rough idea is to find "enough holomorphic functions" on the torus. You shouldn't take this too literally, though, because a function from $\mathbf{C}/L$ to $\mathbf{C}$ is the same as the following:

Definition: A function $f : \mathbf{C} \to \mathbf{C}$ is **doubly periodic** (with respect to L) if $f(z) = f(z + \lambda)$ for all $z \in \mathbf{C}$ and all $\lambda \in L$.

and if you insist that they be holomorphic, you run into:

Lemma 5.1: The only doubly periodic holomorphic functions are the constant functions.

Proof: Let $P \subset \mathbf{C}$ be a closed bounded subset which containes a fundamental domain for the lattice. For example, let P be the closed region bounded by the parallelogram joining 0 and 1, 1 and $\tau + 1$, $\tau + 1$ and τ, and τ and 0. Then $f(\mathbf{C}) = f(P)$, because f is doubly periodic, so that if f is also continuous, then it is bounded. But the only bounded holomorphic functions are the constants, by Liouville's theorem.

So if we are looking to find a map $\Phi : \mathbf{C}/L \to \mathbf{P}^n_{\mathbf{C}}$ by stringing together $n+1$ holomorphic functions $\phi_0, ..., \phi_n$ on $\mathbf{C}/L$ in the following way:

$$\Phi = (\phi_0 : ... : \phi_n) : \mathbf{C}/L \to \mathbf{P}^n_{\mathbf{C}}$$

then all the ϕ_i have to be constants (and one of them has to be nonzero) so Φ has to be the constant map, which is not a terribly interesting map.

The key to constructing interesting maps is to realize that the ϕ_i do not actually have to be doubly periodic functions on $\mathbf{C}$ in order to determine a well-defined map Φ. Let's define:

Definition: $f : \mathbf{C} \to \mathbf{C}$ is **almost doubly periodic with multiplier** η if $f(z+1) = f(z)$ and $f(z+\tau) = e^{2\pi i\eta}f(z)$ for all $z \in \mathbf{C}$.

Observation: If ϕ and ψ are doubly periodic with multipliers η and η' respectively, then $\phi\psi$ is doubly periodic with multiplier $\eta+\eta'$ and $\frac{\phi}{\psi}$ is double periodic (and meromorphic) with multiplier $\eta - \eta'$.

In particular, it follows that if ϕ_i and ϕ_j are both almost doubly periodic with the **same** multiplier, then the **ratio** ϕ_i/ϕ_j is **doubly** periodic (defined wherever $\phi_j \neq 0$). It follows that the map Φ defined as above will be well-defined as a map: $\Phi : \mathbf{C}/L \to \mathbf{P}^n_{\mathbf{C}}$ (though not as a map to $\mathbf{A}^{n+1}_{\mathbf{C}}$) if the ϕ_i are chosen to be almost doubly periodic with the same multiplier and do not share a common zero.

So let's try to find some of these functions. First, recall that we can Fourier expand all "singly periodic" holomorphic functions:

Proposition 5.2: Suppose f is holomorphic and satisfies $f(z+1) = f(z)$ for all $z \in \mathbf{C}$. Then there are uniqely determined constants c_n for each $n \in \mathbf{Z}$ such that:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n z}.$$

Conversely, suppose that $\{c_n\}, n \in \mathbf{Z}$ are complex numbers. Then the "Fourier series" $f(z) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n z}$ converges to a holomorphic function on $\mathbf{C}$ if and only if:

$$\lim_{n\to+\infty} \sup |c_n| r^n = 0 = \lim_{n\to-\infty} \sup \frac{|c_n|}{r^n}$$

for all $r \in \mathbf{R}^+$.

Proof: Consider the exponential function:

$$exp : \mathbf{C} \to \mathbf{C}^* \text{ defined by } \ exp(z) = e^{2\pi i z}.$$

The log function $log(\frac{w}{2\pi i})$ is a holomorphic inverse to exp, but can only be consistently defined on simply connected open subsets of $\mathbf{C}^*$.

