Course Notes for Math 737 (Algebraic Geometry)

1. Introduction. What is Algebraic Geometry? We need to begin by defining some notation and getting familiar with some of the basic objects.

k will denote a field and $\overline{k}$, its algebraic closure.

Definition (of Affine Space and its Functions):

(a) $\mathbf{A}^n_k := k^n$ is **affine** n **space over** k (or just affine space).

(b) $k[\mathbf{A}^n_k] := k[x_1, ..., x_n]$ is the **ring of regular functions on** $\mathbf{A}^n_k$, or it is sometimes called the coordinate ring of $\mathbf{A}^n_k$.

(c) $k(\mathbf{A}^n_k) := k(x_1, ..., x_n)$ is the **field of rational functions on** $\mathbf{A}^n_k$.

Remarks: A regular function $f \in k[\mathbf{A}^n_k]$ is an "honest" function:

$$f : \mathbf{A}^n_k \to k; \quad (a_1, ..., a_n) \mapsto f(a_1, ..., a_n)$$

A rational function $\phi(x) \in k(\mathbf{A}^n_k)$ is an "almost-everywhere-defined" honest function. If $\phi = \frac{f}{g}$ is in lowest terms, then:

$$\phi : \mathbf{A}^n_k - - > k; \quad (a_1, ..., a_n) \mapsto \frac{f(a_1, ..., a_n)}{g(a_1, ..., a_n)}$$

and this is well-defined whenever $g(a_1, ..., a_n) \neq 0$. We use a broken arrow to indicate that the domain is not necessarily all of $\mathbf{A}^n_k$.

Definition (of Projective Space and its "Functions"):

(a) $\mathbf{P}^n_k := \{\text{lines through the origin in } k^{n+1}\}$ is **projective** n**-space over** k (or just projective space). Alternatively, $\mathbf{P}^n_k = k^{n+1} - \{(0, ..., 0)\}/\sim$ where $(a_0, ..., a_n) \sim (b_0, ..., b_n)$ iff there is a $\lambda \in k^*$ such that $b_i = \lambda a_i$ for all i. The notation $(a_0 : ... : a_n)$ denotes a point of $\mathbf{P}^n_k$ in the equivalence class containing $(a_0, ..., a_n)$

(b) $k[\mathbf{P}^n_k] := k[x_0, ..., x_n] = \oplus_{d=0}^{\infty} k[x_0, ..., x_n]_d$ (the polynomial ring as a graded ring) is the **homogeneous coordinate ring of** $\mathbf{P}^n_k$.

(c) $k(\mathbf{P}^n_k) := k(x_0, ..., x_n)_0$ (the field of "balanced" fractions) is the **field of rational functions on** $\mathbf{P}^n_k$.

Explanation and Remarks: A graded ring is a ring R together with a direct-sum decomposition of abelian groups: $R = \oplus_{d=0}^{\infty} R_d$ with the property that $F \in R_d$ and $G \in R_e$ implies $FG \in R_{d+e}$. An element $F \in R_d$ is called **homogeneous of degree** d. If $F \in R$ is an arbitrary element, then we write $F = F_0 + ... + F_d$ for its decomposition as a sum of homogeneous elements.

In the case above, where $R = k[x_0, ..., x_n]$, this decomposition is by degree. In other words, a **homogeneous polynomial of degree** d is a polynomial F which is a sum of monomials, all of which have degree d. In this case, each $k[\mathbf{P}_k^n]_d$ is a vector space over $k = k[\mathbf{P}_k^n]_0$ and the multiplication maps:

$$m : k[\mathbf{P}_k^n]_d \times k[\mathbf{P}_k^n]_e \to k[\mathbf{P}_k^n]_{d+e}$$

are all bilinear.

The field of balanced fractions of a graded integral domain R is:

$$k(R)_0 := \{\frac{f}{g} \mid f \text{ and } g \text{ are homogeneous of the } \textbf{same} \text{ degree}\}/\sim$$

where $\sim$ is the ordinary equivalence relation on fractions. You check that this is a subfield of the ordinary field of fractions $k(R)$.

