Course Notes for Math 738 (Algebraic Geometry)

15. Varieties. Dimension. One of the most basic notions in geometry is that of dimension. We will find several equivalent formulations of the dimension of a variety over an algebraically closed field, and prove they agree with the degree of the Hilbert polynomial in case $X \subseteq \mathbf{P}_k^n$ is a projective variety. We prove that projective varieties in the same projective space intersect when they "ought to"and explain the upper-semicontinuity property of the fiber dimensions of a regular map.

We begin by generalizing the definition of dimension from §1:

Definition: The **dimension** of a variety X over k is the transcendence degree (over k) of the field $k(X)$.

Examples: (a) If $\Phi : X \to Y$ is an open immersion, then $\dim(X) = \dim(Y)$. If Φ is dominant, then $\dim(X) \geq \dim(Y)$. If Φ is a closed immersion or a finite map, then $\dim(X) \leq \dim(Y)$. The dimension of $X \times Y$ is $\dim(X) + \dim(Y)$ (it suffices to check this for affine X and Y).

(b) If $X \subset \mathbf{A}_k^n$ is a nonempty hypersurface over an algebraically closed field, then $I(X)$ is principal, so $k[X] \cong k[x_1, ..., x_n]/\langle f \rangle$ for a nonconstant f and $\dim(X) = n - 1$. The same holds for projective hypersurfaces.

Throughout this section, we assume k is algebraically closed.

Lemma 15.1: The only variety over k of dimension 0 is a point.

Proof: If X is a point, then $k(X) = k$. Conversely, if X is not a point, choose an open immersion $V \hookrightarrow X$ from an affine V (also not a point). If $p, q \in V$ are distinct, there is an $f \in k[V]$ such that $f(p) \neq f(q)$, and it follows that $k \subset k(V) = k(X)$ is a nontrivial field extension, so it must have positive transcendence degree since k is assumed to be algebraically closed.

Exercise: Let $X \subset \mathbf{P}_k^n$ be a projective variety and $C(X) \subset \mathbf{A}_k^{n+1}$ be the affine cone over X. Show that $\dim(C(X)) = \dim(X) + 1$.

(Hint: Find an isomorphism: $\{C(X) - V(x_i)\} \cong \{\mathbf{A}_k^1 - \{0\}\} \times \{X - V(x_i)\}$.)

In Example (b) above, the dimension of a hypersurface is one less than the dimension of the ambient affine space because "obviously" a single polynomial relation reduces the transcendence degree of the field of fractions by one. This is less obvious but still true in an arbitrary variety. Namely:

Theorem 15.2: (Krull's Principal Ideal Theorem) If X is a variety over k of dimension n and $f \in \mathcal{O}_X(U)$ is nonconstant, then each irreducible component $Z \subseteq V(f) := \{x \in U | f(x) = 0\} \subset U$ has dimension exactly $n-1$.

Proof: Given X, f and Z, choose $V \subset U$ to be an affine open subset such that $V \cap Z \neq \emptyset$, but $V \cap Z' = \emptyset$ for all other components $Z' \subset V(f)$. (We can do this because $U - \cup Z'$ is open, hence covered by open affines!) Thus $\dim(V) = n$, $\dim(V \cap Z) = \dim(Z)$ and $f \in k[V]$ has the property that $Z \cap V = V(f)$, so $\mathrm{rad}(f) = I(Z \cap V)$ is a prime ideal.

Let $\pi : V \to \mathbf{A}_k^n$ be the finite dominant map from the geometric Noether normalization (§12) and regard π^* as an inclusion $k[x_1, ..., x_n] \subset k[V]$. The idea behind the proof is as follows: the image of $V(f) \subset V$ under π is an irreducible closed subvariety of $\mathbf{A}_k^n$ (because π is finite). If we show that it is a hypersurface and has the same dimension as $V(f)$, then we are done by Example (b) above.

Let $K = k(x_1, ..., x_n)$ and $L = k(V)$. Let $g = \mathrm{Nm}_{L/K}(f)$ be the image of f under the norm map. Since $\mathrm{Nm}(f)$ is always a power of the constant term in the minimal polynomial of f, the coefficient lemma (§13) tells us that $g \in k[x_1, ..., x_n]$. Moreover, suppose $y^m + a_{m-1}y^{m-1} + ... + a_m = 0$ is the minimal polynomial and $g = a_m^k$. Then we have:

$$g = -a_m^{k-1} f(f^{m-1} + a_1 f^{m-2} + ... + a_{m-1})$$

so $g \in \mathrm{rad}(f)$ since the $a_i \in k[x_1, ..., x_n]$, and we therefore have an inclusion: $\mathrm{rad}(g) \subseteq \mathrm{rad}(f) \cap k[x_1, ..., x_n]$ of ideals in $k[x_1, ..., x_n]$. On the other hand, if $h \in \mathrm{rad}(f) \cap k[x_1, ..., x_n]$, then $h^M = bf$ for some positive M and $b \in k[V]$, so $h^{M[L:K]} = \mathrm{Nm}_{L/K} h^M = (\mathrm{Nm}_{L/K} b)g$. Thus $h \in \mathrm{rad}(g)$, and we get $\mathrm{rad}(g) = \mathrm{rad}(f) \cap k[x_1, ..., x_n]$.