Analytic continuation arguments show that the log function is, however, well-defined as a function from $\mathbf{C}^*$ to $\mathbf{C}/\mathbf{Z}$, which means that if f is singly periodic, then the composition

$$f \circ log : \mathbf{C}^* \to \mathbf{C}$$

is well-defined and holomorphic. Such functions are given by Laurent series $\sum_{n=-\infty}^{\infty} c_n w^n$ where the c_n are uniquely determined (as residues) and the $|c_n|$ satisfy the "lim sup" properties above. Thus $f = f \circ log \circ exp$ has the desired form. Conversely, if $g = \sum_{n=-\infty}^{\infty} c_n w^n$ and the lim sup properties are satisfied, then g is holomorphic, and letting $f = g \circ exp$ produces the desired singly periodic function.

Now let's look for some almost doubly periodic functions:

Proposition 5.3: If f is almost doubly periodic and holomorphic with (constant) multiplier $\eta \in \mathbf{C}$, then either:

(i) $\eta = n\tau + m \in L$ and $f(z) = ce^{2\pi i n z}$ for a constant c or else:

(ii) $\eta \notin L$, and $f \equiv 0$.

Proof: We "match" the Fourier series:

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n z} e^{2\pi i n \tau} = f(z+\tau) = e^{2\pi i \eta} f(z) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n z} e^{2\pi i \eta}$$

Because of the uniqueness in Proposition 5.2, the only way these Fourier series can given the same function is if $f \equiv 0$ or else $e^{2\pi i n \tau} = e^{2\pi i \eta}$, i.e. $\eta - n\tau \in \mathbf{Z}$ and all the coefficients other than c_n are equal to zero. In this case, $\eta \in L$, and $f(z) = c_n e^{2\pi i n z}$.

Whoops. Introducing the multiplier η has only made matters worse if $\eta \notin L$, and even if $\eta \in L$, any map built out of almost doubly periodic functions as:

$$\Phi = (\phi_0, ..., \phi_n) = (c_0 e^{2\pi i n z}, ..., c_n e^{2\pi i n z}) : \mathbf{C}/L \to \mathbf{P}^n_{\mathbf{C}}$$

is still the constant map! Things get much more interesting when we consider multipliers that are **linear** functions of z.

Definition: For each $m \in \mathbf{Z}^{>0}$ and $\eta \in \mathbf{C}$, let $V(m,\eta)$ be the vector space of holomorphic functions which are doubly periodic with multiplier $\eta - mz$.

Proposition 5.4: $V(m,\eta)$ has dimension m.

Proof: Once again, we match Fourier series:

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n z} e^{2\pi i n \tau} = f(z+\tau) = e^{2\pi i(\eta - nz)} f(z) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i(n-m)z} e^{2\pi i \eta}$$

But this time, we can specify the coefficients $c_0, ..., c_{m-1} \in \mathbf{C}$ freely, after which the rest of the coefficients are constrained by:

$$c_{n+m} = c_n e^{2\pi i(n\tau - \eta)}.$$

Recall that $\mathrm{Im}(\tau) > 0$, and $|e^{a+bi}| = |e^a|$ (for $a, b \in \mathbf{R}$) so that for each integer k and $0 \leq n < m$, then $|c_{n+km}| = |c_n| e^{p(k)}$ where $p(k)$ is a quadratic polynomial in k with leading term $-k^2 \ \mathrm{Im}(\tau)$. But from this it follows that the lim sup conditions for convergence are automatically satisfied for every initial choice of $c_0, ..., c_{m-1}$, and we get the proposition.

Note: If $p \in \mathbf{C}/L$ and ϕ is meromorphic with multiplier $\eta - mz$, then it does not make sense to ask for the value of $\phi(p)$ because of the multiplier. Since the multiplier is a nowhere vanishing function, at least the zeroes and poles of ϕ (and multiplicities) are well-defined on $\mathbf{C}/L$! There is the usual convention that $p \in \mathbf{C}/L$ is a zero of multiplicity zero if ϕ has neither a zero nor a pole at p. Notice that although a nonzero ϕ may have infinitely many zeroes and poles on the complex plane, it must have only finitely many on the torus because the zeroes and poles are discrete and $\mathbf{C}/L$ is compact.