An element $F \in k[\mathbf{P}_k^n]$ is **not** in general an "honest" function as in the affine case, because in order to be well-defined at a point $(a_0 : ... : a_n)$, it would have to satisfy $F(a_0, ..., a_n) = F(\lambda a_0, ..., \lambda a_n)$ for all $\lambda \in k^*$. However, if F is homogeneous of degree d, then $F(\lambda a_0, ..., \lambda a_n) = \lambda^d F(x_0, ..., x_n)$, so $F(a_0, ..., a_n) = 0$ if and only if $F(\lambda a_0, ..., \lambda a_n) = 0$ for all $\lambda \in k^*$. Thus **if** F **is homogeneous**, then at least the statements "$F(a_0 : ... : a_n) = 0$" and "$F(a_0 : ... : a_n) \neq 0$" make sense, and can be tested on any representative.

Finally, notice that each $\phi \in k(\mathbf{P}_k^n)$ has a representative $\frac{F}{G}$ in lowest terms, and that since F and G have the same degree, it follows that for each $(a_0 : ... : a_n) \in \mathbf{P}_k^n$, the value $\frac{F(a_0,...,a_n)}{G(a_0,...,a_n)}$ does not depend upon the representative (the λ^d factors cancel!) provided that $G(a_0, ..., a_n) \neq 0$. Thus the function:

$$\phi : \mathbf{P}_k^n - - > k; \quad (a_0 : ... : a_n) \mapsto \frac{F(a_0, ..., a_n)}{G(a_0, ..., a_n)}$$

is an "honest" function at the points $(a_0 : ... : a_n)$ where $G(a_0 : ... : a_n) \neq 0$.

Consider the following subsets of $\mathbf{P}^n_k$:

$$U_i := \{(a_0 : ... : a_n) \in \mathbf{P}^n_k \mid a_i \neq 0\}, \text{ and}$$

$$Z_i := \{(0 : ... : 0 : a_i : ... : a_n) \in \mathbf{P}^n_k \mid a_i \neq 0\}.$$

Here are three simple properties of projective space:

Property 1. (Cover of $\mathbf{P}^n_k$ by copies of $\mathbf{A}^n_k$):

(i) $\mathbf{P}^n_k = \cup_{i=0}^n U_i$ and

(ii) Each map $f_i : \mathbf{A}^n_k \to \mathbf{P}^n_k$ defined by

$$f_i(a_1, ..., a_n) = (a_1 : ... : a_i : 1 : a_{i+1} : ...a_n)$$

is a bijection onto U_i.

Property 2. (Partition of $\mathbf{P}^n_k$ by smaller affine spaces):

(i) $\mathbf{P}^n_k = \coprod_{i=0}^n Z_i$ (disjoint union), and

(ii) Each map $g_i : \mathbf{A}^{n-i}_k \to \mathbf{P}^n_k$ defined by

$$g_i(a_1, ..., a_{n-i}) = (0 : ... : 0 : 1 : a_1 : ... : a_{n-i})$$

is a bijection onto Z_i.

For Property 3, we assume that either $k = \mathbf{R}$ (the real numbers) or $k = \mathbf{C}$ (the complex numbers), in which case we have the ordinary Euclidean topology on the vector spaces k^n.

Property 3. (Compactness of $\mathbf{P}^n_k$): $\mathbf{P}^n_k$ is compact.

Proof: In these two cases, $\mathbf{P}^n_k$ is a **manifold** (see §0.2). It is a C^∞ manifold when $k = \mathbf{R}$ and a complex manifold when $k = \mathbf{C}$. In particular, $\mathbf{P}^n_k$ is a topological space (the U_i are now open sets, and the Z_i are locally closed with respect to this topology). Moreover, the map $k^{n+1} - \{(0, ..., 0)\} \to \mathbf{P}^n_k$ is continuous with respect to this topology and factors through teh retraction: $k^{n+1} - \{(0, ..., 0)\} \to S$ to the unit sphere in k^{n+1}. ($S = S^n$ when $k = \mathbf{R}$ and $S = S^{2n+1}$ when $k = \mathbf{C}$). Since spheres are compact, projective space is compact. For fun, notice that the map $S^n \to \mathbf{P}^n_{\mathbf{R}}$ is a covering space of degree s and that the fibers of the map $S^{2n+1} \to \mathbf{P}^n_{\mathbf{C}}$ are circles.

