Course Notes for Math 737 (Algebraic Geometry)

2. Introduction. Eine Kleine Linear Algebra. We define subspaces and projections, prove that projective space over an infinite field is irreducible, and investigate a quadric hypersurface in $\mathbf{P}^3_k$. First, a basis-free:

Redefinition (of projective space): Let V be a finite-dimensional vector space over k. Then:

(i) $\mathbf{P}(V)$ is the set of lines in V.

(ii) $k[\mathbf{P}(V)]$ is the (graded) symmetric algebra $\oplus_{d=0}^{\infty} \operatorname{Sym}^d(V^*)$ over k.

Explanation and Remarks: If $\dim(V) = n+1$ and a basis $e_0, ..., e_n$ for V is chosen, let $x_0, ..., x_n$ be the dual basis for V^*. Then the natural maps $k[x_0, ..., x_n]_d \to \operatorname{Sym}^d(V^*)$ are isomorphisms of vector spaces over k, and putting them all together gives a map:

$$k[x_0, ..., x_n] \to \oplus_{d=0}^{\infty} \operatorname{Sym}^d(V^*)$$

which is an isomorphism of graded k-algebras. So this definition is the same as the previous one! Similarly, we obtain basis-free redefinitions of projective algebraic subsets and varieties, which coincide with the old definitions once a basis is chosen.

So why introduce this basis-free definition? For one thing, it is easier to state and use results in terms of the new definition (though proofs almost always begin by choosing bases!). For another, it points out a very important observation. Namely, suppose that bases $e_0, ..., e_n$ and $f_0, ..., f_n$ are chosen for V. Suppose that $F \in \operatorname{Sym}^d(V^*)$. Then with respect to the basis $x_0, ..., x_n$ dual to the e_i's, we identify F with a homogeneous polynomial $F' \in k[x_0, ..., x_n]_d$, whereas with respect to the basis $x_0, ..., x_n$ dual to the f_i's, we identify F with a homogeneous polynomial $F'' \in k[x_0, ..., x_n]_d$. Now for the observation:

Observation: There is no qualitative difference between F' and F''.

Definition: We say that F' above is transformed into F'' under the change of basis taking $e_0, ..., e_n$ to $f_0, ..., f_n$.

Example: Any nonzero linear form $L' \in k[x_0, ..., x_n]_1$ is transformed into any other nonzero linear form $L'' \in k[x_0, ..., x_n]_1$ by an appropriate change of basis. So there is no qualitative difference among linear forms.

Lemma 2.1: Suppose $W \subset V$ is a sub-vector space over k. Then:

$\iota_W : \mathbf{P}(W) \to \mathbf{P}(V)$, the injective map sending lines $l \subset W$ to $l \subset V$, and

$\iota_W^* : k[\mathbf{P}(V)] \to k[\mathbf{P}(W)]$, the surjective "pull-back" homomorphism of graded k-algebras given by the transpose maps $\mathrm{Sym}^d V^* \to \mathrm{Sym}^d W^*$, satisfy the following compatibility property for algebraic sets:

$$\iota_W^{-1}(V(\{F_s\}_{s\in S})) = V(\{\iota_W^*(F_s)\}_{s\in S}) \subset \mathbf{P}(W)$$

so that in particular, ι_W^{-1} takes algebraic subsets to algebraic subsets.

Proof: Let's choose bases $e_0, ..., e_m$ for W, extended to $e_0, ..., e_n$ for V to see this more clearly. Then with respect to the dual basis $x_0, ..., x_n$ for V^*, the pull-back $\iota_W^* : k[x_0, ..., x_n] \to k[x_0, ..., x_m]$ is obtained by setting $x_{m+1} = ... = x_n = 0$. Also, with respect to these bases,

$$\iota_W(a_0 : ... : a_m) = (a_0 : ... : a_m : 0 : ... : 0)$$

and the compatibility property readily follows, because if $F_s \in k[\mathbf{P}(V)]_d$, then $\iota_W^*(F_s) = F_s(x_0, ..., x_m, 0, ..., 0)$ and $(a_0 : ... : a_m) \in \iota_W^{-1}(V(\{F_s\}))$ iff $F_s(a_0 : ... : a_m : 0 : ... : 0) = 0$ for all the F_s.

