Course Notes for Math 737 (Algebraic Geometry)

0.3 Pre-Introduction. More Category Theory. Products and, more generally, fibred products are defined, with numerous examples. Category-theoretic versions of the properties "Hausdorff" and "compact" are defined. It is observed, in particular, that using fibred products, these can be thought of as properties of **morphisms**.

Definition (of Products): Suppose $\mathcal{C}$ is a category and $A, B, C \in \mathrm{Ob}(\mathcal{C})$. We say C is a product of A and B if it satisfies the following two properties:

(Existence of Projections) There exist two "projection" morphisms $\pi_A : C \to A$ and $\pi_B : C \to B$ with the following:

(Universal Property) To each $C' \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $f_A : C' \to A$ and $f_B : C' \to B$, there is a **unique** morphism $f : C' \to C$ such that $f_A = \pi_A \circ f$ and $f_B = \pi_B \circ f$.

Lemma 0.3.1: A product, **if it exists**, is unique up to isomorphism.

Proof: Suppose C and C' are two products of A and B and that the maps π_A and π_B, f_A and f_B, defined as above, are the projections. Then because C and C' are both products, let $f : C' \to C$ and $g : C \to C'$ be the morphisms produced by the universal property. Then consider the composition $f \circ g : C \to C$. It has the property that $\pi_A \circ f \circ g = \pi_A$ and $\pi_B \circ f \circ g = \pi_B$, which happens to also be the case when $f \circ g$ is replaced by id(!) But since C is a product, the uniqueness in the universal property implies $f \circ g =$ id and similarly $g \circ f =$ id. In other words, g is a two-sided inverse to f, so f and g are isomorphisms(!!)

Examples: (a) The Cartesian product of sets is a product in $\mathcal{S}$. The very notation (a, b) for an element of $A \times B$ gives us our map $C' \to A \times B$ defined by $f(p) = (f_A(p), f_B(p))$. Of course, any set bijective to $A \times B$ can be thought of as a product (just get the projections right!).

(b) In the category $\mathcal{T}$, the Cartesian product of A and B, together with the product topology (i.e. the topology on $A \times B$ defined by letting products of open sets be a basis for all the open sets) is a product. This is because the map $C' \to A \times B$ from (a) is continuous for the product topology. (Why?)

(c) Ordinary products of groups, rings, etc. are indeed products.

(d) Let R be a commutative ring with 1, and consider the category $\mathcal{C}(R)$ whose objects are the commutative algebras over R (i.e. commutative rings containing R as a subring with the same 1) and whose morphisms are the R-homomorphisms (i.e. ring homomorphisms which are the identity on R). Then the tensor product $A \otimes_R B$ is **not** a product in this category, because it satisfies all the rules, but **with the arrows reversed(!)** This is called a **coproduct**. Note: products in $\mathcal{C}$ are coproducts in its dual $\mathcal{C}^*$.

(e) Recall we let $\mathcal{P}(S)$ be the category of subsets of S with inclusions as the morphisms. In this category, both products and coproducts exist. Intersections are the products and unions are the coproducts!

Exercises: (a) In which of the following categories are ordinary products also coproducts? Sets, abelian groups, vector spaces, comm. rings with 1.

(b) Once "pairwise" products exist in a category, then products of three or more objects exist (what's the definition?), and agree up to isomorphism with the products formed two-at-a-time.

Proposition: Products of differentiable and holomorphic manifolds exist.

Proof: (1) We first prove that if $B \subset \mathbf{R}^n$ and $B' \subset \mathbf{R}^m$ are open subsets, then $B \times B' \subset \mathbf{R}^{n+m}$ is the product **in the category of differentiable manifolds**. To do this, I claim first that $f = (f_1, ..., f_n) : M \to B$ is a differentiable map if and only if each $f_i \in \mathcal{C}_M^\infty(M)$. One direction: if f is differentiable, then $f_i = f^*(x_i) \in \mathcal{C}_M^\infty(M)$. The other direction: any $\phi(x_1, ..., x_n) \in \mathcal{C}_B^\infty(U)$ is by definition a $\mathcal{C}^\infty$ function of the variables $x_1, ..., x_n$, so $f^*(\phi) = \phi(f_1, ..., f_n)$ is C^∞ on $f^{-1}(U)$ by the chain rule (checked locally!). But now (1) is easy. Given $f : M \to B$ and $g : M \to B'$, the (unique!) lift to $(f_1, ..., f_n, g_1, ..., g_m) : M \to B \times B'$ is a differentiable map.