Let's consider next the special case $k = \mathbf{C}$ and $n = 1$.

$k[\mathbf{A}^1_{\mathbf{C}}] = \mathbf{C}[z]$ and $k(\mathbf{A}^1_{\mathbf{C}}) = \mathbf{C}(z)$.

Notice that the regular functions are holomorphic functions and the rational functions are meromorphic functions on $\mathbf{C}$. Of course, there are more holomorphic and meromorphic functions than these, as the holomorphic functions include all power series which converge on $\mathbf{C}$ (for instance e^z). One important qualitative difference is the fact that polynomials may have only finitely many roots (zeroes) whereas holomorphic functions may have infinitely many zeroes (for instance $sin(z)$). Similarly, a rational function has only finitely many zeroes and poles, whereas a meromorphic function may have infinitely many of each (for instance $tan(z)$).

$\mathbf{P}^1_{\mathbf{C}}$ is the Riemann sphere, which you can think of as two copies of $\mathbf{C}$ (variables z and w) patched together along $\mathbf{C} - \{0\}$ by the patching function $z = \frac{1}{w}$ (Property 1). Or you can think of it as obtained from $\mathbf{C}$ by adding one point "at infinity" (Property 2) which makes it a compact complex manifold (Property 3).

$k[\mathbf{P}^1_{\mathbf{C}}]_0 = \mathbf{C}$, the constant functions, which happen to also be the only holomorphic functions on the Riemann sphere!

$k(\mathbf{P}^1_{\mathbf{C}}) = \mathbf{C}(z,w)_0 \cong \mathbf{C}(z)$ (see HW # 4). This is the same as the field of meromorphic functions on the Riemann sphere. (The additional condition that the function be meromorphic at ∞ forces it to be a rational function!)

This observation: that on $\mathbf{P}^1_{\mathbf{C}}$ the rings of rational functions and meromorphic functions are the same, is true in great generality. It is known to algebraic geometers as the GAGA principle (after a paper of Serre, not because it makes us go gaga!)

Of course, algebraic geometry is not just the study of affine and projective space. So let's introduce the major players...affine and projective varieties.

Definition (of algebraic sets): (Affine) A subset $V \subset \mathbf{A}^n_k$ is called an **affine algebraic set** if there is a collection of regular functions $f_s \in k[\mathbf{A}^n_k]$ indexed by a set S such that:

$$V = V(\{f_s\}_{s\in S}) := \{p \in \mathbf{A}^n_k \mid f_s(p) = 0 \text{ for all } s \in S\}.$$

(Projective) A subset $V \subset \mathbf{P}^n_k$ is called a **projective algebraic set** if there is a collection of homogeneous polynomials $F_s \in k[\mathbf{P}^n_k]$ (not necessarily of the same degree) indexed by a set S such that:

$$V = V(\{F_s\}_{s\in S}) := \{p \in \mathbf{P}^n_k \; | F_s(p) = 0 \text{ for all } s \in S\}.$$

Examples: (a) Affine algebraic sets you studied in high school, for instance:

Parabolas: $V(y - x^2) \subset \mathbf{R}^2$. **Hyperbolas:** $V(xy - 1)$ or $V(x^2 - y^2 - 1)$.

Saddle Surfaces: $V(xy - z) \subset \mathbf{R}^3 = \mathbf{A}^3_{\mathbf{R}}$.

(b) (Determinantal Loci) Suppose that M is a matrix of linear forms in the variables $x_0, ..., x_n$. For example, suppose $n = 5$ and

$$M = \begin{pmatrix} x_0 & x_1 & x_3 \\ x_1 & x_2 & x_4 \\ x_3 & x_4 & x_5 \end{pmatrix}$$

Then given a point $p \in \mathbf{P}^n_k$, we can evaluate the entries of M at a representative for p, obtaining a matrix of scalars which is well-defined up to multiplication of all the entries by λ. Thus, the rank of M at p is well-defined, and the **determinantal loci associated to** M are the subsets:

$$D_r(M) := \{p \in \mathbf{P}^n_k \mid \text{rank}(M(p)) \le r\} \subset \mathbf{P}^n_k.$$

These are algebraic subsets because they are the zero loci of the $r+1 \times r+1$ minors of M, which are homogeneous (and in this case of the same degree).