Definitions: (a) A **projective line** in $\mathbf{P}(V)$ is by definition the image of ι_W where $W \subset V$ has dimension 2. In other words, it is the image of a $\mathbf{P}^1_k$. If $p \neq q \in \mathbf{P}(V)$, then those two lines in the vector space V span a plane $W \subset V$, hence a projective line in $\mathbf{P}(V)$, which we will denote by $l_{p,q}$.

(b) A **projective subspace** or plane $\Lambda \subset \mathbf{P}(V)$ is any subset of $\mathbf{P}(V)$ which is equal to the image of ι_W for some $W \subset V$ (which is uniquely determined by Λ!). Points $p_1, ..., p_m \in \mathbf{P}(V)$ determine a projective subspace by letting $W \subset V$ be the span of the corresponding lines in V. This projective subspace, denoted $\Lambda_{p_1,...,p_m}$ is called the **span** of the points.

Here's an application of the Lemma:

Proposition 2.2: If k is an infinite field, then:

(i) $\mathbf{P}^n_k$ is a projective variety, and

(ii) $I(\mathbf{P}^n_k) = \{0\}$.

Proof: You have already done the case $n = 1$ as homework in §1.

(Irreducibility) Let $\mathbf{P}^n_k = \mathbf{P}(V) = V' \cup V''$ where V is a vector space over k of dimension $n+1$, and $V', V'' \subset \mathbf{P}(V)$ are projective algebraic subsets. If $V' \neq \mathbf{P}(V)$ and $V'' \neq \mathbf{P}(V)$, then we can find $p \in V' - V''$ and $q \in V'' - V'$. Thus if $W \subset V$ is the subspace such that $\iota_W(\mathbf{P}(W)) = l_{p,q}$, then

$$\mathbf{P}(W) = \iota_W^{-1}(V') \cup \iota_W^{-1}(V'')$$

where the first does not contain $\iota_W^{-1}(q)$ and the second does not contain $\iota_W^{-1}(p)$. By Lemma 2.1, these are algebraic subsets of $\mathbf{P}(W)$, which violates the irreducibility of $\mathbf{P}(W) = \mathbf{P}^1_k$.

(ii) Suppose that $0 \neq F \in I(\mathbf{P}^n_k)$ is homogeneous of minimal degree d. We proceed by induction on n and on d. Suppose $\Lambda = \iota_W(\mathbf{P}(W)) \subset \mathbf{P}^n_k$ is the plane associated to the subspace $W \subset k^{n+1}$ spanned by $e_0, ..., e_{n-1}$. Using the Lemma again, we see that $V(\iota_W^*(F)) = \iota_W^{-1}(V(F)) = \mathbf{P}(W)$. By induction on n, we can conclude that $\iota_W^*(F) = 0$. But if that is the case, then since $\iota_W^*(F)$ was obtained by setting $x_n = 0$, it follows that $x_n | F$. Thus we can write $F = x_n G$, and it follows that $\mathbf{P}^n_k = V(F) = V(x_n) \cup V(G)$. Now clearly, $V(x_n) \neq \mathbf{P}^n_k$, so since $\mathbf{P}^n_k$ is irreducible by (i), it follows that $V(G) = \mathbf{P}^n_k$, so that $G \in I(\mathbf{P}^n_k)$ is also nonzero of degree $d-1$, violating the minimality of the degree of F.

Next, let's classify all the algebraic subsets defined by linear equations:

Proposition 2.3: Suppose V is a vector space of dimension $n+1$ over k. Then:

(a) Every projective subspace $\Lambda = \iota_W(\mathbf{P}(W)) \subset \mathbf{P}(V)$ is an algebraic subset. In fact, if we let $r = \dim(V) - \dim(W)$, then there are elements $L_1, ..., L_r \in V^*$ such that

$\Lambda = V(L_1, ..., L_r)$. Moreover, if k is infinite, then:

(i) Λ is irreducible,

(ii) $I(\Lambda)$ is the ideal generated by $L_1, ..., L_r$, and

(iii) $k[\Lambda] \cong k[x_0, ..., x_{n-r+1}]$, so in particular $\dim(\Lambda) = n - r$.