(2) Start with the gluing data defining manifolds M and M'. That is, assume we are given open subsets $B_i \subset \mathbf{R}^n$ and $B'_{i'} \subset \mathbf{R}^m$, as well as open subsets $U_{i,j}, U'_{i',j'}$ and gluing maps $\phi_{i,j}$ and $\phi'_{k',l'}$ satisfying all the rules. Then we form the new manifold from the following data:

(a) For each (i, i'), the products: $B_i \times B'_{i'}$

(b) For each (i, i') and (j, j'), the open subset $U_{i,j} \times U_{i',j'} \subset B_i \times B_{i'}$.

(c) Gluing functions $\phi_{(i,i'),(j,j')}(p, q) = (\phi_{i,j}(p), \phi'_{i',j'}(q))$.

Then using the gluing construction, we obtain a premanifold which is homeomorphic to the Cartesian product $M \times N$ with the product topology (hence a manifold(!)), for which the (Cartesian) projections to M and N are differentiable maps. Furthermore, this is a product in the category of differentiable manifolds because by (1), the lifted map $f : P \to M \times N$ restricts to a differentiable map on the preimage of each $B_i \times B'_{i'}$, hence is differentiable... (fill in the details...one uses the sheaf properties of $\mathcal{C}^\infty_{M\times N}$).

Remark: Thus $M \times N$ is the Cartesian product (as a set) with the product topology (as a topological space). It may almost seem as though we don't need the gluing construction at all!. But remember, although the topological space $M \times N$ is determined, one needs to construct the correct sheaf $C^\infty_{M\times N}$. You are not going to be able to do this without "building it from scratch"!

Next, let $\mathcal{T}$ be the category of topological spaces and recall the following:

Definitions (a) $T \in \mathrm{Ob}(\mathcal{T})$ is **Hausdorff** if, for all distinct points $p, q \in T$, there exist open subsets $U, V \subset T$ such that $p \in U$, $q \in V$ and $U \cap V = \emptyset$.

(b) $T \in \mathrm{Ob}(\mathcal{T})$ is **compact** if each open cover $T = \cup_{\lambda\in\Lambda} U_\lambda$ admits a finite subcover $T = U_{\lambda_1} \cup ... \cup U_{\lambda_n}$.

Examples: (i) In §0.2, we saw that premanifolds need not be Hausdorff. Of course, open subsets of $\mathbf{R}^n$ are Hausdorff, as are manifolds, by definition.

(ii) Any topological space T with a generic point (see §0.1) is never Hausdorff, because **every** open subset contains a generic point, by definition. Thus, if R is an integral domain with more than one prime ideal, then $\mathrm{Spec}(R)$ is not Hausdorff.

(iii) Closed, bounded subsets of $\mathbf{R}^n$ are compact, by Heine-Borel. If R is a **Noetherian** ring, then $\mathrm{Spec}(R)$ is compact. (See §11).

Next, suppose that $\mathcal{C}$ is a category for which:

(i) Each $A \in \mathrm{Ob}(\mathcal{C})$ is a topological space (among other things) and

(ii) Each $f \in \mathrm{Mor}(\mathcal{C})$ is continuous (among other things) and

(iii) Products exist (i.e. any two objects have a product).

Aside from $\mathcal{T}$ itself, we have only found one interesting example so far: differentiable or holomorphic manifolds. There will be others(!!)

Then we make the following definitions:

Definition: (a) For each $A \in \mathrm{Ob}(\mathcal{C})$, consider the map $\delta : A \to A \times A$ which comes from the universal property applied to the two identity maps. A is a set, in particular, and δ is an injective mapping, so we can let Δ be the image of δ. Δ is called the **diagonal**. Then we say A is **separated** if $\Delta \subset A \times A$ is a closed subset.

(b) For each $A \in \mathrm{Ob}(\mathcal{C})$, we say that A is **proper** if all the projection maps $\pi_B : B \times A \to B$ for all $B \in \mathrm{Ob}(\mathcal{C})$ are closed maps.

Lemma: Suppose $\mathcal{C}$ is a category satisfying (i)-(iii). Then an object X is separated if and only if for all objects Y and morphisms $f, g : Y \to X$, the subset:

$$\{f = g\} := \{y \in Y \mid f(y) = g(y)\} \subset Y \text{ is closed}$$

Proof: Notice that $\Delta = \{\pi_1 = \pi_2\} \subset X \times X$, so one direction of the Lemma is trivial. For the other, notice that the unique morphism:

$$(f, g) : Y \to X \times X \text{ (by definition of the product)}$$

has the property that $\{f = g\} = (f, g)^{-1}(\Delta)$. If Δ is closed, then this is closed since the morphisms are continuous.

