Course Notes for Math 737 (Algebraic Geometry)

0.2 Pre-Introduction. Sheaves, Manifolds and Vector Bundles. We define sheaves, and investigate the two most important examples from geometry: the sheaf (of commutative rings with 1) of differentiable or holomorphic functions on a manifold and the sheaf (of abelian groups) of differentiable or holomorphic sections of a vector bundle.

What is a sheaf? Suppose that T is a topological space and that $\mathcal{F}$ is a presheaf of rings (or abelian groups) on T. The we say that $\mathcal{F}$ is a **sheaf** if it satisfies $\mathcal{F}(\emptyset) = \{0\}$ and the following additional two properties:

Suppose $\cup_{\lambda\in\Lambda} U_\lambda = U$ is an open cover of an open subset of T. Then:

(i) If $s \in \mathcal{F}(U)$ and $\rho_{U,U_\lambda}(s) = 0$ for all $\lambda \in \Lambda$, then $s = 0$.

(ii) If there exist $s_\lambda \in \mathcal{F}(U_\lambda)$ for all $\lambda \in \Lambda$ such that for all $\lambda, \mu \in \Lambda$, we have: $\rho_{U_\lambda,U\lambda\cap U_\mu}(s_\lambda) = \rho_{U_\mu,U_\lambda\cap U_\mu}(s_\mu)$, then there exists an $s \in \mathcal{F}(U)$ (necessarily unique by (i)!) such that $\rho_{U,U\lambda}(s) = s_\lambda$ for all $\lambda \in \Lambda$.

Example: The presheaf $\mathcal{C}_{T,R}$ of continuous functions from T to a continuous ring R (or continuous abelian group) is a sheaf. That's because:

(i) If $s : U \to R$ and $s|_{U_\lambda} \equiv 0$ for all λ, then s is the zero function!

(ii) If $s_\lambda : U_\lambda \to R$ are functions for all λ, and if two such functions restrict to the same function on the overlaps, then they glue together to define a function $s : U \to R$. (The value of s at a point is well-defined, as it doesn't depend upon which open set U_λ contains that point!)

Lest you think every presheaf is a sheaf:

Nonexamples: (i) Suppose T is a topological space with the property that there exists an open cover $T = \cup U_\lambda$ by **proper** open subsets of T. (Most topological spaces have this property!) Suppose that R is a nonzero ring, and let F be the presheaf defined by

$$F(U) = \begin{cases} 0 \text{ if } U \neq T \\ R \text{ if } U = T \end{cases}$$

and all the $\rho_{U,V}$ are the zero homomorphism (except for $\rho_{T,T}$, of course!) Then F violates (i) for the cover $T = \cup U_\lambda$, so it isn't a sheaf.

(ii) Suppose T is a topological space which contains disjoint (nonempty!) open subsets U_1 and U_2 (some interesting topological spaces do not have this property!) and R is a nonzero ring. Let F be the presheaf defined by:

$$F(U) = \begin{cases} R \text{ if } U \neq \emptyset \\ 0 \text{ if } U = \emptyset \end{cases}$$

and let the $\rho_{U,V}$ all be the identity, except for $\rho_{U,\emptyset}$, which is forced to be zero. Then if we choose $s_1 = 0 \in F(U_1)$ and $s_2 \neq 0 \in F(U_2)$, we cannot find an $s \in F(U_1 \cup U_2)$ satisfying (ii). (Do note, though, that (i) is satisfied!)

On to some more interesting examples:

What is a differentiable manifold? An n-dimensional differentiable manifold is built out of open subsets of $\mathbf{R}^n$ and "gluing" diffeomorphisms:

(The Gluing construction) Given open subsets $B_1, B_2, ... \subseteq \mathbf{R}^n$ and open subsets $U_{i,j} \subset B_i$ such that each $U_{i,i} = B_i$, and further, that each $U_{i,j}$ is related to $U_{j,i}$ by a diffeomorphism: $\phi_{i,j} : U_{i,j} \xrightarrow{\sim} U_{j,i}$ with the following properties:

(i) $\phi_{i,i}$ is the identity map for all i.