(b) Conversely, if $\{L_s\}_{s \in S} \subset V^*$, then $V(\{L_s\}_{s \in S}) = \Lambda$ for some projective subspace Λ, and the $L_1, ..., L_r$ from (a) may be chosen from among the $\{L_s\}$.

Proof: (a) Let $e_0, ..., e_m$ be a basis for W, extended to a basis $e_0, ..., e_n$ for V, and let $x_0, ..., x_n$ be the dual basis for V^*. With respect to this basis, it is clear that $\Lambda = V(x_{m+1}, ..., x_n)$, and indeed $r = n - m$. Suppose k is infinite and Λ is reducible. Then $\Lambda = V' \cup V''$, and as in the proof of Proposition 2.2, we get a cover $\mathbf{P}(W) = \iota_W^{-1}(V') \cup \iota_W^{-1}(V'')$ by algebraic subsets (using Lemma 2.1) which violates the irreducibility of $\mathbf{P}(W)$. Also, if $F \in I(\Lambda)$, and F is not in the ideal generated by $x_{m+1}, ..., x_n$, then $0 \neq \iota_W^*(F) \in k[\mathbf{P}(W)]$ because $\iota_W^*(F)$ is obtained by setting $x_{m+1} = ... = x_n = 0$. But $\iota_W^*(F) \in I(\mathbf{P}(W))$ using Lemma 2.1 again, contradicting Proposition 2.2. So $I(\Lambda)$ is the ideal generated by $x_{m+1}, ..., x_n$. Finally, $k[\Lambda] \cong k[\mathbf{P}(V)]/I(\Lambda) \cong k[x_0, ..., x_m]$.

(b) Think of the L_s as vectors in the dual space V^*. Let $T \subset V^*$ be the span of these vectors. Let $r = \dim(T)$, and choose $L_{s_1}, ..., L_{s_r}$ to be a basis for T. Of course it is clear that $V(\{L_s\}_{s\in S}) \subseteq V(L_{s_1}, ..., L_{s_r})$, but in fact equality holds because each L_s is a linear combination of $L_{s_1}, ..., L_{s_r}$. Now let $W \subset V$ be the kernel of the transpose map $V \to T^*$. Then $\dim(V) - \dim(W) = r$, and it is easy to see that to see that $\Lambda := \iota_W(\mathbf{P}(W)) = V(L_{s_1}, ..., L_{s_r})$.

Note: In the proof of (b), the set of vectors $\{L_s\}_{s\in S} \subset V^*$ was replaced by a **finite** subset without changing the algebraic set it defines. This can be done with arbitrary sets of homogeneous polynomials (see §5).

Here's an easy corollary:

Corollary 2.4: Suppose k is infinite and $\Lambda, \Lambda' \subset \mathbf{P}_k^n$ are projective subspaces of codimensions r and r' respectively. Then:

(a) $\Lambda \cap \Lambda'$ is another projective subspace (maybe empty, but)

(b) if $r + r' \leq n$, then $\Lambda \cap \Lambda' \neq \emptyset$, and $\dim(\Lambda \cap \Lambda') \geq n - (r + r')$.

Proof: If $\mathbf{P}_k^n = \mathbf{P}(V)$ and $W, W' \subset V$ correspond to Λ and Λ', then $\Lambda \cap \Lambda'$ is the image of $\iota_{W\cap W'}$. If $W \cap W' = \{0\}$, then this is empty, but if $r + r' \leq n$, then the dimensions $n - r + 1$ and $n - r' + 1$ of W and W' respectively satisfy $(n - r + 1) + (n - r' + 1) > n + 1 = \dim(V)$, so that the two subspaces intersect in more than zero(!) In fact, they intersect in a subspace of dimension at least $n - (r + r') + 1$, hence (b).

Note: So for instance, there are no parallel lines in $\mathbf{P}_k^2$ (!) We'll eventually see that a version of part (b) of the Corollary still holds when Λ and Λ' are replaced by arbitrary projective varieties(!)