For the next three propositions, suppose that $\mathcal{C}$ satisfies (i)-(iii) and:

(iv) All products of objects in $\mathcal{C}$ are Cartesian products with the product topology (among other things).

Proposition: A is Hausdorff if and only if A is separated.

Proof: Suppose that $p, q \in A$ and $U, V \subset A$ are open neighborhoods of p and q respectively with empty intersection. Then $(p, q) \in U \times V \subset A \times A - \Delta$. Thus Hausdorff implies separated.

Next, suppose that $(p, q) \in A \times A - \Delta$. If W is an open neighborhood of (p, q) which does not intersect Δ, then we can find an open neighborhood of (p, q) of the form $U \times V$ which is contained in W. Thus $U \cap V = \emptyset$, and $p \in U$ and $q \in V$. So separated implies Hausdorff.

Thus "separated" and "Hausdorff" are equivalent for the objects in any category satisfying (i) -(iv). The analogy between "proper" and "compact" is not quite so perfect:

Proposition: If A is compact, then A is proper.

Proof: Given B and a closed subset $Z \subset A \times B$, we need to show that $\pi_B(Z) \subset B$ is closed. Pick $b \in B - \pi_Z(B)$. Then for each point $a \in A$, we can find a product neighborhood $U_a \times V_a$ of (a,b) which is disjoint from the closed set Z. Thus $A = \cup_{a \in A} U_a$, so that since A is compact, a finite number $U_{a_1}, ..., U_{a_r}$ will also cover, and it follows that $p \in \cap_{i=1}^r V_{b_i}$, and moreover that this open subset of B is in the complement of $\pi_B(Z)$. Thus $\pi_B(Z)$ is closed.

The converse needs extra conditions, however. For instance:

Proposition: Suppose that every object in $\mathcal{C}$ has a countable basis of open sets, and suppose that there is at least one object $B \in \mathrm{Ob}(\mathcal{C})$ which has in it a decreasing chain of open sets $B \supseteq V_1 \supseteq V_2 \supseteq ...$ whose intersection V is not open. Then A is compact iff A is proper.

Proof: We need to prove proper implies compact.

Given A and an open cover $A = \cup_{\lambda \in \Lambda} U_\lambda$, there is always a countable subcover $A = \cup_{n=1}^\infty U_{\lambda_n}$, because A has a countable basis of open sets. Suppose that A is not compact, and such a cover is given which does not admit a finite subcover.

Let $W_n = U_{\lambda_1} \cup ... \cup U_{\lambda_n}$. Then in the product $A \times B$ (where B and V_n are chosen as in the statment of the Proposition) the subset $\cup_{n=1}^\infty W_n \times V_n \subset A \times B$ is open. Let Z be the complement of this set. I claim that $\pi(Z) = B - V$, which is therefore not closed, hence A is not proper.

To see this, suppose that $p \in V$. Then for each $a \in A$, choose a W_n from the cover of A so that $a \in W_n$. Then $(a,p) \in W_n \times V_n$ so that $(a,p) \notin Z$, so $p \notin \pi(Z)$. On the other hand, if $p \notin V_n$ for some (minimal) n, then since the original cover did not admit a finite subcover, it follows that $W_n \neq A$, thus there is an $a \in A$ such that $a \notin W_n$. Since the W_n are increasing, it follows that $a \notin W_m$ for all $m \leq n$, so that for this choice of a, it follows that $(a,p) \notin W_n \times V_n$ for all n. Thus $(a,p) \in Z$ and $p \in \pi(Z)$.

Example: A manifold M (differentiable or holomorphic) is always separated. It is compact if and only if it is proper, because manifolds always have a countable basis of open sets (from the gluing construction!) and if we let $B = \mathbf{R}^n$ (or $\mathbf{C}^n$), then the balls of radius $\frac{1}{n}$ about the origin give a collection of open subsets satisfying the requirements of the last Proposition.

Note that property (iv) is essential for the propositions!

Funny Example: Let T be a topological space. Give every subset $S \subset T$ the induced topology. This is a topology making all inclusions into continuous maps, so the category $\mathcal{P}(T)$ satisfies (i),(ii) and (iii). But! the product $A \times B$ in this category is the intersection, which certainly does not agree with the Cartesian product. And indeed, the conclusions of the propositions above do not hold if we make suitable choices for T:

(a) Every object of $\mathcal{P}(T)$ is separated(!) because $A \times A = A \cap A = A$. If we choose T to be non-Hausdorff, then T is separated but not Hausdorff.