(ii) $\phi_{j,i} = \phi_{i,j}^{-1}$ for all i, j and

(iii) $\phi_{i,j}(U_{i,j} \cap U_{i,k}) = U_{j,i} \cap U_{j,k}$ for all i, j, k and:

$$\phi_{j,k} \circ \phi_{i,j} = \phi_{i,k} \text{ as functions from } U_{i,j} \cap U_{i,k} \text{ to } B_k.$$

Then we build a topological space M as follows:

$$M = \coprod_{i=1}^{\infty} B_i / \sim \ \text{ where } p \sim q \Leftrightarrow p \in U_{i,j}, q \in U_{j,i} \text{ and } \phi_{i,j}(p) = q.$$

The conditions (i)-(iii) imply that $\sim$ is an equivalence relation and that the natural inclusions $B_i \hookrightarrow \coprod B_i$ induce inclusions $B_i \hookrightarrow M$. We will be sloppy and identify each B_i with its image. We put the topology on M generated by the open subsets of the B_i. The fact that the $\phi_{i,j}$ are homeomorphisms implies that $U \subset M$ is open if and only if $U \cap B_i$ is open in B_i for all i. Finally, observe that $U_{i,j} = B_i \cap B_j$, when the intersection is thought of as a subset of B_i.

Forming a topological space M in this way allows us to define C^∞ functions on each open subset of M. This information is packaged as a sheaf:

Definition: The **presheaf of C^∞ functions** on a topological space M constructed as above is denoted by $\mathcal{C}_M^\infty$ and is defined as follows:

(a) $\mathcal{C}_M^\infty(U) = \{f : U \to \mathbf{R} \mid f|_{U \cap B_i} \text{ is } C^\infty \text{ on } U \cap B_i \subset B_i \text{ for each } i.\}$

(b) $\rho_{U,V}$ is the restriction of functions.

$\mathcal{C}_M^\infty$ is a **sheaf** of commutative rings with 1. This is proved exactly as in the earlier example of the sheaf $\mathcal{C}_{T,\mathbf{R}}$ of continuous functions.

Definition: A **differentiable premanifold** is a topological space M together with a sheaf $\mathcal{C}_M^\infty$ which is constructed as above.

So why don't we call these things manifolds? It's because of the:

Bad Example: Suppose $B_1 = \mathbf{R} = B_2$, $U_{1,2} = U_{2,1} = \mathbf{R} - \{0\}$, and $\phi_{1,2} = \text{ id } = \phi_{2,1}$. Then the gluing construction gives an M which looks like $\mathbf{R}$ with two origins. This isn't Hausdorff!

A manifold is a Hausdorff premanifold. More precisely, suppose:

(i) N is a Hausdorff topological space,

(ii) $\mathcal{C}_N^\infty$ is a sheaf of commutative rings with 1 which is a subsheaf of $\mathcal{C}_{N,\mathbf{R}}$. (By this I mean that each $s \in \mathcal{C}_N^\infty(U)$ is a continuous function $s : U \to \mathbf{R}$ and the restriction map on $\mathcal{C}_N^\infty$ is just restriction of functions.)

Then:

Defintion: The pair $(N, \mathcal{C}_N^\infty)$ is a **differentiable manifold** if there is a premanifold M and a homeomorphism $h : N \to M$ such that for every open subset $U \subset M$, the natural "pull-backs" $h^* : \mathcal{C}_{M,\mathbf{R}}(U) \to \mathcal{C}_{N,\mathbf{R}}(h^{-1}(U))$ restrict to **isomorphisms**:

$$h^* : \mathcal{C}_M^\infty(U) \xrightarrow{\sim} \mathcal{C}_N^\infty(h^{-1}(U))$$

Note:: It is certainly not a given that the image of $\mathcal{C}_M^\infty(U)$ lies in $\mathcal{C}_N^\infty(h^{-1}(U))$ under h^*, though it is a given that the restriction of h^* to $\mathcal{C}_M^\infty(U)$ is injective (but not that it is surjective!).

Notice also that the essential data in defining a manifold consists of a topological space **and** a sheaf.

Definition: If $(N, \mathcal{C}^\infty_N)$ and $(N', \mathcal{C}^\infty_{N'})$ are differentiable manifolds, then a map $f : N \to N'$ is called a **differentiable map** if

(i) f is continuous, and

(ii) the image of each $\mathcal{C}^\infty_{N'}(U)$ under the pull-back f^* is in $\mathcal{C}^\infty_N(f^{-1}(U))$ (informally: "f^* pulls back C^∞ functions to C^∞ functions".)