As in the proof of Proposition 12.4, this implies that $V(g) = \pi(V(f))$. So it only remains to prove that these varieties have the same dimension. But $k[V]/\mathrm{rad}(f)$ is integrally dependent upon $k[x_1, ..., x_n]/\mathrm{rad}(g)$, so the extension of fields: $k(V(g)) \subset k(V(f))$ is finite, hence they have the same transcendence degree over k, and we are done.

Remark: By induction, we see that if X is a variety of dimension n and $f_1, ..., f_r \in \mathcal{O}_X(U)$, then each component of $V(f_1, ..., f_r)$ has dimension at least $n - r$. We may not have equality because some f_i may be zero on an irreducible component of $V(f_1, ..., f_{i-1})$.

Definition: The **topological codimension** of a closed irreducible subset $Z \subset X$ is the supremum of the lengths r of chains:

$$Z \subset Z_1 \subset ... \subset Z_r = X$$

of closed irreducible sets, each properly contained the next.

Remark: It follows easily from the Noetherianness of $k[V]$ for an affine V that one cannot have a chain of infinite length. But much more is true:

Corollary 15.3: The topological codimension of a closed irreducible $Z \subset X$ is equal to $\dim(X) - \dim(Z)$. It is realized by a chain:

$$Z \subset Z_1 \subset ... \subset Z_r = X$$

such that each $\dim(Z_i) = \dim(Z) + i$.

Proof: Given $Z \subset X$, then either $Z = X$, or else there is an open affine $V \subset X$ such that $V \cap Z \neq V$. Then $V \cap Z \subset V$ corresponds to a nonzero ideal $I \subset k[V]$. In particular, there is a nonzero $f \in k[V]$ such that $f(Z \cap V) \equiv 0$. By the theorem, $Z \cap V$ is contained in a component $W \subseteq V(f)$, which satisfies $\dim(W) + 1 = \dim(V)$. Taking closures, we see that Z is contained in a closed irreducible subset $Y \subset X$ such that $\dim(Y) + 1 = \dim(X)$. If we repeat the process with X replaced by Y and continue, then we eventually reach $Z = Y$, producing a chain of irreducible subsets of length $\dim(X) - \dim(Z)$ with the desired properties. On the other hand, if we are given any other chain, then the argument above implies that the dimension must drop with each proper inclusion, and the corollary follows.

Special Case: The topological codimension of a point is the **topological dimension** of X, which mercifully does not depend upon the point, since it is equal to $\dim(X)$. Thus we have proved that $\dim(X)$, which was defined in terms of $k(X)$, actually only depends upon the Zariski topology of X(!)

If Z is a closed, irreducible subset of a variety X of (topological) codimension r, it is natural to ask whether there are exactly r regular functions on X whose common zero locus is Z. This is a ridiculous question if X is projective since the only regular functions there are the constants (14.3). Even in the affine case, where there are lots of regular functions the answer is often no. But we do have the following:

Proposition 15.4: If V is an affine variety and $Z \subset V$ is closed and irreducible of codimension r, then there are regular functions $f_1, ..., f_r \in k[V]$ such that Z is a **component** of $V(f_1, ..., f_r)$. Moreover, $f_1, ..., f_r$ may be chosen so that **every** component of $V(f_1, ..., f_r)$ has codimension r.

Proof: If $Z = V$, the Proposition is trivial. Otherwise, there is a nonzero $f_1 \in I(Z) \subset k[V]$. By Krull's theorem, each component of $V(f_1)$ has codimension 1 in V. Suppose $Z_1, ..., Z_k \subset V(f_1)$ are all the components.

Inductively, suppose that $Z \subset V(f_1, ..., f_q)$, and that every component of $V(f_1, ..., f_q)$ has codimension q in V. Let $Z_1, ..., Z_k \subset V(f_1, ..., f_q)$ be all the irreducible components. If we can find a polynomial $f_{q+1} \in I(Z)$ such that $f_{q+1} \notin I(Z_i)$ for all $i = 1, ..., k$, then $Z \subset V(f_1, .., f_{q+1})$, and by applying Krull's theorem to each Z_i, we see that every component of $V(f_1, ..., f_{q+1})$ has codimension $q+1$, so we may continue the induction, and ultimately, we will get Z as a component of $V(f_1, ..., f_r)$. The existence of such an f_{q+1} is a consequence of the following algebraic lemma:

Lemma 15.5: Suppose $P, P_1, ..., P_k \subset A$ are prime ideals in a ring A such that $P \not\subset P_i$ for all i. Then $P \not\subset \cup_{i=1}^{k} P_k$.

