Course Notes for Math 738 (Algebraic Geometry)

14. Varieties. Projective Varieties. We show how projective varieties are abstract varieties (i.e. separated prevarieties). We prove that a product of projective varieties is again projective, using the Segre embedding, and prove that the "basic" open subsets $U_F \subset X$ of a projective variety are affine, using the Veronese embedding. Finally, we show that a projective variety over an algebraically closed field is proper.

Suppose $X \subset \mathbf{P}^n_k$ is a projective variety over k.

Definition: The topology on X is the Zariski topology, and the sheaf $\mathcal{O}_X$ is defined as follows:

$$\mathcal{O}_X(U) := \{\phi \in k(X) \mid U \subset \ \mathrm{dom}(\phi)\}$$

with the inclusions of rings as restriction maps.

Exercise: We've already seen in §1 that $\phi \in \mathcal{O}_X(U)$ can be thought of as a function from U to k. Prove that it is continuous, when k has the cofinite topology, and that $\phi \in k(X)$ is determined by the corresponding function.

Proposition 14.1: $\mathbf{P}^n_k$ is a prevariety over k.

Proof: For $i = 0, ..., n$ let $U_{x_i} := \{(p_0 : ... : p_n) \in \mathbf{P}^n_k \mid p_i \neq 0\}$ and let

$h_i : U_{x_i} \to V_i := V(y_i - 1) \subset \mathbf{A}^{n+1}_k$ be given by $h_i(p_0 : ... : p_n) = (\dfrac{p_0}{p_i}, ..., \dfrac{p_n}{p_i})$.

Clearly h_i is a bijection. If $f \in k[y_0, ..., y_n] = k[\mathbf{A}^{n+1}_k]$ has degree $\leq d$, then the "homogenized" $F := x_i^d f(\frac{x_0}{x_i}, ..., \frac{x_n}{x_i})$ is homogeneous of degree d and $h_i^{-1}(V_i \cap V(f)) = U_{x_i} \cap V(F)$, so h_i^{-1} takes closed sets to closed sets. If $F \in k[x_0, ..., x_n]_d$, then because $x_i^d F(\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}) = F(x_0, ..., x_n)$, we have $h_i(U_{x_i} \cap V(F)) = V_i \cap V(F(y_0, ..., y_n))$, so h_i is a homeomorphism.

Next, consider the isomorphism of fields $h_i^* : k(V_i) \xrightarrow{\sim} k(\mathbf{P}^n_k)$ given by:

$$\frac{f(y_1, ..., y_{i-1}, 1, y_{i+1}, ..., y_n)}{g(y_1, ..., y_{i-1}, 1, y_{i+1}, ..., y_n)} \mapsto \frac{x_i^d f(\frac{x_0}{x_i}, ..., \frac{x_n}{x_i})}{x_i^d g(\frac{x_0}{x_i}, ..., \frac{x_n}{x_i})}$$

for any $d \geq \ \max\{\deg(f),\ \deg(g)\}$. If $\phi \in \mathcal{O}_{V_i}(U)$, then $h_i^*(\phi) \in \mathcal{O}_{\mathbf{P}^n_k}(h_i^{-1}(U))$ and as a function from $h_i^{-1}(U)$ to k this agrees with the usual definition of the pull-back. Thus $\mathbf{P}^n_k$ is a prevariety and $\{U_{x_i}, h_i\}$ is an affine atlas.

Corollary 14.2: $X \subset \mathbf{P}_k^n$ with sheaf $\mathcal{O}_X$ is a prevariety over k, and if $U_{x_i} := X - V(x_i)$, then $\{U_{x_i}, h_i|_{U_{x_i}}\}$ is an affine atlas.

Proof: In the proof that a closed irreducible subset $Z \subset Y$ of a prevariety over k is another prevariety (§13), we showed in particular that if Y is affine, then the construction of a topology and sheaf on Z reproduced the topology and sheaf of Z as an affine variety. The same proof applies in case Y is a projective variety (replacing ratios of polynomials by ratios of homogeneous polynomials of the same degrees). Thus we get the corollary from Proposition 14.1 since we have already seen in §13 that $(X, \mathcal{O}_X)$ is a prevariety with the indicated affine atlas.