Next, let $\Lambda \subset \mathbf{P}(V)$ be a projective subspace, let $W \subset V$ be the corresponding sub-vector space, and consider the canonical surjective linear map

$$A : V \to V/W.$$

Lemma 2.5: The following (which will be called the projection from Λ):

$\pi_\Lambda : \mathbf{P}(V) --> \mathbf{P}(V/W)$, the surjective "map" sending a line $l \subset V$ to the line $A(l) \subset V/W$, and

$\pi_\Lambda^* : k[\mathbf{P}(V/W)] \to k[\mathbf{P}(V)]$, the injective homomorphism given by the maps $\mathrm{Sym}^d A^* :\ \mathrm{Sym}^d(V/W)^* \to\ \mathrm{Sym}^d V^*$, satisfy the same compatibility as in Lemma 2.1.

Whoa! Something's fishy in the definition of π_Λ. Namely, π_Λ isn't defined everywhere! In fact, it's not defined at any of the points of Λ because the lines in W are sent to zero! On the other hand, since $W =\ \ker(A)$, the map π_Λ is well-defined on the complement $U := \mathbf{P}(V) - \Lambda$, (and U is, intuitively, "almost all of $\mathbf{P}(V)$" because Λ has smaller dimension). Such a "map" will be called a **rational map** (analogous to rational functions, which were also not defined everywhere, and are also denoted by a broken arrow). U is the domain of π_Λ, when thought of as an honest map. That's all well and good, but now the compatibility assertion needs to be rewritten:

$$\pi_\Lambda^{-1}(V(\{F_s\}_{s\in S})) = V(\{\pi_\Lambda^*(F_s)\}_{s\in S}) \cap U.$$

Proof: Choose compatible bases for V and V/W. That is, let $e_0, ..., e_n$ be a basis for V, and $f_0, ..., f_m$ be a basis for V/W such that $A(e_i) = f_i$ for $i \leq m$. Then π_Λ^* is the injective homomorphism (with respect to dual bases $x_0, ..., x_n$ for V^* and $y_0, ..., y_n$ for V/W^*) $\pi_\Lambda^* : k[y_0, ..., y_m] \to k[x_0, ..., x_n]$ defined by sending $\pi_\Lambda^*(y_i) = x_i$. Moreover,

$$\pi_\Lambda(a_0 : ... : a_n) = (a_0 : ... : a_m)$$

But again, the choices of bases makes the lemma a triviality because the preimage $\pi_\Lambda^{-1}(V(\{F_s\}_{s\in S})) \subset U$ is the set of points $(a_0 : ... : a_n)$ such that $(a_0 : ... : a_m) \neq (0 : ... : 0)$ and such that $F_s(a_0, ..., a_m) = 0$ for all $s \in S$.

Remarks: Fix a line $l \subset V/W$ and corresponding point $p \in \mathbf{P}(V/W)$. Then the fiber $\pi_\Lambda^{-1}(p)$ is the set of lines $l' \subset V$ with the property that $A(l') = l$. But such a line is constrained to lie in the subspace $W' := A^{-1}(l)$ of dimension equal to $\dim(W) + 1$, and conversely, any line $l' \subset W'$ maps to l, provided that it doesn't lie in $W \subset W'$. Thus if we let $\Lambda' = \iota_{W'}(\mathbf{P}(W'))$, then:

$$\pi_\Lambda^{-1}(p) = \Lambda' \cap U$$

where $\Lambda \subset \Lambda'$ and $\dim(\Lambda') = \dim(\Lambda) + 1$. For this reason, we call this the projection from Λ.

Example: Suppose $p \in \mathbf{P}^2$. Then via "projection from p":

$$\pi_p : \mathbf{P}_k^2 - \{p\} \to \mathbf{P}_k^1$$

we get a bijection between the set of projective lines through p in $\mathbf{P}_k^2$ and the set $\mathbf{P}_k^1$. (Each such line is sent to its image point). We can, of course, generalize this example to produce bijections between the set of lines through a point in $\mathbf{P}_k^n$ and the set $\mathbf{P}_k^{n-1}$.