Definitions: (a) A formal sum:

$$D := \sum_{i=1}^{n} d_i p_i$$

is called a **divisor** on $\mathbf{C}/L$ if each $d_i \in \mathbf{Z}$ and each $p_i \in \mathbf{C}/L$. It is called an **effective** divisor if each $d_i \geq 0$.

(b) The **degree** of the divisor $D = \sum_{i=1}^{n} p_i d_i$ is the integer

$$\deg(D) := \sum_{i=1}^{n} d_i$$

(c) The **evaluation** of the divisor $D = \sum_{i=1}^{n} p_i d_i$ is the **point**

$$\mathrm{ev}(D) := d_1 p_1 + d_2 p_2 + \ldots + d_n p_n \in \mathbf{C}/L$$

(recall that $\mathbf{C}/L$ is an abelian group!)

Proposition 5.5: If $0 \neq \phi \in V(m, \eta)$, define an effective divisor

$$(\phi) := \sum_{p \in \mathbf{C}/L} d_p p$$

by letting d_p be the order of zero of ϕ at p. Then:

(a) $\deg((\phi)) = m$. In particular, ϕ has at most m distinct zeroes.

(b) if $0 \neq \psi \in V(m, \eta)$, then $\mathrm{ev}((\phi)) = \mathrm{ev}((\psi))$.

Proof: We use residues. The zeroes of $\phi : \mathbf{C} \to \mathbf{C}$ are discrete, so we can find a parallelogram $P(w)$ joining $w, w+1, w+1+\tau, w+\tau$ for some $w \in \mathbf{C}$ which does not pass through any of the zeroes of ϕ. Consider the integral:

$$\frac{1}{2\pi i} \int_{P(w)} \frac{\phi'(z)}{\phi(z)} dz.$$

Because $\frac{\phi'(z)}{\phi(z)}$ is singly periodic, two of the sides cancel, and the multiplier on ϕ becomes an "adder" $\frac{\phi'(z+\tau)}{\phi(z+\tau)} = \frac{\phi'(z)}{\phi(z)} - 2\pi i m$, which gives a contribution of m from the remaining two sides. Since this residue integral counts the number of zeroes of ϕ in $P(w)$, we get (a).

For (b), let $f(z) = \frac{\phi(z)}{\psi(z)}$ and consider the complex number:

$$c = \frac{1}{2\pi i} \int_{P(w)} z \frac{f'(z)}{f(z)} dz$$

where $P(w)$ does not contain any zeroes or poles of $f(z)$. This integral computes the sum of the **locations** of the zeroes and poles of $f(z)$ in $P(w)$, so to prove (b), it suffices to show that $c \in L$. But the (meromorphic) $f(z)$ is doubly periodic with respect to L, so after cancelling the common contributions from oppoisite sides, the integral becomes: $\frac{1}{2\pi i} \int_{w+1}^{w+1+\tau} \frac{f'(z)}{f(z)} dz + \frac{1}{2\pi i} \int_{w+1+\tau}^{w+\tau} \tau \frac{f'(z)}{f(z)} dz$, so indeed $c = a + b\tau$ (where the a and b are winding numbers, hence integers).

Thus for each fixed $m > 0$, we get a well-defined map:

$$\rho : \mathbf{C} \to \mathbf{C}/L; \quad \eta \mapsto \text{ ev}((\phi))$$

where ϕ is any nonzero element of $V(m, \eta)$. We need to understand this map:

Proposition 5.6: (a) $\overline{\rho} : \mathbf{C}/L \to \mathbf{C}/L$; $\overline{\eta} \mapsto$ ev$((\phi))$ is well-defined.

(b) $\overline{\rho}$ is injective and holomorphic, hence an isomorphism.

Proof: If $\eta' - \eta = a + b\tau$ and $\phi \in V(m, \eta)$, then $e^{2\pi i(a+bz)}\phi \in V(m, \eta')$. Since $e^{2\pi i(a+bz)}$ has no zeroes or poles, part (a) is immediate.