Proof: We may assume that $P_i \not\subset P_j$ for $i \neq j$. By induction, we may therefore assume that $P_i \not\subset \cup_{j \neq i} P_j$. Let $f_i \in P_i - \cup_{j \neq i} P_j$ for all $i = 1, ..., k$, and let $g_i \in P - \cup_{j \neq i} P_j$ (which we may also do by induction). Then

$$f = \sum_{j=1}^{k} (\prod_{i \neq j} g_i f_i)$$

is in $P - \cup_{i=1}^{k} P_i$, as desired.

Exercise: The question motivating Proposition 15.4 is not so ridiculous for projective varieties after all. If $X \subset \mathbf{P}_k^n$ is a projective variety and $Z \subset X$ is a closed irreducible subset of codimension r, show that there are r homogeneous elements $F_1, ..., F_r \in k[X]$ such that Z is a component of $V(F_1, ..., F_r)$ and all components of $V(F_1, ..., F_r)$ have codimension r in X.

The case of codimension 1 in affine and projective space is quite nice.

Proposition 15.6: Every closed, irreducible subset of codimension 1 in either $\mathbf{A}_k^n$ or $\mathbf{P}_k^n$ is a hypersurface.

Proof: By Proposition 15.4, if $V \subset \mathbf{A}_k^n$ has codimension 1, then it is an irreducible component of of a hypersurface $V(g)$. Since $k[x_1, ..., x_n]$ is a UFD, we may write $g = f_1^{d_1} ... f_r^{d_r}$, where the f_i are distinct irreducible polynomials, and the $V(f_i)$ are the irreducible components of $V(g)$. One of these must be V. One uses the exercise in the same way for projective space.

In case $X \subset \mathbf{P}_k^n$ is a projective variety, we can measure the dimension of X by its Hilbert polynomial:

Proposition 15.7: The dimension of X is equal to the degree of $\mathrm{H}_X(d)$.

Proof: By two applications of Corollary 15.3, we have a chain:

$$\{p\} \subset Z_1 \subset ... \subset Z_m = X \subset ... \subset Z_n = \mathbf{P}_k^n$$

of closed irreducible subsets, where $p \in X$ is a point and $i = \dim(Z_i)$.

Moreover, by the exercise above, there are homogeneous polynomials $F_n, ..., F_1 \in k[x_0, ..., x_n]$ such that each Z_{i-1} is a component of $V(F_i) \cap Z_i$. Let $M_i = k[Z_i]/\langle F_i \rangle$, thought of as a graded $k[x_0, ..., x_n]$-module. By Lemma 3.5, we have $\deg(\mathrm{H}_{M_i}(d)) = \deg(\mathrm{H}_{Z_i}(d)) - 1$. But $k[Z_{i-1}]$ is a quotient of M_i, so it follows that:

$$\deg(\mathrm{H}_{Z_{i-1}}(d)) \leq \deg(\mathrm{H}_{Z_i}(d)) - 1$$

for all i.

Since $\deg(\mathrm{H}_p(d)) = 0$ and $\deg(\mathrm{H}_{\mathbf{P}_k^n}(d)) = n$ (see §3), the inequalities above must be **equalities** for all i, and in particular, $\deg(\mathrm{H}_X(d)) = m = \dim(X)$.

Next, we consider intersections in affine and projective space.

Proposition 15.8: Suppose $V, W \subset \mathbf{A}_k^n$ are affine varieties of codimension r and s, respectively. Then every component of $V \cap W$ has codimension $r + s$ or less. (Of course, $V \cap W$ may be empty!)

Proof: An easy case. Suppose that $V = V(f_1, ..., f_r)$. Then we have $V \cap W = V(f_1, ..., f_r) \cap W$ and the Proposition follows immediately from the remark following Krull's theorem.

The general case. Let $k[\mathbf{A}_k^n \times \mathbf{A}_k^n] \cong k[\mathbf{A}_k^{2n}] \cong k[x_1, ..., x_n, y_1, ..., y_n]$, let $\Delta \subset \mathbf{A}_k^{2n}$ be the diagonal, and let $p : \Delta \xrightarrow{\sim} \mathbf{A}_k^n$ be the isomorphic restriction of the first projection.

Let $V \times W \subset \mathbf{A}_k^{2n}$ be the natural inclusion. Then from the equivalences:

$$(v, w) \in \Delta \Leftrightarrow v = w \Leftrightarrow v \in V \cap W$$

it follows that $\{V \times W\} \cap \Delta$ is mapped homeomorphically to $V \cap W$ under the isomorphism p. In particular, the irreducible components of $\{V \times W\} \cap \Delta$ and of $V \cap W$ have the same dimension(!)