Definition: A prevariety X is **projective** if it is isomorphic to a projective variety, i.e. if there is a closed immersion $\Phi : X \hookrightarrow \mathbf{P}_k^n$. X is **quasi-projective** if there is an open immersion $\Psi : X \hookrightarrow Y$ where Y is projective.

Example: An affine variety (or more generally, a quasi-affine variety) over k is also quasi-projective. Given an affine variety V, let $V \subset \mathbf{A}_k^n$ be a particular embedding, and consider the projective variety $\overline{V} \subset \mathbf{P}_k^n$ defined as in §4. Then by construction, the first chart of the atlas on $\overline{V}$ is the homeomorphism $h_0 : U_{x_0} \to V$, which yields $h_0^{-1} : V \hookrightarrow \overline{V}$ as an open immersion. In the case of a quasi-affine variety, simply observe that a composition of open immersions is an open immersion.

But can a projective variety be quasi-affine? Certainly, this can happen over non-algebraically closed fields. For instance an ellipse in $\mathbf{R}^2$ is both affine and projective (see §4). Before we answer this for algebraically closed fields, we need a definition of "balanced" localizations:

Definition: If A is a graded commutative ring with 1, if $F \in A_d$, and if $\mathcal{P} \subset A$ is a homogeneous prime ideal, define:

$$(A_F)_0 := \{\frac{G}{F^n} \mid n \geq 0 \text{ and } G \in A_{dn}\}/\sim \quad \text{and}$$

$$(A_{\mathcal{P}})_0 := \{\frac{G}{H} \mid G, H \text{ are homogeneous of the same degree and } H \notin \mathcal{P}\}/\sim$$

These are commutative rings (not graded!). $(A_{\mathcal{P}})_0$ is a local ring with maximal ideal consisting of the fractions with numerator belonging to $\mathcal{P}$. If A is an integral domain, then the field of fractions of each localization is the balanced field of fractions $(k(A))_0$ of §1.

Proposition 14.3: Suppose $X \subset \mathbf{P}^n_k$ is a projective variety, $p \in X$, and let $U_F := X - V(F)$ for a homogeneous $F \in k[X]$. Then

(a) $k(X)$ is the same field, whether defined as the field of rational fractions on X (as in §1) or as in §13 using the fact that X is a prevariety.

(b) $\mathcal{O}_{X,p} = (k[X]_{m_p})_0 \subset k(X)$, where $m_p := I(p)$

And if the field k is algebraically closed, then:

(c) If F is homogeneous of positive degree, then $\mathcal{O}_X(U_F) = (k[X]_F)_0$.

(d) $\mathcal{O}_X(X) = k = k[X]_0$.

Proof: By definition, each $\mathcal{O}_X(U)$ is a subring of the field of rational fractions of $k[X]$, so we can interpret

$$k(X) := \lim_{\rightarrow\, U \subset X} \mathcal{O}_X(U) \quad \text{and } \mathcal{O}_{X,p} := \lim_{\rightarrow\, p \in U \subset X} \mathcal{O}_X(U)$$

as unions in that field, from which (a) is clear and (b) follows as in 13.1(a).

Given $\phi \in k(X)$, let I be the homogeneous ideal generated by the denominators of ϕ. If $G \in I$ is homogeneous, then G is a denominator of ϕ. And as before, $\mathrm{dom}(\phi) = X - V(I)$. Now suppose F is homogeneous of positive degree and $\phi \in \mathcal{O}_X(U_F)$. By the projective Nullstellensatz (§12), we have $F^n \in I$, hence $\mathcal{O}_X(U_F) \subseteq (k[X]_F)_0$, which is the non-obvious part of (c).

We cannot conclude (d) similarly because the projective Nullstellensatz only tells us that if $\phi \in \mathcal{O}_X(X)$, then $I = k[X]$ **or** $\mathrm{rad}(I) = m$, where m is the image in $k[X]$ of the irrelevant maximal ideal. Instead, observe that $\phi \in \mathcal{O}_X(U_{x_i}) = (k[X]_{x_i})_0$ for all $i = 0, ..., n$ by (c), so we may write:

$$\phi = \frac{F_0}{x_0^{d_0}} = ... = \frac{F_n}{x_n^{d_n}}$$

for homogeneous elements $F_i \in k[X]_{d_i}$. If we let $d \geq \sum_{i=0}^n d_i$, it follows that if $G \in k[X]_d$, then $G\phi \in k[X]_d$. This is because any monomial summand of a representative of G in $k[x_0, ..., x_n]$ must contain at least one x_i to the power d_i or more, by assumption on d. Iterating, we get $G\phi^m \in k[X]_d$ for all $m \geq 0$. Then for any nonzero $G \in k[X]_d$, we obtain $k[X][\phi]$ as a sub-module of the (graded) $k[X]$-module $G^{-1}k[X] := \{\frac{f}{G} \mid f \in k[X]\}$, which is finitely generated (in fact, principally generated).