So maybe you like your maps to be well-defined everywhere and therefore prefer projective subspaces to projections. But let me point out:

Observation: The projections determine "pull-back" injections of **fields** of rational functions:

$$\pi_\Lambda^* : k(\mathbf{P}(V/W)) \hookrightarrow k(\mathbf{P}(V)),\ \pi_\Lambda^*\left(\frac{F}{G}\right) = \frac{\pi_\Lambda^*(F)}{\pi_\Lambda^*(G)}$$

the key being the fact that $\pi_\Lambda^* : k[\mathbf{P}(V/W)] \to k[\mathbf{P}(V)]$ is **injective**, so that $\pi_\Lambda^*(G) \neq 0$. You should check yourself that this π_Λ^* is well-defined, and that ι_W^* "defined" in a similar way would not, in fact, be well-defined(!) In fact, check that under the isomorphisms $k(\mathbf{P}_k^n) \cong k(\mathbf{A}_k^n) \cong k(x_1, ..., x_n)$ from §1, the homomorphism π_Λ^* is just the inclusion of fields:

$$k(x_1, ..., x_m) \subset k(x_1, ..., x_n).$$

There will be a class of rational maps, called **dominant** maps, defined by the property that the map is (almost) surjective and the pull-back on "functions" (actually, on germs of functions(!)) is injective, which are precisely the maps which produce inclusions of fields as above.

Let's think about a new example! Consider the quadric hypersurface:

$$Q := V(xy - zw) \subset \mathbf{P}^3_k$$

(where x, y, z, w is the dual basis to the standard basis for k^4).

Basic Observation: There is a bijection:

$$f : \mathbf{P}^1_k \times \mathbf{P}^1_k \to Q$$

defined by $f((a : b), (\lambda : \mu)) = (a\lambda : b\mu : a\mu : b\lambda)$. Let's check some of this:

Well-defined. If $a\lambda = b\mu = a\mu = b\lambda = 0$ then either $a = b = 0$ or $\lambda = \mu = 0$. Also, $f((\gamma a : \gamma b), (c\lambda : c\mu)) = (\gamma c a\lambda : ... : \gamma c b\lambda)$, so indeed f is well-defined, at least as a map to $\mathbf{P}^3_k$.

Image is in Q. $(a\lambda)(b\mu) - (a\mu)(b\lambda) = 0$.

Injective. $f((1 : b), (1 : \mu)) = (1 : b\mu : \mu : b)$ so it is clearly injective on points of this form. One checks the other points similarly.

Surjective. Exercise.

Next Observation: For each $(a : b) \in \mathbf{P}^1_k$ and $(\lambda : \mu) \in \mathbf{P}^1_k$, the images:

$$l'(a : b) := f(\{(a : b)\} \times \mathbf{P}^1_k) \text{ and } l(\lambda : \mu) := f(\mathbf{P}^1_k \times \{(\lambda : \mu)\})$$

are **projective lines** in Q. One can see this, for example, by noting:

$$l'(a : b) = V(bx - az, ay - bz) \text{ and } l(\lambda : \mu) = V(\mu x - \lambda z, \lambda y - \mu w).$$

Thus $l'(a : b)$ and $l(\lambda : \mu)$ are two *families* of lines in Q, each parametrized by $\mathbf{P}^1_k$. (On the Cartesian product, the $l(\lambda : \mu)$ are the "horizontal" $\mathbf{P}^1_k$'s and the $l'(a : b)$ are the "vertical" $\mathbf{P}^1_k$'s.) Since we are talking about lines in $\mathbf{P}^3_k$, there is no reason a priori why two of them should intersect (i.e. "skew" lines in space don't intersect, even at infinity(!)). But as a consequence of the discussion so far, we have:

(1) Lines from different families intersect in a point, and

(2) Lines from the same family are either the same line, or else they do not intersect (i.e. they are skew).