Remark: One needs to check, of course, that this is well-defined. That is, if $(N, \mathcal{C}^\infty_N)$ is a differentiable manifold in more than one way, for example, via homeomorphisms $h : N \to M$ and $h' : N \to M'$ to premanifolds, then the definition of a differentiable map is unaffected, and similarly for N'. But this is easily checked! Also notice that in spite of its importance, we will often leave the sheaf out of our notation, usually referring to a manifold as "M".

Extension: If the subsets B_i are in $\mathbf{R}^{2n} = \mathbf{C}^n$ and the $\phi_{i,j}$ are holomorphic, then one defines M in the same way, and the sheaf $\mathcal{O}_M$ of **holomorphic functions** on M analogously. Complex premanifolds, **complex manifolds** and **holomorphic maps** of complex manifolds are all defined analogously.

Definition: (a) The collection of differentiable manifolds M together with differentiable maps is often called "the differentiable (or C^∞) category".

(b) The collection of complex manifolds M together with holomorphic maps is often called "the holomorphic category".

Observation: If M is a differentiable manifold, then each open $U \subset M$ admits a sheaf of C^∞ functions which makes it a manifold. Its topology is the induced topology, and its sheaf of C^∞ functions is the sheaf $\mathcal{C}^\infty_U$ defined by:

$$\mathcal{C}^\infty_U(V) := \mathcal{C}^\infty_M(V) \text{ for all open subsets } V \subset U.$$

The inclusion map $U \hookrightarrow M$ becomes a differentiable map, of course.

The same method works in the holomorphic category. Please do notice (or remember) that there is **no** analogous method for making closed subsets of M into manifolds (even though they do inherit an induced topology).

We can define "germs " of functions on a manifold as a special case of:

Definition: Suppose $\mathcal{F}$ is a presheaf of rings (or abelian groups) on T and that $p \in T$. Then the **stalk** of $\mathcal{F}$ at p is the direct limit:

$$\mathcal{F}_p := \lim_{\rightarrow} F(U)$$

over all open sets U containing p.

That is, a pair $(U, s \in \mathcal{F}(U))$ determines an element of $\mathcal{F}_p$, and two pairs (U, s) and (V, t) determine the same element of the stalk iff there is an open subset $W \subset U \cap V$ such that $\rho_{U,W}(s) = \rho_{V,W}(t)$.

Germs: If M is a differentiable (resp. complex) manifold, then an element of a stalk $\mathcal{C}^\infty_{M,p}$ (resp. $\mathcal{O}_{M,p}$) is called a **germ** of functions at p.

Observations: (1) The **evaluation** at p of a germ of functions at p is well-defined. In the C^∞ case, this is a real number, while in the holomorphic case, it is a complex number. This is obvious from the definition.

(2) The rings of germs: $\mathcal{C}^\infty_{M,p}$ or $\mathcal{O}_{M,p}$ are **local** rings. The maximal ideal is the set of germs that evaluate to 0 at p. Any other germ has a "germ" reciprocal, because a function which is nonzero at p is nonzero in a neighborhood of p (continuity!).

What is a Vector Bundle? Suppose M is a C^∞ manifold of dimension n.

Suppose that $M = \cup_{i=1}^\infty U_i$ is an open cover of M by countably many open subsets, and that for each i, j, there are C^∞ "transition functions":

$$g_{i,j} : U_i \cap U_j \to \ \mathrm{GL}(r, \mathbf{R}).$$

(To be C^∞ means that each coordinate of the matrix is in $\mathcal{C}^\infty_M(U_i \cap U_j)$.) Suppose that the maps satisfy:

(i) $g_{i,i}$ is the constant function mapping to the identity matrix for all i,

(ii) $g_{i,j}(p) = g_{j,i}^{-1}(p)$ for all i, j and $p \in U_i \cap U_j$ and

(iii) For all i, j, k and $p \in U_i \cap U_j \cap U_k$, we have:

$$g_{j,k}(p) * g_{i,j}(p) = g_{i,k}(p).$$

Then we build a space E and map $\pi : E \to M$ as follows. As a set,

$$E = \coprod_{i=1}^\infty U_i \times \mathbf{R}^r / \sim$$

where $(p, v) \in U_i \times \mathbf{R}^r$ and $(q, w) \in U_j \times \mathbf{R}^r$ are equivalent if and only if $p = q \in U_i \cap U_j$, and $g_{i,j}(p)(v) = w$. Notice that $\sim$ is an equivalence relation because of (i),(ii) and (iii). Moreover, notice that the map:

$$\pi : E \to M, \text{ given by } \pi\overline{(p, v)} = p$$

is well-defined. As in the gluing construction of M, the sets $U_i \times \mathbf{R}^r$ are naturally subsets of E, and E is a topological space, with $U \subset E$ open if and only if $U \cap (U_i \times \mathbf{R}^r)$ is open for all i. You check that E is Hausdorff.

If $V \subset U_i \times \mathbf{R}^r$, we'll say $f : V \to \mathbf{R}$ is a C^∞ function if $f|_{V \cap (B_j \times \mathbf{R}^r)}$ is C^∞ for all B_j. Then define the "sheaf of C^∞ functions on E" by letting:

$$\mathcal{C}^\infty_E(U) = \{f : U \to \mathbf{R} \mid f|_{U \cap (U_i \times \mathbf{R}^r)} \text{ is a } C^\infty \text{ function for all } i\}.$$

Lemma: E together with its sheaf $\mathcal{C}^\infty_E$ of C^∞ functions is a differentiable manifold and $\pi : E \to M$ is a differentiable map.

Proof: Exercise.

Definition: Suppose M and F are differentiable manifolds and $\sigma : F \to M$ is a C^∞ map. Then we say that the pair (F, σ) is a **(geometric) vector bundle over** M if there is a pair (E, π) built out of transition functions as above, and a diffeomorhism $h : F \to E$ such that $\pi \circ h = \sigma$.

The Tangent Bundle. (The most important example!) Let M be a manifold constructed out of the gluing functions:

$$\phi_{i,j} : U_{i,j} \to U_{j,i}$$

and let $\phi^k_{i,j}$ denote the k-th component of $\phi_{i,j}$, thought of as a map to $\mathbf{R}^n$. Let $M = \cup_{i=1}^\infty U_i$ be the open cover by subsets $U_i = B_i$ (identified with their images in M), so that $U_i \cap U_j = U_{i,j}$, and let:

$$g_{i,j} = \left(\frac{\partial \phi^k_{i,j}}{\partial x_l} \right) : U_i \cap U_j \to \ \mathrm{GL}(n, \mathbf{R}).$$

Then the functions $g_{i,j}$ satisfy all the desired properties for transition functions. This is because of the **chain rule**. The vector bundle built out of these transition functions is called the **tangent bundle** and is denoted by T_M (the map to M is almost always denoted by π).

Suppose M is a differentiable manifold and $\pi : E \to M$ is a vector bundle.

Definition: If $U \subset M$ is an open set, then a C^∞ **section of** E **over** U is a differentiable map:

$$s : U \to \pi^{-1}(U)$$

with the property that $\pi|_{\pi^{-1}(U)} \circ s : U \to U$ is the identity.

Let $\mathcal{E}(U)$ be the set of C^∞ sections of E over U. Define operations:

$$+ : \mathcal{E}(U) \times \mathcal{E}(U) \to \mathcal{E}(U) \quad \text{and} \quad * : \mathcal{C}^\infty_M(U) \times \mathcal{E}(U) \to \mathcal{E}(U)$$

as follows. For each $p \in U$, choose a some U_i such that $p \in U_i$, and consider $U_i \times \mathbf{R}^r \hookrightarrow E$. It follows that for each $s, t \in \mathcal{E}(U)$, we can write $s(p) = (p, v)$ and $t(p) = (p, w)$ for some vectors $v, w \in \mathbf{R}^n$. If in addition, $f \in \mathcal{C}^\infty_M(U)$, then define:

$$(s + t)(p) := (p, v + w) \text{ and } fs(p) = (p, f(p) \cdot v)$$

Of course, this definition involved a choice of U_i. But since the transition functions $g_{i,j}$ take values in $\mathrm{GL}(r, \mathbf{R})$, it follows that this definition is independent of the choice of U_i. Moreover, it clearly takes sections to sections and preserves the C^∞-ness of the section since that is locally defined, anyway(!)