Thus, since $k[X]$ is a Noetherian ring, it follows that $k[X][\phi]$ is also a finitely generated graded $k[X]$-module. But then it follows that $k[X][\phi]$ is integrally dependent upon the subring $k[X]$, so ϕ must satisfy an monic polynomial:

$$\phi^m + f_1\phi^{m-1} + ... + f_m = 0; \quad f_i \in k[X]$$

and we may assume that the $f_i \in k[X]_0$ because ϕ has degree 0. That is, the f_i are **constants**. But k is algebraically closed and ϕ is continuous, so $\phi \in k$, which gives the non-obvious inclusion in (d).

Corollary 14.4: Suppose k is algebraically closed.

(a) If X is projective and quasi-affine, then X is a point.

(b) More generally, if $\Phi : X \to V$ is a regular map from a projective prevariety to an affine variety, then Φ is the constant map.

Proof: Suppose Φ is a map as in (b). Choose a particular $V \subset \mathbf{A}_k^n$. Then the coordinate functions $x_i \in \mathcal{O}_V(V)$ pull back to $\Phi^*(x_i) \in \mathcal{O}_X(X)$, which must be constants by 14.3(d). Thus the map is the constant map. (a) is a special case where the map Φ is an open immersion.

Remark: If we regard rational functions on projective varieties as analogues of meromorphic functions on compact, complex manifolds, then 14.3(d) is an version of the maximum principle, which tells us that the only holomorphic functions on a compact, complex manifold are the constants.

Next suppose $X \subset \mathbf{P}_k^n$ and and $F_0, ..., F_m \in k[X]_d$ are homogeneous of the same degree. Then although the F_i are not functions from X to k, they almost define a map to $\mathbf{P}_k^m$. In fact, the "map":

$$\Phi : X - - > \mathbf{P}_k^m \text{ given by } p \mapsto (F_0(p) : ... : F_m(p))$$

is well-defined and continuous on the locus $U := X - V(F_0, ..., F_m)$. But it may be possible to extend the definition of Φ beyond U. Namely, we may have a $q \notin U$ and $G_0, ..., G_m \in k[X]_e$ such that $(G_0(q) : ... : G_m(q))$ is well-defined and $(G_0(p) : ... : G_m(p)) = (F_0(p) : ... : F_m(p))$ for all p where both are defined. It is a consequence of continuity and the Zariski topology that provided such G's exist, they uniquely determine an extension of Φ to a continuous function whose domain includes an open neighborhood of q.

Definition: A **rational map** $\Phi : X - - > Y$ between projective varieties $X \subset \mathbf{P}_k^n$ and $Y \subset \mathbf{P}_k^m$ is a continuous function from a nonempty open subset of X to Y determined by $F_0, ..., F_m \in k[X]_d$ as above (with the additional restriction that the image of Φ should lie in Y). The **domain** of Φ is the largest set to which the definition may be extended.

Examples: (a) Any rational map $\Phi : \mathbf{P}_k^1 - - > Y$ has all of $\mathbf{P}_k^1$ as its domain(!) If we are given $F_0, ..., F_m \in k[x, y]_d$ and $(p : q) \in V(F_0, ..., F_m)$, then we can factor $xq - yp$ out of each F_i, and continue until we obtain $G_0, ..., G_m \in k[x, y]_e$ which determine the same map as the F's and satisfy $V(G_0, ..., G_m) = \emptyset$.