(b) A subset $A \subset T$ is closed iff $A \cap B \subset B$ is closed for all $B \subset T$ (why?). Thus, the closed subsets of T are the proper objects. If we pick T to be non-compact, then T is proper but not compact.

It's time to raise the level of abstraction.

Definition (of Fibred Products): Suppose $f : A \to X$ and $g : B \to X$ are morphisms in a category $\mathcal{C}$. Then an object C is called a **fibred product** of A and B **over** X, (and denoted by $A \times_X B$) if:

(Existence of Projections) There exist two "projection" morphisms $\pi_f : C \to B$ and $\pi_g : C \to A$ such that the following diagram commutes:

$$\begin{array}{ccc} C & \stackrel{\pi_f}{\to} & B \\ \pi_g \downarrow & & g \downarrow \\ A & \stackrel{f}{\to} & X \end{array}$$

and π_f, π_g satisfy the following

(Universal Property) To each object C' and morphisms $h_f : C' \to C$ and $h_g : C' \to B$ making the analogous diagram commute, there is a unique map $h : C' \to C$ such that $h_f = \pi_f \circ h$ and $h_g = \pi_g \circ h$.

Lemma 0.3.2 Fibred products are unique up to isomorphism.

Proof: Same as Lemma 0.3.1.

Examples: (a) Fibred products exist in the category of sets. Indeed, the subset of $A \times B$ defined by:

$$A \times_X B := \{(a, b) \in A \times B \mid f(a) = g(b)\}$$

is a fibred product of A and B over X. In particular, you should check that in the category of sets:

(i) If $f : A \hookrightarrow X$ and $g : B \hookrightarrow X$ are **injective**, then the intersection of their images $f(A) \cap f(B) \subset X$ is a fibred product of A and B over X.

(ii) If $B = \{p\}$ is a one-point set, then let $q = g(p) \in X$. Then the fiber $f^{-1}(q) \subset A$ is a fibred product of A and B over X.

(iii) More generally, the preimage of $x \in X$ under $g \circ \pi_f = f \circ \pi_g : A \times_X B \to X$ is just the ordinary Cartesian product $f^{-1}(x) \times g^{-1}(x)$. So this map to X has the products of the fibres as its fibres, hence the name: fibred product(!)

(b) In the category $\mathcal{T}$, we can give $A \times_X B$ from (a) the induced topology from $A \times B$, and it becomes a fibred product in $\mathcal{T}$.

(c) In the category $\mathcal{C}$ of commutative rings with 1, the tensor product $A \otimes_X B$ is a fibred **co**product. Thus in the dual category $\mathcal{C}^*$ (which we will see is equivalent to the category of affine schemes), fibred products exist.

Fibred products of manifolds often do not exist! Two examples:

(a) Consider the differentiable map $f : \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x, y) = xy$. Then the fiber product with the inclusion $\iota : \{0\} \hookrightarrow \mathbf{R}$ of the origin in $\mathbf{R}$ is, by earlier remarks, the preimage $f^{-1}(0)$, which is the x-axis union the y-axis with the induced topology. There is no way to make this a manifold!

Note: By the implicit function theorem, we at least know that the fibre of a C^∞ map $f : \mathbf{R}^n \to \mathbf{R}^m$ is a manifold nearby a point where df has rank m.

(b) Although intersections of open subsets of $\mathbf{R}^n$ are still open subsets of $\mathbf{R}^n$, intersections of arbitrary manifolds are bad, too. For instance, in §2 we saw the following example: $Q = V(xy - zw) \subset \mathbf{P}^3_{\mathbf{C}}$ (a manifold...why?) and a projective subspace $\Lambda \subset \mathbf{P}^3_{\mathbf{C}}$ (a manifold, differomorphic to $\mathbf{P}^2_{\mathbf{C}}$) intersect in a pair of projective lines l, l' which intersect in a point. Topologically, this intersection is a pair of spheres intersecting at a point, and indeed, if the fibred product $Q \times_{\mathbf{P}^3} \Lambda$ were to exist, it would have to have this topology. We reach the same problem as in (a).

If $\mathcal{C}$ is a category satisfying (i),(ii) and in which fibred products always exist, then intuitively, a morphism is separated if all its fibers are separated, and likewise for properness. Precisely:

Definitions: (a) Given a morphism $f : A \to X$, let $\Delta \subset A \times_X A$ be the image of A under the (injective!) morphism δ determined by the two identity morphisms. Then f is **separated** if Δ is closed.

(b) A morphism $f : A \to X$ is **proper** if for all $g : B \to X$, the induced morphism $\pi_f : A \times_X B \to B$ is a closed map.