Next Observation: The two surjective maps:

$$\pi_l := \pi_1 \circ f^{-1} : Q \to \mathbf{P}^1_k \text{ and } \pi_{l'} := \pi_2 \circ f^{-1} : Q \to \mathbf{P}^1_k$$

are extensions of projection maps in the following sense:

Extension of the Projection: Choose any line $l'(a:b) \subset Q$. I claim that $\pi_{l'}$ is the same map as the restriction of the projection $\pi_{l'(a:b)} : \mathbf{P}^3_k --> \mathbf{P}^1_k$ to a rational map from Q to $\mathbf{P}^1_k$ **away from the points of** $l'(a:b)$ (because the projection was only defined away from those points). What this means is that all the projections $\pi_{l'(a:b)}$ become the same map (where defined) when they are restricted to Q, and by moving $(a:b)$, the domains of these maps fill up all of Q. So the map we get by "gluing" all these maps together is $\pi_{l'}$ (and I hope the notation is now clear!). The same argument applies to π_l.

Proof: $l'(a:b)$ corresponds to the plane $W \subset k^4$ spanned by the vectors $(a,0,0,b)$ and $(0,b,a,0)$ (letting $(\lambda:\mu) = (1:0)$ and $(0:1)$, respectively). Moreover, we can choose a basis for V/W such that a vector $(\lambda, 0, \mu, 0) \in k^4$ is taken to (λ, μ) under the projection $A : V \to V/W$. Thus, $\pi^{-1}_{l'(a:b)}(\lambda:\mu)$ is $\Lambda' - l'(a:b)$ where Λ' corresponds to the subspace spanned by W and $(\lambda, 0, \mu, 0)$ (unless $b = 0$, in which case we need to use $(0, \mu, 0, \lambda)$ (why?)). The fiber of the restriction to Q is $(\Lambda' \cap Q) - l'(a:b)$. Thus to see that the fibers of the projection are the lines $l(\lambda:\mu)$, we need to show (exercise!) that:

$$\Lambda' \cap Q = l'(a:b) \cup l(\lambda:\mu).$$

Last Observation: If k is an infinite field, then

(i) Q is a projective variety and

(ii) $I(Q)$ is the ideal $\langle xy - zw \rangle$.

Proof: Suppose $Q = V' \cup V''$. Since each $l'(a:b)$ is irreducible, it follows by the usual compatibility that either $l'(a:b) \subset V'$ or else $l'(a:b) \subset V''$. Let $W' \subset \mathbf{P}^1_k$ be the set of points $(a:b)$ such that $l'(a:b) \subset V'$ and similarly for W''. If we can prove that W' and W'' are algebraic subsets, then the irreducibility of $\mathbf{P}^1_k$ will imply that either $V' = Q$ or else $V'' = Q$, which proves irreducibility. So how do we prove this? Just by writing things down carefully! Namely,

$$V' = V(\{F_s(x,y,z,w)\}_{s \in S}) \subset \mathbf{P}^3_k$$

for some homogeneous polynomials F_s so

$$W' = V(\{F_s(\lambda x, \mu y, \mu x, \lambda y)\}_{s\in S,(\lambda:\mu)\in \mathbf{P}^1_k}) \subset \mathbf{P}^1_k$$

and similarly for W''.

Now, suppose that $F \in I(Q)$. I want to prove that $xy - zw$ divides F. Let's assume not and that $F \in k[x, y, z, w]_d$ is homogeneous with minimal degree among the polynomials that are in $I(Q)$ and not divisible by $xy - zw$. Then because F pulls back to zero on each of the lines $V(x, z)$, $V(x, w)$, $V(y, z)$ and $V(y, w)$, it follows that no term in F is of the form $y^a w^b$, $y^a z^b$, $x^a w^b$ or $x^a z^b$ (why?). But then either xy or zw divides every term of F, so we can write:

$$F = P_1 \cdot (xy) + P_2 \cdot (zw)$$

and so $F = P_1(xy - zw) + (P_1 + P_2)(zw)$. I claim that $P_1 + P_2 \in I(Q)$ and certainly $xy - zw$ does not divide it, so we contradict the assumption that F has minimal degree(!) Why is $P_1 + P_2 \in I(Q)$? Because $I(Q)$ is a **prime** ideal, and $zw \notin I(Q)$ (For example, $V(zw)$ does not contain the point $(1 : 1 : 1 : 1) \in Q$.)

Second Homework Assignment

1. $\mathbf{P}^n_k$ isn't irreducible if k is finite (say $k = \mathbf{F}_q$, the finite field with $q = p^m$ elements) because in that case, $\mathbf{P}^n_k$ is finite(!) How many points are there? And what is $I(\mathbf{P}^n_k)$?