(b) Suppose $(F_0 : ... : F_m)$ and $(G_0 : ... : G_m)$ determine the same rational map $\Phi : X - - > Y$. Then $F_i G_j = F_j G_i$ for all i and j since $\frac{F_i}{F_j} = \frac{G_i}{G_j}$ as rational functions. It follows that if $k[X]$ is a UFD (which turns out to be quite rare!) and $F_0, ..., F_m \in k[X]_d$ do not all share an irreducible factor, then each F_i **divides** G_i above, and Φ cannot be extended beyond $X - V(F_0, ..., F_m)$. Unlike (a), the domain may not always be all of X. For instance, since $k[\mathbf{P}_k^n]$ is a UFD, and since the projection maps π_Λ of §2 are determined by linearly independent elements of $k[\mathbf{P}(V)]_1$ which therefore share no common factor, we see that the domain of π_Λ cannot be extended to any points of $\Lambda \subset \mathbf{P}(V)$. (One could also argue this by continuity.)

(c) Recall the basic conic $X = V(xy - z^2) \subset \mathbf{P}_k^2$ and the rational map $\pi : X - - > \mathbf{P}_k^1$ given by $\pi(p : q : r) = (p : q)$ or $\pi(p : q : r) = (q : r)$. Notice $X = \text{dom}(\pi)$, but neither of these expressions captures the entire domain.

Compare the following with the corresponding example in §13.

Proposition 14.5: A regular map (in the category of prevarieties) of projective varieties is the same as a rational map which is everywhere defined.

Proof: Suppose $X \subset \mathbf{P}_k^n$ and $Y \subset \mathbf{P}_k^m$ are projective varieties. Suppose $\Phi : X \to Y$ is a regular map and $\Phi(p) \in U_{y_i} \subset Y$. By definition, $\Phi^*(\frac{y_j}{y_i}) = \phi_j \in k(X)$ where $\phi_j \sim \frac{F_j}{G_j}$ for $F_j, G_j \in k[X]_{d_j}$ and $G_j(p) \neq 0$. It follows that the map:

$$(F_0 \prod_{j \neq 0} G_j : ... : F_m \prod_{j \neq m} G_j)$$

is defined at p (the ith coordinate is nonzero) and moreover agrees with Φ on the locus where it is defined. Each different choice of p gives maps which agree on an open subset of X, hence must agree everywhere by continuity, so indeed Φ is a rational map which is everywhere defined.

Conversely, suppose $\Phi \sim (F_0 : ... : F_m)$ is defined at p, say, $\Phi(p) = q$. Then $\Phi^*(\frac{A}{B}) = \frac{A(F_0,...,F_m)}{B(F_0,...,F_m)}$ and the denominator does not vanish identically provided $B(q) \neq 0$. From this it follows that $\Phi^*(\mathcal{O}_Y(U)) \subseteq \mathcal{O}_X(\Phi^{-1}(U))$ for all U containing q. Thus if Φ can be defined at all p, it follows that Φ is regular as a map of prevarieties.

Here is a fundamental example:

Projective Linear Transformations: Suppose $A = (a_{i,j}) \in \mathrm{GL}(n+1,k)$ (where we index i and j from 0 to n). Then A determines the regular map

$$\Phi_A : \mathbf{P}^n_k \to \mathbf{P}^n_k \text{ given by } (p_0 : ... : p_n) \mapsto (\sum_{i=0}^n a_{0,i}p_i : ... : \sum_{i=0}^n a_{n,i}p_i)$$

which is an **isomorphism** with inverse $\Phi_{A^{-1}}$. Moreover, it is easy to check that $\Phi_A = \Phi_B$ if and only if $B = \lambda A$ for some $\lambda \in k^*$, hence the group $\mathrm{PGL}(n+1,k) := \mathrm{GL}(n+1,k)/k^*$ is a subgroup of the automorphism group $\mathrm{Aut}(\mathbf{P}^n_k)$ (in fact, we will see that it is the entire automorphism group).

Proposition 14.6: A quasi-projective variety is separated.

Proof: It suffices to show that projective varieties are separated (§13), and it even suffices to show that $\mathbf{P}^n_k$ is separated (also §13). So the proposition follows if we can show that for each prevariety X over k and regular maps $\Phi, \Psi : X \to \mathbf{P}^n_k$, the set $\{\Phi = \Psi\} \subset X$ is closed (see §0.3).

Given $p, q \in \mathbf{P}^n_k$, we can always find a linear form $L = \sum_{i=0}^n a_i x_i$ such that $p, q \notin V(L) \subset \mathbf{P}^n_k$. A suitable linear transformation takes $V(L)$ to $V(x_0)$, hence $U_L \cong U_{x_0}$ which is one of the affines in our atlas for $\mathbf{P}^n_k$. Thus any pair of points in $\mathbf{P}^n_k$ are contained in an open affine subvariety.