For (b), let's first prove that $\overline{\rho}$ is holomorphic. For each fixed m, let $\phi(z, \eta)$ be the function whose Fourier expansion (as a function of z) consists of $c_0 = 1$, $c_1 = \ldots = c_{m-1} = 0$, and the rest are determined. Then it is an exercise for you to see that ϕ is holomorphic, both as a function of z and as a function of η. But then for each η, we can evaluate by a residue integral:

$$\overline{\rho}(\eta) = \frac{1}{2\pi i} \int_{P(w)} z \frac{\frac{\partial \phi(z,\eta)}{\partial z}}{\phi(z, \eta)} dz$$

which is holomorphic in η, because we can differentiate under the integral.

Next, let's prove injectivity when $m = 1$. In this case, ev$((\phi))$ is the unique zero of ϕ, so that if $\phi \in V(1, \eta)$ and $\psi \in V(1, \eta')$ have the same evaluation, then they have the same zero with multiplicity one, so that the fraction ψ/ϕ is holomorphic. Moreover, it has multiplier $\eta' - \eta$, which, by Proposition 5.3, must therefore be in L. Thus $\overline{\rho}$ is injective. But now a holomorphic map is an open map, and an injective and holomorphic map from $\mathbf{C}/L$ to iself is an isomorphism. In particular, $\overline{\rho}$ is surjective.

For general m, suppose $\phi \in V(m, \eta)$ and $\psi \in V(m, \eta')$ and they evaluate to the same point in $\mathbf{C}/L$. Choose $\alpha \in \mathbf{C}/L$ so that $m\alpha = \eta \in \mathbf{C}/L$. If $\theta \in V(1, \alpha)$, it follows that θ has a unique zero at some mth root of ev$((\phi))$ because $\theta^m \in V(m, \eta)$. It follows from the injectivity in case $m = 1$ that there exist m^2 functions $\theta_i^m \in V(m, \eta)$ with zeroes of multiplicity m at each of the points β satisfying $m\beta =$ ev$((\phi))$. But of course the same is true for ψ, and as in the previous paragraph we can consider the ratio of two such with zeroes at the same point to conclude injectivity and the isomorphism.

Embedding Theorem: Choose a basis $\phi_1, ..., \phi_m$ for $V(m, \eta)$ and let:

$$\Phi = (\phi_1 : ... : \phi_m) : \mathbf{C}/L \to \mathbf{P}_{\mathbf{C}}^{m-1}$$

be the resulting map to projective space. Then:

(a) If $m \geq 2$, then Φ is well-defined.

(b) If $m \geq 3$, then Φ is injective and an embedding.

Proof: For (a), we have to show that there does not exist a $p \in L$ such that $\phi(p) = 0$ for all $\phi \in V(m, \eta)$. Suppose that there was such a p. By Proposition 5.6(b), we could find a $\psi \in V(1, \eta')$ such that ψ has a zero at p of multiplicity one. Now consider the functions ϕ_i/ψ. They are holomorphic, linearly independent, and all belong to $V(m-1, \eta - \eta')$ which contradicts the dimension of this vector space. Do notice that we need $m \geq 2$(!)

For the injectivity in (b), it suffices to show that for each $p, q \in \mathbf{C}/L$, there exists a $\phi \in V(m, \eta)$ such that $\phi(p) = 0$ but $\phi(q) \neq 0$. That's because if we write $\phi = \sum_{i=1}^{n} c_i \phi_i$, then $\Phi(p) = \Phi(q)$ implies $\phi_i(p) = \lambda \phi_i(q)$ for all i and some nonzero λ, hence $\phi(p) = \lambda \phi(q)$, which is of course impossible if $\phi(p) = 0$ and $\phi(q) \neq 0$. Consider the subspace $W \subset V(m, \eta)$ defined by:

$$W := \{\phi \in V(m, \eta) \mid \phi(p) = 0\}.$$

Choose $\psi \in V(1, \eta')$ with a zero at p (and not at q). Then multiplication by ψ gives an injective map of vector spaces: $V(m-1, \eta - \eta') \to W$. But now we are done, because $m - 1 \geq 2$, so by (a) there is a $\theta \in V(m-1, \eta - \eta')$ such that $\theta(q) \neq 0$, so that $\phi = \psi\theta$ has the desired property.