In the example, $D_3(M) = \mathbf{P}^5_k$, $D_2(M) = V(\det(M))$ and $D_1(M) = V(x_0x_2 - x_1^2, x_0x_4 - x_1x_3, x_0x_5 - x_3^2, x_1x_4 - x_2x_3, x_1x_5 - x_3x_4, x_2x_5 - x_4^2)$.

Remark: An algebraic set $V(f)$ defined by a single polynomial is called a **hypersurface**. Every algebraic set (affine or projective) is an intersection of hypersurfaces: $V(\{f_s\}) = \cap_{s\in S} V(f_s)$.

Definition (of reducibility): An affine or projective algebraic set V is **reducible** if:

$$V = V' \cup V'' \text{ where } V', V'' \text{ are algebraic sets, } V' \neq V \text{ and } V'' \neq V.$$

Otherwise, it is **irreducible**.

Examples: (i) A finite set of points is always an algebraic subset. It is irreducible iff it consists of (at most) one point.

(ii) If k is infinite, then $\mathbf{A}^1_k, \mathbf{P}^1_k$ (and more generally all the affine and projective spaces..see §2) are irreducible...can you see why?

(iii) The union of two hypersurfaces $V(f)$ and $V(f')$ is another hypersurface, $V(ff')$ (why?). Conversely, if k is algebraically closed and f (or F) is an irreducible polynomial, then $V(f)$ (or $V(F)$) is irreducible...if you can see why, my hat's off to you!

Definition (of geometric ideals): (Affine) An ideal $I \subset k[\mathbf{A}^n_k]$ is called a **geometric ideal** if there exists a subset $V \subset \mathbf{A}^n_k$ such that:

$$I = I(V) := \{f \in k[\mathbf{A}^n_k] \mid f(p) = 0 \text{ for all } p \in V\}.$$

(Easy exercise: $I(V)$ is always an ideal!)

(Projective) A homogeneous ideal $I \subset k[\mathbf{P}^n_k]$ is called a **homogeneous geometric ideal** if there is a subset $V \subset \mathbf{P}^n_k$ such that:

$$I = I(V) := \{F \in k[\mathbf{P}^n_k] \ | F(a_0, ..., a_n) = 0 \text{ for all } (a_0 : ... : a_n) \in V\}.$$

(A homogeneous ideal in a graded ring is an ideal that may be generated by homogeneous elements.)

Examples: (i) The (maximal!) ideal in $k[\mathbf{A}^n_k]$ generated by the polynomials $x_1 - a_1, ..., x_n - a_n$ is equal to $I(\{(a_1, ..., a_n)\})$.

(ii) The homogeneous ideal in $k[\mathbf{P}^n_k]$ generated by all polynomials of the form $x_i a_j - x_j a_i$ is equal to $I(\{(a_0 : ... : a_n)\})$.

(iii) Suppose k is algebraically closed and $V = V(F) \subset \mathbf{P}^n_k$. Then $I(V)$ is the principal ideal generated by G, where G is obtained from F as follows. Factor $F = aP_1^{d_1}...P_N^{d_N}$ as a product of powers of irreducible (homogeneous) polynomials where no P_i divides a P_j. ($k[x_0, ..., x_n]$ is a unique factorization domain!) Then $G = P_1...P_N$. Seems pretty intuitive, but how would you prove it?

OK. Now for the punchline:

Lemma 1.1: If V is an irreducible affine algebraic set, then $I(V)$ is a **prime** (geometric) ideal.

Proof: By definition, $V = V(\{f_s\}_{s\in S})$ for some polynomials $f_s \in k[\mathbf{A}_k^n]$. Suppose that $I(V)$ is not prime. Then there are polynomials $g, h \in k[\mathbf{A}_k^n]$ such that $(gh)(p) = g(p)h(p) = 0$ for all $p \in V$ and there exist points $q, r \in V$ such that $g(q) \neq 0$ and $h(r) \neq 0$. Let:

$$V' = V(\{g, f_s\}), \text{and } V'' = V(\{h, f_s\})$$

Then $V = V' \cup V''$ is reducible(!)