Now, suppose $\Phi(p) \neq \Psi(p)$. Then we can find an open affine $V \subset \mathbf{P}^n_k$ such that $\Phi(p), \Psi(p) \in V$ by the above discussion, and since affine varieties are separated, there must be an open neighborhood $p \in U \subset \Phi^{-1}(V) \cap \Psi^{-1}(V)$ such that $\Phi(q) \neq \Psi(q)$ for all $q \in U$. Thus $\{\Phi = \Psi\} \subset X$ is closed.

Given positive integers n and m, let $N = (n+1)(m+1) - 1$ and consider the **Segre embedding** $s_{n,m} : \mathbf{P}^n_k \times \mathbf{P}^m_k \to \mathbf{P}^N_k$ given by:

$$s_{n,m}((p_0 : ... : p_n), (q_0 : ... : q_m)) = (p_0 q_0 : p_1 q_0 : ... : p_i q_j : ... : p_n q_m)$$

Proposition 14.7: The Segre embedding is a closed immersion.

Proof: $s_{n,m}$ is well-defined and injective as a mapping of sets (see §2). Consider the homogeneous prime ideal $I \subset k[x_{00}, x_{10}, ..., x_{ij}, ..., x_{nm}] = k[\mathbf{P}^N_k]$ defined as the kernel of the ring homomorphism $k[\mathbf{P}^N_k] \to k[\mathbf{P}^n_k] \otimes k[\mathbf{P}^m_k]$ determined by $x_{ij} \mapsto x_i \otimes y_j$. Then clearly $s_{n,m}(\mathbf{P}^n_k \times \mathbf{P}^m_k) \subseteq X := V(I)$. Each of the quadrics $Q_{i,j,k,l} := x_{ij}x_{kl} - x_{il}x_{kj}$ belongs to I, so in particular $X \subseteq V(\{Q_{i,j,k,l}\})$. Suppose we are given $(r_{00} : ... : r_{nm}) \in V(\{Q_{i,j,k,l}\})$ with some $r_{ij} \neq 0$. Then setting $p_k = r_{kj}/r_{ij}$ and $q_l = r_{il}$, we get $r_{kl} = p_k q_l$ for all k and l, so $X \subseteq V(\{Q_{i,j,k,l}\}) \subseteq s_{n,m}(\mathbf{P}^n_k \times \mathbf{P}^m_k)$, hence we have equalities. Since I is prime, we see that X is a projective variety.

Consider the open subsets $U_{x_{ij}} \subset X$. Since $p_i q_j \neq 0$ if and only if $p_i \neq 0$ and $q_j \neq 0$, the Segre embedding restricts to maps:

$$s_{n,m} : U_{x_i} \times U_{y_j} \to U_{x_{ij}}$$

If V_i, W_j and V_{ij} are the corresponding affine varieties via the affine atlases, then the induced map $V_i \times W_j \to V_{ij}$ is:

$$((a_0, ..., 1, ..., a_n), (b_0, ..., 1, ..., b_m)) \mapsto (a_0 b_0, ..., 1, ..., a_n b_m)$$

which is a regular map of affine varieties (recall that $V_i \times W_j \subset \mathbf{A}^{n+m+2}_k$). It follows from Lemma 13.3(a) that $s_{n,m}$ is a regular map. But since each a_k and b_l appears on the right of the expression (as the kj and il terms respectively), we see that the projection onto these coordinates gives a regular inverses to the maps, and it follows by the same lemma that $s^{-1}_{n,m} : X \to \mathbf{P}^n_k \times \mathbf{P}^m_k$ is regular, hence $s_{n,m}$ a closed immersion.

Corollary 14.8: A product of quasiprojective varieties is projective.

Proof: Suppose $U \subseteq X \subset \mathbf{P}^n_k$ and $V \subseteq Y \subset \mathbf{P}^m_k$ are quasiprojective varieties. Then the inclusion $U \times V \subseteq X \times Y \subset \mathbf{P}^n_k \times \mathbf{P}^m_k$ is an open immersion followed by a closed immersion, and since a composition of closed immersions is a closed immersion, it follows that $U \times V$ is isomorphic to its image under the Segre embedding, which is a quasiprojective variety.