Remark: Notice that there is a well-defined **zero section** of E over each open $U \subset M$, but there is no a priori reason why, for instance, the group $\mathcal{E}(M) \neq \{0\}$ (and it often is in the holomorphic category!).

Propostion: The assignments:

$U \mapsto \mathcal{E}(U)$ together with "+" and ordinary restriction of sections

defines a **sheaf** of abelian groups (which we denote by $\mathcal{E}$.) This is called the **sheaf of** C^∞ **sections of** E.

Proof: Everything is nearly obvious. The sheaf property follows just as in the earlier examples of sheaves of functions, because as in the case of functions, a section is zero if it is zero locally, and it is defined if it is defined locally in such a way that the local definitions agree on the overlaps!!

But we can do a little better than this!

Definition: Suppose T is a topological space, $\mathcal{O}$ is a sheaf of commutative rings with 1 on T, and $\mathcal{F}$ is a sheaf of abelian groups on T. Then $\mathcal{F}$ is called a **sheaf of $\mathcal{O}$-modules** if:

(i) Each $\mathcal{F}(U)$ has the structure of a $\mathcal{O}(U)$-module such that:

(ii) For each $V \subset U$ and $s \in \mathcal{F}(U), f \in \mathcal{O}(U)$, we have:

$$\rho_{U,V}(f) \cdot \rho_{U,V}(s) = \rho_{U,V}(f \cdot s).$$

Proposition: Given E, then the structure given above for $\mathcal{E}(U)$ as a $\mathcal{C}^\infty_M(U)$-module makes $\mathcal{E}$ into a sheaf of $\mathcal{C}^\infty_M$-modules.

Proof: Obvious from the definitions.

A final remark/definition. If $U \subset U_i$ for some U_i, then $\mathcal{E}(U)$ is isomorphic to the **free** module $(\mathcal{C}^\infty_M(U_i))^{\oplus r}$ of rank r. That's because when we indentify $U_i \times \mathbf{R}^r$ with $\pi^{-1}(U_i)$, then the C^∞ sections can be identified with the C^∞ maps:

$$s = (s_1, ..., s_r) : U_i \to \mathbf{R}^r$$

In general, if $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules with this property (i.e. that there is a cover of T by open subsets U_i such that each $\mathcal{F}(U_i)$ is free), then we will say that $\mathcal{F}$ is a **locally free** sheaf of $\mathcal{O}$-modules.

Definition: Θ_M is the sheaf of C^∞ sections of T_M.

And all this can be similarly defined in the holomorphic category!

Some Manifold and Vector Bundle Examples

Example 1: Projective Space. (Compare with Property 1 in §1) We are given $B_0, ..., B_n \subset \mathbf{R}^{n+1}$ defined as: $B_i := \{(a_0, ..., a_{i-1}, 1, a_{i+1}, ..., a_n)\}$. Of course, these are not open in $\mathbf{R}^{n+1}$, but they are each **equal** to $\mathbf{R}^n$, when we just forget the ith coordinate. The open subsets are:

$$U_{i,j} := \{(a_0, ..., a_n) \in B_i \ |a_j \neq 0\}$$

and the gluing functions are:

$$\phi_{i,j}(a_0, ..., a_n) = (\frac{a_0}{a_j}, ..., \frac{a_n}{a_j})$$

which do indeed map $U_{i,j}$ to $U_{j,i}$ are are quite definitely C^∞ mappings (and holomorphic if we started with $B_i \subset \mathbf{C}^{n+1}$ with the same definition).

Form $\mathbf{P}^n_\mathbf{R}$ by the gluing construction.

Exercises: (a) $\mathbf{P}^n_\mathbf{R}$ is Hausdorff. Hence a C^∞ manifold.

(b) The map $\pi : \mathbf{R}^{n+1} - (0, ..., 0) \to \mathbf{P}^n_\mathbf{R}$ defined by:

$$\pi(a_0, ..., a_n) = (\frac{a_0}{a_i}, ..., \frac{a_n}{a_i}) \in B_i \text{ if } a_i \neq 0$$

is well-defined and differentiable, as is the induced map from S^n to $\mathbf{P}^n_\mathbf{R}$, which proves that $\mathbf{P}^n_\mathbf{R}$ is compact. (Property 3 from §1).

(c) Convince yourself that all this works for $\mathbf{R}$ replaced by $\mathbf{C}$, giving rise to the complex manifold $\mathbf{P}^n_\mathbf{C}$ which is likewise compact.