2. Suppose that $l \subset \mathbf{P}(V)$ is a projective line and $F \in k[\mathbf{P}(V)]_d$.

(a) Prove that if the number of points in $l \cap V(F)$ is greater than d, then $l \subset V(F)$.

(b) Suppose k is algebraically closed. Is the number of points exactly d if $l \not\subset V(F)$? Is it possible that $V(F) \neq \emptyset$ but $l \cap V(F) = \emptyset$?

3. More on the quadric Q.

(a) Prove the stuff left as exercises, i.e.

Prove $f : \mathbf{P}^1_k \times \mathbf{P}^1_k \to Q$ is surjective.

Prove that $\Lambda' \cap Q = l'(a : b) \cup l(\lambda : \mu)$.

(b) Consider the restricted projection $\pi_{(0:0:0:1)} : Q \to \mathbf{P}^2_k$. Find all the fibers of this map. Prove that unlike π_l and $\pi_{l'}$, there is no way to "continuously" extend $\pi_{(0:0:0:1)}$ to make it well-defined on all of Q.

(c) Find $k(Q)$ and the dimension of Q (assuming k is infinite).

(d) Prove that every line contained in Q is either an $l'(a : b)$ or an $l(\lambda : \mu)$.

(e) Suppose that l_1, l_2, l_3 are lines in $\mathbf{P}^3_k$ with the property that no two of them intersect. Find a basis for k^4 such that with respect to this basis, l_1, l_2 and l_3 are all members of the same family of lines in the quadric Q. Now, suppose that l_4 is another line with the property that $l_4 \cap Q$ consists of exactly d points (and $d \leq 2$ by exercise 2(a)). Prove that there are exactly d lines $l \subset \mathbf{P}^3_k$ which intersect the four given lines(!) Describe them. What happens if $l_4 \subset Q$?

4. Quadrics in general. Suppose k is a field of characteristic not equal to 2.

(a) Suppose $F \in k[\mathbf{P}(V)]_2$ (i.e. a homogeneous quadratic polynomial). Prove that given a basis $e_0, ..., e_n$ for V, there is a unique symmetric matrix $M(F)$ with coefficients in k and with the property that:

$$F = (x)^t M(F)(x), \text{ where } (x) = \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}.$$

Suppose that $f_0, ..., f_n$ is another basis for V and $A = (a_{i,j})$ is the change-of-basis matrix, i.e. $A(e_i) = f_i$. Then what is the matrix representing F in the new basis?

Definition: F is **degenerate** if $\det(M(F)) = 0$.

(b) Transform $xy - zw$ into $x^2 - y^2 + z^2 - w^2$ under a suitable change of basis for k^4. (What happens if the characteristic is equal to 2?)

Definition: $F \sim F'$ if there is a change of basis transforming F into F'.

(c) Prove that if $F \sim F'$ then F is degenerate iff F' is degenerate.

Prove that any F is equivalent to $a_0x_0^2 + ... + a_nx_n^2$ for some $a_0, ..., a_n \in k$.

Prove that under the equivalence relation $\sim$, there is one equivalence class of nondegnerate $F \in k[\mathbf{P}(V)]_2$ if k is algebraically closed.

Find all the equivalence classes when $k = \mathbf{R}$ (and $\dim(V) = n$).

(d) For each degenerate $F \subset k[\mathbf{P}(V)]_2$, prove that the projective subspace $\Lambda(F)$ associated to $\ker(M(F)) \subset V$ is contained in $V(F)$. If k is algebraically closed, prove that $F \sim F'$ if and only if $\dim(\Lambda(F)) = \dim(\Lambda(F'))$.

(e) Given a degenerate F, let $W = \ker(M(F))$ and prove that the image of $V(F)$ under projection from $\Lambda(F)$ is $V(F')$, where $F' \in k[\mathbf{P}(V/W)]_2$ is nondegenerate.

(f) We saw that $Q = V(xy - zw)$ has lots of projective lines on it. Suppose that k is algebraically closed and F is nondegenerate. Then what is the largest dimension of a projective subspace $\Lambda \subset V(F)$?