Next, given positive integers n and d, let $N = \binom{n+d}{d} - 1$ and consider the **Veronese embedding** (a.k.a. the d-uple embedding): $v_{n,d} : \mathbf{P}^n_k \to \mathbf{P}^N_k$ defined as follows. Choose your favorite ordering of the degree d monomials $M_I(x_0, ..., x_n) = x_0^{i_0}...x_n^{i_n}$ multi-indexed by $I = (i_0, ..., i_n)$. (You should check that $N+1$ is the total number of such monomials). Then

$$v_{n,d}(p_0 : ... : p_n) = (\{M_I(p_0 : ... : p_n)\}) \text{ (with your chosen ordering)}$$

is a regular map by Proposition 14.5 since, of course, $V(\{M_I\}) = \emptyset$.

Example: The rational normal curves of §3 are the images:

$$v_{1,d}(p : q) = (p^d : p^{d-1}q : ... : q^d)$$

Proposition 14.9: The Veronese embeddings are closed immersions.

Proof: Let $\mathcal{P} \subset k[\mathbf{P}^N_k]$ be the homogeneous, prime ideal which is the kernel of the ring homomorphism $k[\mathbf{P}^N_k] \to k[\mathbf{P}^n_k]$ given by $x_I \mapsto M_I$ for each multi-index I. Then $v_{n,d}(\mathbf{P}^n_k) \subseteq X := V(\mathcal{P})$, and X is a projective variety. For each quadruple of multi-indices I, J, K, L satisfying $I + J = K + L$, we obtain a quadric $Q_{I,J,K,L} := x_I x_J - x_K x_L$ which belongs to $\mathcal{P}$. It is left as an exercise to check that if $(\{r_I\}) \in V(\{Q_{I,J,K,L}\})$ and some $r_J \neq 0$, then there is a point $(p_0 : ... : p_n)$ so that $r_I = M_I(p_0 : ... : p_n)$ for all multi-indices I. Once this is accomplished, we conclude that $X = V(\{Q_{I,J,K,L}\}) = v_{n,d}(\mathbf{P}^n_k)$.

Next, suppose that $I_0 = x_0^d, I_1 = x_0^{d-1}x_1, ..., I_n = x_0^{d-1}x_n$. Then the projection map $\pi : v_{n,d}(\mathbf{P}^n_k) - - > \mathbf{P}^n_k$ given by $\pi(\{r_I\}) \mapsto (r_{I_0} : ... : r_{I_n})$ extends to a regular map (why?) which is the inverse to $v_{n,d}$.

Corollary 14.10: If $X \subset \mathbf{P}^n_k$ is a projective variety and $F \in k[X]$ is homogeneous of positive degree, then $U_F := X - V(F)$ is an **affine** variety.

Proof: Write $F = \sum a_I M_I$ as a sum of distinct monomials of degree d. Consider the image $Y := v_{n,d}(X) \subset \mathbf{P}^N_k$ under the Veronese embedding. Then Y is a projective variety isomorphic to X because of the Proposition and the fact that a composition of closed immersions is a closed immersion. Moreover, if we let $L = \sum a_I x_I \in k[Y]_1$, then $v_{n,d}^{-1}(U_L) = U_F$, so it suffices to show that $U_L \subset Y$ is an affine variety. But a suitable linear transformation takes $V(L)$ to $V(x_0)$ and Y to an isomorphic $Y' \subset \mathbf{P}^N_k$, so $U_L \cong U_{x_0} \subseteq Y'$ which is affine by 14.2.

Remark: If k is algebraically closed, then since U_F is affine, we know further that $k[U_F] \cong (k[X]_F)_0$ by 14.3 and 12.2.

As some motivation for the next theorem, consider the central question of the classical subject of elimination theory:

Question: Given polynomials (over an algebraically closed field k):

$$f_1(x_0, ..., x_n; y_1, ..., y_m), ..., f_r(x_0, ..., x_n; y_1, ..., y_m)$$

with the property that each f_i is homogeneous in the variables $x_0, ..., x_n$, then do there exist auxiliary polynomials

$$g_1(y_1, ..., y_m), ..., g_s(y_1, ..., y_m)$$

also with coefficients in k such that $g_1(p) = ... = g_s(p) = 0$ if and only if $f_1(a_0, ..., a_n; p) = ... = f_r(a_0, ..., a_n; p) = 0$ for some $(a_0, ..., a_n) \neq (0, ..., 0)$? (I.e. can we "eliminate" the homogeneous variables?)