Specific Example: The Riemann Sphere. You may have learned about the Riemann Sphere in another context, namely, as $\mathbf{C} \cup \{\infty\}$, where:

(i) neighborhoods of ∞ are complements of closed, bounded sets in $\mathbf{C}$,

(ii) $f(z)$ is holomorphic at ∞ if $f(\frac{1}{z})$ is holomorphic at 0, etc.

Notice that this is the **same** information provided by $\mathbf{P}^1_\mathbf{C}$ with its sheaf $\mathcal{O}_{\mathbf{P}^1_\mathbf{C}}$ of holomorphic functions. In this case, $\mathbf{P}^1_\mathbf{C}$ is formed from $B_0, B_1 = \mathbf{C}$ with gluing functions $\phi_{0,1}(z) = \frac{1}{z}$ and $\phi_{1,0}(w) = \frac{1}{w}$. Notice also:

(a) $\mathcal{O}_{\mathbf{P}^1_{\mathbf{C}}}(\mathbf{P}^1_{\mathbf{C}}) = \mathbf{C}$, the constant functions. Suppose $s \in \mathcal{O}_{\mathbf{P}^1_{\mathbf{C}}}$. Then the two restrictions $s_0 := \rho_{\mathbf{P}^1,B_0}(s)$ and $s_1 := \rho_{\mathbf{P}^1,B_1}(s)$ must be holomorphic functions of z and w, respectively. Thus,

$$s_0(z) = a_0 + a_1 z + a_2 z^2 + ... \text{ and } s_1(w) = b_0 + b_1 w + b_2 w^2 + ...$$

But because they "glue together" to the global section s, it follows that $s_1(\frac{1}{z}) = s_0(z)$ on the overlap, i.e. $a_0 + a_1 z + ... = b_0 + b_1 z^{-1} + ...$ so that $a_0 = b_0$ and all the other terms are zero(!)

(b) $\Theta_{\mathbf{P}^1_{\mathbf{C}}}(\mathbf{P}^1_{\mathbf{C}}) \cong \mathbf{C}^3$, as a vector space over the constant functions(!) (Recall that Θ_M is a sheaf of $\mathcal{O}_M$-modules, so that when $\mathcal{O}_M(M)$ is a field, then $\Theta_M(M)$ is a vector space.) Suppose that $t \in \Theta_{\mathbf{P}^1_{\mathbf{C}}}(\mathbf{P}^1_{\mathbf{C}})$. Then the restrictions t_0, t_1 to B_0 and B_1, repsectively, may be thought of as functions $t_0 : B_0 \to \mathbf{C}$ and $t_1 : B_1 \to \mathbf{C}$ that satisfy the following on the overlap:

$$t_1(\frac{1}{z}) = \left(\frac{\partial w}{\partial z}\right) t_0(z) = -\frac{1}{z^2} t_0(z)$$

or, if $t_0(z) = a_0 + a_1 z + ...$ and $t_1(w) = b_0 + b_1 w + ...$, then:

$$a_0 z^{-2} + a_1 z^{-1} + a_2 + a_3 z + ... = b_0 + b_1 z^{-1} + b_2 z^{-2} + b^3 z^{-3} + ...$$

so that $a_0 = b_2$, $a_1 = b_1$, $a_2 = b_0$ and all other terms are zero(!)

This is the first case of the following important "rigidity" properties of compact complex manifolds (which we will state without proof). Namely, let M be a compact complex manifold. Then:

Rigidity 1: $\mathcal{O}_M(M) = \mathbf{C}$, the constant functions.

Rigidity 2: Let $\pi : E \to M$ be **any** holomorphic vector bundle on M. Then $\mathcal{E}(M)$ is a finite-dimensional vector space over $\mathbf{C}$.

Note that this rigidity does NOT apply to differentiable manifolds M. (Hint: Think about bump functions!)

Final Exercise: Prove by hand (as above) that:

(a) $\mathcal{O}_{\mathbf{P}^n_{\mathbf{C}}}(\mathbf{P}^n_{\mathbf{C}}) = \mathbf{C}$ and (b) $\Theta_{\mathbf{P}^n_{\mathbf{C}}}(\mathbf{P}^n_{\mathbf{C}}) \cong \mathbf{C}^{n^2+2n}$