Suppose that $p \in \mathbf{C}/L$ and choose a $\phi_i \in V(m, \eta)$ such that $\phi_i(p) \neq 0$. Φ is an embedding at p iff for some $\phi \in V(m, \eta)$, the function $f(z) := \frac{\phi(z)}{\phi_i(z)}$ (which is holomorphic in a neighborhood of p) satisfies $f'(p) \neq 0$. But:

$$f'(z) = \frac{\phi'(z)\phi_i(z) - \phi(z)\phi_i'(z)}{\phi_i^2(z)}$$

so that it suffices to find a $\phi \in V(m, \eta)$ with the property that $\phi(p) = 0$ but $\phi'(p) \neq 0$. In other words, we want ϕ to vanish at p with multiplicity one. But this is not hard to arrange! Namely, letting $\psi \in V(1, \eta')$ be as before, if we choose $\theta \in V(m-1, \eta - \eta')$ such that $\theta(p) \neq 0$, then let $\phi = \psi\theta$. QED.

But what has all this to do with algebraic geometry? I'm glad you asked:

Degree 3 Embeddings: Let $\Phi : \mathbf{C}/L \to \mathbf{P}^2_{\mathbf{C}}$ be the embedding determined by $m = 3$ and $\eta \in \mathbf{C}$. Let C be the image of Φ. Then:

(a) Every projective line $l \subset \mathbf{P}^2_{\mathbf{C}}$ intersects C in exactly three points (counted with multiplicity on $\mathbf{C}/L$).

(b) The linear maps $e_d : \mathbf{C}[x,y,z]_d \to V(3d, d\eta); \quad x^a y^b z^c \mapsto \phi_1^a \phi_2^b \phi_3^c$ are surjective for all d and any basis ϕ_1, ϕ_2, ϕ_3 for $V(3, \eta)$.

(c) There is an irreducible $F \in \mathbf{C}[x,y,z]_3$ such that $C = V(F)$.

(d) $I(C)$ is prime and equal to $\langle F \rangle$, and $h_C(d) = 3d$.

So the embedded torus is an irreducible plane cubic!

Proof: A projective line is of the form $l = V(L)$, where $L = ax+by+cz$ is a linear form on $\mathbf{P}^2_{\mathbf{C}}$. Thus $l \cap C = \{p \in \mathbf{C}/L \mid a\phi_1(p) + b\phi_2(p) + c\phi_3(p) = 0\}$. Letting $\phi = a\phi_1 + b\phi_2 + c\phi_3$, then this is just the set of zeroes of ϕ, and (a) is equivalent to $\deg((\phi)) = 3$, which we proved in Proposition 5.5(a).

To prove (b), we observe that there exists a point $p \in \mathbf{C}/L$ and a basis $\phi_1, \phi_2, \phi_3 \in V(3, \eta)$ such that the multiplicities of ϕ_1, ϕ_2, ϕ_3 at p are $3, 1, 0$ respectively (exercise!). Then there are monomials of degree d in the ϕ_i with multiplicity $3d, 3d-2, 3d-3, ..., 0$ at p (write them down!) which are necessarily linearly independent in $V(3d, d\eta)$, hence form a basis by Proposition 5.4. Thus e_d is surjective for this choice of ϕ_1, ϕ_2, ϕ_3, hence for any choice.

When $d = 3$, we have $\dim(\mathbf{C}[x,y,z]_3) = \binom{5}{2} = 10$ and $\dim(V(9, 3\eta)) = 9$, so e_3 has a one-dimensional kernel. Let $F \neq 0$ be an element of the kernel, and observe that we at least have $C \subseteq V(F)$ by definition! Moreover, computing dimensions in case $d = 1, 2$ shows that there is NO nontrivial kernel in these cases. But if $e_2(Q) = \psi \neq 0$ and $e_1(L) = \theta \neq 0$, then $e_3(QL) = \psi\theta \neq 0$, so that F must be irreducible(!)