There is a projective version of this, which holds when k is infinite. Its proof is left as an exercise.

Definition (of varieties and their functions): (Affine) An irreducible algebraic subset $V \subset \mathbf{A}_k^n$ is called an **affine variety**. The quotient ring $k[V] := k[\mathbf{A}_k^n]/I(V)$ is called the **ring of regular functions** on V. It is an integral domain by Lemma 1.1. Its field of fractions, which we denote by $k(V)$, is called the **field of rational functions on** V.

(Projective) Assume that k is infinite. Then $V \subset \mathbf{P}_k^n$ is called a **projective variety** if it is an irreducible algebraic subset. The quotient ring $k[V] := k[\mathbf{P}_k^n]/I(V)$ is a naturally *graded* integral domain (because $I(V)$ is homogeneous). The graded ring $k[V] = \oplus_{d=0}^{\infty} k[V]_d$ is called the **homogeneous coordinate ring of** V. The field of balanced fractions of $k[V]$ is denoted by $k(V)$ and is called the **field of rational functions on** V.

Remarks: If V is an affine variety, then an element $f \in k[V]$ really is a function on V. That's because if $f, g \in k[\mathbf{A}_k^n]$ are representatives for f, then $f - g \in I(V)$ by definition, so that $f(p) - g(p) = 0$ for all $p \in V$. Similarly, all the remarks we made about the "functions" on affine and projective space hold for affine and projective varieties.

These, then, are the objects we want to study. Our aim is to study their "geometric" properties, but our definitions and tools will come mostly from commutative algebra(!) For example:

Definition (of dimension): The dimension of an affine or projective variety V is the transcendence degree over k of the field $k(V)$.

First Homework Assignment

1. Prove Properties 1 and 2 of projective space.

2. Prove that if k is infinite and $f \in k[\mathbf{P}^n_k]$ is an "honest" function on $\mathbf{P}^n_k$, then f is a constant polynomial.

3. Suppose that R is a graded integral domain. Prove that the field of balanced fractions $k(R)_0$ is indeed well-defined and is a subfield of the ordinary field of fractions $k(R)$.

4. Find an isomorphism $k(\mathbf{A}^n_k) \cong k(\mathbf{P}^n_k)$.

5. Prove that a finite set of points in $\mathbf{A}^n_k$ or $\mathbf{P}^n_k$ is an algebraic subset.

6. If k is infinite, prove that $\mathbf{A}^1_k$ and $\mathbf{P}^1_k$ are irreducible, and that $I(\mathbf{A}^1_k) = \{0\}$ and $I(\mathbf{P}^1_k) = \{0\}$.

7. (a) If $I \subset R$ is an ideal in a graded commutative ring R, then I was defined to be homogeneous if it could be generated by homogeneous elements. Prove that I is homogeneous if and only if $I = \oplus_{d=0}^{\infty}(I \cap R_d)$, as abelian groups. In this case, we let $I_d := I \cap R_d$.

(b) If k is infinite and $V \subset \mathbf{P}^n_k$ is any subset, prove that if a polynomial $F = F_0 + ... + F_d \in I(V)$, then each $F_i \in I(V)$. Conclude that the ideal $I(V)$ is always homogeneous.

(c) If $I \subset R$ is a proper homogeneous ideal in an integral domain, then find a natural grading $R/I = \oplus_{d=0}^{\infty}(R/I)_d$ making R/I into a graded commutative ring with 1.

(d) Prove the following projective version of Lemma 1:

"If k is an infinite field and $V \subset \mathbf{P}^n_k$ is an irreducible algebraic subset, then $I(V)$ is a homogeneous prime ideal."

8. Prove that:

(a) $I(\{a_1, ..., a_n)\}) = \langle x_1 - a_1, ..., x_n - a_n\rangle$.

(b) If k is infinite, then $I(\{(a_0 : ... : a_n)\}) = \langle x_i a_j - x_j a_i\rangle_{i,j=0}^n$,

as was asserted on page 6. What goes wrong in (b) if k is finite?