The zero loci $V(f_1, ..., f_r) \subset \mathbf{P}^n_k \times \mathbf{A}^m_k$ are closed subsets (why?), and the question is equivalent to asking whether the image of $V(f_1, ..., f_r)$ under the projection map:

$$\pi_2 : \mathbf{P}^n_k \times \mathbf{A}^m_k \to \mathbf{A}^m_k$$

is also closed. Thus this is a special case of the theorem below.

Remark: Before we get too smug and congratulate ourselves on being better mathematicians than our nineteenth century counterparts, let me point out that they were also interested in **how** to construct the g's from the f's, a question about which the theorem below says nothing.

Theorem 14.11: Projective varieties are proper if k is algebraically closed.

Proof: (due to Grothendieck) We want to prove that given a projective variety $X \subset \mathbf{P}^n_k$ and a prevariety Y, the map $\pi_2 : X \times Y \to Y$ is closed. By HW # 3(c), it suffices to assume $X = \mathbf{P}^n_k$. And if $Y = \cup U_i$ is an open cover, then π_2 is closed if and only if $\pi_2 : \mathbf{P}^n_k \times U_i \to U_i$ is closed for all i, so that we may additionally assume that Y is affine.

Let $S := k[Y]$, and cover $\mathbf{P}^n_k \times Y$ by open affines $U_i := U_{x_i} \times Y$ whose coordinate rings therefore satisfy $k[U_i] \cong (k[\mathbf{P}^n_k]_{x_i})_0 \otimes_k S \cong S[\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}]$. Finally, let $A := S[x_0, ..., x_n]$ be graded by degree in the x_i's.

If $Z \subset \mathbf{P}^n_k \times Y$ is a closed subset, let:

$$I_d := \{F \in A_d \mid F(\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}) \in I(Z \cap U_i) \text{ for all } i\}$$

and define $I(Z) := \sum I_d$, noting that this is a homogeneous ideal in A. We have $\pi_2(Z) \subseteq V(I_0) \subset Y$ by definition and we need to prove $\pi_2(Z) = V(I_0)$.

Claim: The zero locus $V(I(Z)) \subset \mathbf{P}^n_k \times Y$ is equal to Z.

Proof of the claim: Suppose $f \in I(Z \cap U_i)$. For large enough d, we have $G := x_i^d f \in A_d$, and each $f_j := G(\frac{x_0}{x_j}, ..., \frac{x_n}{x_j}) = \frac{G}{x_j^d}$ vanishes on $Z \cap U_i \cap U_j$. Thus even if $f_j \notin I(Z \cap U_j)$, then $\frac{x_i}{x_j} f_j \in I(Z \cap U_j)$. So any $F := x_i^e f \in I_e$ for $e > d$ has the property that $F(\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}) = f$. If $p \notin Z$ and $p \in U_i$, then there is an $f \in I(Z \cap U_i)$ such that $f(p) \neq 0$, and any $F \in I_e$ constructed above has the same property, so $Z = V(I(Z))$.

Next, suppose $q \notin \pi_2(Z)$ and let $m_q \subset S$ be the corresponding maximal ideal. Consider $\pi_2^{-1}(q) = \mathbf{P}^n_k \times \{q\}$. This closed set is disjoint from Z, so intersecting with each U_i gives disjoint closed sets $Z \cap U_i$ and $U_{x_i} \times \{q\}$ in U_i with ideals $I(Z \cap U_i)$ and $m_q S[\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}]$ respectively. Thus

$$I(X \cap U_i) + m_q S[\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}] = S[\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}]$$

by the Nullstellensatz. In other words, for each $i = 0, ..., n$ there exist elements $f_i \in I(X \cap U_i), m_{i,j} \in m_q$ and $a_{i,j} \in S[\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}]$ such that:

$$f_i + \sum m_{i,j} a_{i,j} = 1$$

Multiplying through by sufficiently large powers e_i of the x_i, we get:

$$F_i + \sum m_{i,j} H_{i,j} = x_i^{e_i} \text{ for } F_i \in I_{e_i}, H_{i,j} \in A_{e_i}$$