To finish the proof of (c), I would like to argue as follows. For every projective line l, we know that $l \cap V(F)$ consists of three points (HW §2) because F is irreducible, and $l \cap C$ consists of three points, by (a), so that of course $C = V(F)$. The problem lies in the multiplicities, which are calculated differently in the two cases. We have to be more careful.

Suppose $p \in V(F) - C$. If I can find a projective line l through p which meets C in three **distinct** points, then l meets $V(F)$ in at least four points, from which it follows that $l \subset V(F)$, contradicting the irreducibility of F (why?). How to do this? Well, consider the map $\pi_p \circ \Phi : \mathbf{C}/L \to \mathbf{P}^1_{\mathbf{C}}$. This is defined everywhere and holomorphic. This map is surjective, and its differential is necessarily nonzero except for finitely many points of $\mathbf{C}/L$ (the derivative of a holomorphic function has discrete zeroes!). So if $q \in \mathbf{P}^1_{\mathbf{C}}$ stays away from the image of these zeroes, it follows that the line $\pi_p^{-1}(q) \cup \{p\}$ intersects C with multiplicity one at all points, and we're done.

For (d), let $A = \mathbf{C}[x, y, z]$. Then $h_A(d) - h_{A(-3)}(d) = 3d$ if $d > 0$ so that in light of (b), there is no room in $I(C)$ for anything but $\langle F \rangle$. The argument that F is irreducible is readily modified to prove that $I(C)$ is prime.

Finally, let's investigate the addition on this embedded torus:

The Group Law on the Cubic: Suppose $\overline{\rho}(\eta) = 0$, and $\Phi : \mathbf{C}/L \hookrightarrow \mathbf{P}^2_{\mathbf{C}}$ is the embedding associated to $V(3, \eta)$. Let $e = \Phi(0) \in \mathbf{P}^2_{\mathbf{C}}$. Then:

(a) If p, q, r are distinct points of the torus, then $p + q + r = 0 \in \mathbf{C}/L$ iff $\Phi(p), \Phi(q), \Phi(r)$ lie on a projective line.

(b) $\Phi(p + q)$ can be constructed as follows. Let $a = \Phi(p), b = \Phi(q)$ and let c be the third intersection point of $l_{a,b} \cap C$. Then $\Phi(p + q)$ is the third intersection point of $l_{e,c} \cap C$. In case $a = b$, there is a unique "tangent" line to C passing through a with multiplicity > 1, and we use this line in place of $l_{a,b}$. We proceed in the same way if $c = e$.

(c) $x \in C$ is called an inflection point if there is a line $l \subset \mathbf{P}^2_{\mathbf{C}}$ such that $l \cap C$ has multiplicity three at p. There are exactly 9 such points, and e is one of them. Moreover, rule (b) still defines an abelian group structure on C if I replace e by any other inflection point.

Proof: The points $\Phi(p), \Phi(q), \Phi(r)$ lie on a projective line $l = V(L)$ iff they are the zeroes of a function $\phi \in V(3, \eta)$. If $\phi = a\phi_1 + b\phi_2 + c\phi_3$, then $L = ax + by + cz$ and vice versa. Thus if their images lie on a line, then $p + q + r = 0$ by Proposition 5.5(b). But conversely, given p, q, r, by 5.6(b) applied to $m = 1$, I can always find $\phi = \theta_1\theta_2\theta_3 \in V(3, \eta')$ for some η' which vanishes at p, q, r, and by 5.6(b), it follows that $\eta' = \eta$, so that their images lie on a line. This proves (a).

Part (b) is an immediate consequence of (a) unless $a = b$ or $c = e$. In those cases, it is a consequence of the following variant of (a):

$$2p + q = 0 \in \mathbf{C}/L \;\Leftrightarrow\; \Phi(q) \text{ lies on the tangent line to } C \text{ at } \Phi(p)$$

But this is proved exactly as in the previous paragraph.