(the fact that $F_i \in I_{e_i}$ for all sufficiently large e_i was proved in the claim). If we further take $N \geq \sum e_i$, then $I_N + m_q A_N = A_N$ from the above equalities, since every monomial of degree N has contains at least one x_i to the power e_i or more. Now consider the S-module A_N/I_N. We have $m_q(A_N/I_N) = A_N/I_N$, hence by Nakayama 2 (HW # 6), there is an $f \in S - m_q$ that annihilates A_N/I_N, that is, $fA_N \subseteq I_N$. Thus $fx_i^N \in I_N$ for all i and so $f \in I_0$. Thus for an arbitrary $q \notin \pi_2(Z)$, we have found an $f \in I_0$ so that $f(q) \neq 0$. But this implies $V(I_0) = \pi_2(Z)$, as desired.

Ninth Homework Assignment

1. Suppose $X \subset \mathbf{P}^n_k$ and $Y \subset \mathbf{P}^m_k$ are projective varieties.

(a) Show that the homogeneous coordinate ring of the Segre embedding of $X \times Y$ is isomorphic to the graded ring:

$$\oplus_{d=0}^{\infty} k[X]_d \otimes k[Y]_d \subset k[X] \otimes k[Y]$$

Conclude that the Hilbert polynomial (see §3) of the Segre embedding of the product is the product of the Hilbert polynomials.

(b) Show that the image of the Veronese embedding is $V(\{Q_{I,J,K,L}\})$ where the $Q_{I,J,K,L}$ are the quadrics in the proof of Proposition 14.9. Show that the rational map $\pi : v_{n,d}(\mathbf{P}^n_k) - - > \mathbf{P}^n_k$ in that Proposition really is a regular inverse to $v_{n,d}$.

(c) Show that the homogeneous coordinate ring of $v_{n,d}(\mathbf{P}^n_k)$ is isomorphic to

$$\oplus_{e \geq 0} k[\mathbf{P}^n_k]_{de} \subset k[\mathbf{P}^n_k]$$

Conclude that the Hilbert polynomial $\mathrm{H}_{v_{n,d}(X)}(e)$ of the Veronese embedding of X is equal to $\mathrm{H}_X(de)$. Compare the degrees and leading coefficients of the two polynomials.

2. Prove that the addition law on a cubic from §5 is a regular map from $C \times C \to C$.

3. Show that proper varieties have the following properties:

(a) If $\Phi : X \to Y$ is a regular map of varieties and X is proper, then the image of Φ is closed and irreducible in Y, and proper.

(b) If X and Y are proper, then $X \times Y$ is proper.

(c) If X is proper and $Y \subset X$ is closed and irreducible, then Y is proper.

4. Suppose $\Phi : \mathbf{P}^n_k \to \mathbf{P}^m_k$ is a nonconstant regular map and $k = \overline{k}$.

(a) Prove that Φ is a finite map.

(Hint: Projective Nullstellensatz)

(b) Prove that $n \leq m$, and that Φ is dominant if and only if $n = m$. Conclude that $\mathbf{P}^n_k \times \mathbf{P}^m_k \not\cong \mathbf{P}^{n+m}_k$ if m and n are positive integers.

(c)

5. Suppose $X \subset \mathbf{P}_k^n$ and $Y \subset \mathbf{P}_k^m$ are projective varieties over an algebraically closed field. An element $F \in k[X]_d \otimes k[Y]_e$ is called **bihomogeneous of bidegree** (d, e).

(a) Prove that the zero locus $V(F) \subset X \times Y$ of a bihomogeneous function is closed. (We'll call such things "bihomogeneous hypersurfaces".)

(b) Prove that every closed subset in $X \times Y$ is the intersection of finitely many bihomogeneous hypersurfaces.

(c) Prove that a regular map $\Phi : \mathbf{P}_k^n \times \mathbf{P}_k^m \to \mathbf{P}_k^l$ can always be expressed as:

$$\Phi(p, q) = (F_0(p, q) : \ldots : F_l(p, q))$$

where the F_i are bihomogeneous of the same bidegree and $V(F_0, \ldots, F_l) = \emptyset$.

6. Look up or prove on you own:

Nakayama 2: If S is a commutative ring with 1, M is a finitely generated S-module and $I \subset S$ is an ideal such that

$$IM = M$$

then there is an $f \in S$ such that $f - 1 \in I$ and $fM = 0$.

Show how Nakayama 2 implies the Nakayama Lemma of §13.