As a special case of the rule above, we see that $e = \Phi(0)$ must be an inflection point because $0 + 0 + 0 = 0$. There are precisely 9 points $\tau \in \mathbf{C}/L$ which satisfy $3\tau = 0$, which yield the 9 inflection points. Suppose we translate the torus by τ. Then $p + q + r$ and $2p + q$ are translated by $3\tau = 0$, so the rules above are unchanged(!) except that e is replaced by the other inflection point $\Phi(\tau)$, and the group structure defined is simply the group structure on the translation by τ. This proves (c).

Remarks: Some possible generalizations of this discussion come to mind:

(i) The group law, defined in terms of "points lying on a line" makes sense over any field k. Does this always define an abelian group? Answer: Yes, so long as the irreducible cubic is smooth (which guarantees that tangent lines are unique) and contains an inflection point. We'll investigate this later.

(ii) Is every smooth irreducible cubic in $\mathbf{P}^2_{\mathbf{C}}$ the embedded image of a torus? Yes. One can see this by some honest hard work with bases or invoking the Riemann-Roch theorem.

(iii) What can we say about the images of the torus in other projective spaces? Answer: The image is always an irreducible projective variety. It is quite attractive in case $m = 4$, where it is a complete intersection of two smooth quadrics. After that, it's not a complete intersection, and a little harder to study.

(iv) Do we get even more interesting embeddings if we consider other multipliers, for instance multipliers which are quadratic polynomials in z? Answer: Surprisingly, no. The only embeddings of the torus in complex projective space are the ones listed above and compositions with projections. We'll need Chow's lemma to prove this.

Fifth Homework Assignment

1. (a) For any $m > 0$ and $\eta \in \mathbf{C}$, prove that there exists a point $p \in \mathbf{C}/L$ and a basis $\phi_1, ..., \phi_n \in V(m, \eta)$ such that the zero multiplicities of ϕ_i at p are $m, m-2, m-3, ..., 0$, respectively.

(b) Prove that if $m > 2$, then for every $d \geq 0$, the maps:

$$e_d : \mathbf{C}[x_1, ..., x_n]_d \to V(md, d\eta);\; x_1^{i_1}...x_n^{i_n} \mapsto \phi_1^{i_1}...\phi_n^{i_n}$$

are all surjective. What goes wrong if $m = 2$?

2. Suppose that D is an effective divisor on $\mathbf{C}/L$ and that $\deg(D) = m$. Prove that $D = (\phi)$ for a unique $\eta \in \mathbf{C}$ and a unique (up to multiplication by a scalar) $\phi \in V(m, \eta)$.

3. Degree 2 maps:

(a) Prove that the map $\Phi : \mathbf{C}/L \to \mathbf{P}^1_{\mathbf{C}}$ associated to $V(2, \eta)$ is surjective.

(b) Find points $p_1, p_2, p_3, p_4 \in \mathbf{C}/L$ such that Φ is a covering space of degree two (to its image) when restricted to $U = \mathbf{C}/L - \{p_1, p_2, p_3, p_4\}$.

(c) For each p_i, prove that there exist holomorphic isomorphisms from neighborhoods U_i of p_i and V_i of $\Phi(p_i)$ to the unit disk about the origin in $\mathbf{C}$ such that the induced map $U_i \to V_i$ is just $z \mapsto z^2$.

(d) Prove, more generally, that any holomorphic map from one compact complex manifold of dimension one to another is either the constant map or else is surjective. Prove that it is a covering off of finitely many points, and that there are neighborhoods of these points and isomorphisms as in (c) such that the map becomes $z \mapsto z^e$ for some e (which depends upon the point).

4. Degree 4 Embeddings:

(a) Find a basis $\phi_1, \phi_2, \phi_3, \phi_4$ for any $V(4, \eta)$ such that $\phi_1\phi_2 - \phi_3\phi_4 = 0$. Thus $C = \Phi(\mathbf{C}/L)$ is contained in our old friend $Q = V(xy - zw)$.

(b) Prove that every projective line in Q intersects C in two points (counted with multiplicty on $\mathbf{C}/L$).

(c) Find an $F' \in k[x, y, z, w]_2$ such that $C \subset Q' = V(F') \neq Q$.

(d) Prove that $C = Q \cap Q'$, C is a variety, and $I(C) = \langle xy - zw, F' \rangle$.