But now $\dim(\Delta) = n$ and $\dim(V \times W) = 2n - (r+s)$, so the sum of their codimensions is $r + s + n$, and by the easy case applied to $\Delta = V(x_1 - y_1, ..., x_n - y_n)$ and $V \times W$, it follows that each component of the intersection has dimension at least $2n - (r + s + n) = n - (r + s)$, as desired.

Exercise: If $X, Y \subset \mathbf{P}_k^n$ are projective varieties of codimension r and s, show that every component of $X \cap Y$ has codimension at $r + s$ or less.

Corollary 15.9: (The Nonempty Intersection Property of $\mathbf{P}_k^n$) The intersection $X \cap Y$ of projective varieties $X, Y \subset \mathbf{P}_k^n$ is not empty if the sum of their codimensions is n or less.

Proof: Let r and s be the codimensions of X and Y, respectively. Consider the affine cones $C(X), C(Y) \subset \mathbf{A}_k^{n+1}$. By the first exercise of this section, we see that $C(X)$ and $C(Y)$ have codimension r and s in $\mathbf{A}_k^{n+1}$ respectively, so every component of $C(X) \cap C(Y)$ has dimension at least $n + 1 - (r + s) \geq 1$ by the Proposition. But $(0, ..., 0) \in C(X) \cap C(Y)$, so the intersection contains at least one component passing through $(0, ..., 0)$. Such a component must contain points other than $(0, ..., 0)$, since it has positive dimension. But such points determine points in the intersection $X \cap Y$, which is therefore not empty.

Remarks: The corollary is false when projective space is replaced by affine space (think of parallel lines in $\mathbf{A}_k^2$). More seriously, and perhaps counterintuitively, the **proposition** is false for general affine (and projective) varieties. For example, inside the three-dimensional $V = V(xy - zw) \subset \mathbf{A}_k^4$, the cones over lines in the same family of the quadric yield copies of $\mathbf{A}_k^2$ which intersect only at the origin. The proposition would have given this intersection a positive dimension(!) Also notice that if $n, m \geq 1$, then the products $\mathbf{P}_k^n \times \mathbf{P}_k^m$ do not have the nonempty intersection property (why?) giving another proof of the fact that $\mathbf{P}_k^{n+m}$ is not isomorphic to $\mathbf{P}_k^n \times \mathbf{P}_k^m$.

Next, we turn our attention to (dominant) regular maps $\Phi : X \to Y$ of varieties over the algebraically closed field k and more specifically to the dimensions of the fibers. Recall that a finite map has finite fibers (the converse is not always true!). In general the fiber dimension is not constant (where have we already seen an example of this?) Here is another example:

Example: (The variety of kernels) Think of $\mathbf{P}_k^{n^2-1}$ as parametrizing $n \times n$ matrices (modulo scalars) with $\mathrm{PGL}(n,k) \subset \mathbf{P}_k^{n^2-1}$ as an open subset. Then:

$$X := \{(A, v) \mid v \in \ker(A)\} \subset \mathbf{P}_k^{n^2-1} \times \mathbf{P}_k^{n-1}$$

is well-defined, closed and irreducible (exercise), and the first projection restricts to a surjective regular map $p : X \to \mathbf{P}_k^{n^2-1} - \mathrm{PGL}(n,k)$. Let $D_r \subset \mathbf{P}_k^{n^2-1}$ be the closed "degeneracy locus" parametrizing matrices with a kernel of dimension at least r (we will see in §16 that the D_r are irreducible of codimension r^2). Then we obtain a chain:

$$D_{n-1} \subset D_{n-2} \subset \ldots \subset D_1 = \mathbf{P}_k^{n^2-1} - \mathrm{PGL}(n,k)$$

with the property that $p^{-1}(A) = \mathbf{P}(\ker(A)) \cong \mathbf{P}_k^{r-1}$ for all $A \in D_r - D_{r+1}$. In particular, p is injective on $p^{-1}(D_1)$. For our purposes, the relevant observation is that:

$$\{A \in \mathbf{P}_k^{n^2-1} \mid \dim(p^{-1}(A)) \geq r\} = D_{r+1}$$

is closed in X.

Proposition 15.10: If $\Phi : X \to Y$ is a dominant regular map of varieties over k, then:

(a) If $Z \subseteq \Phi^{-1}(p)$ is a component, then $\dim(Z) \geq \dim(X) - \dim(Y)$.

(b) If $W \subseteq Y$ is closed and irreducible, and if $Z \subseteq \Phi^{-1}(W)$ is a component with the property that $\Phi|_Z : Z \to W$ is dominant, then

$$\dim(Z) \geq \dim(W) + (\dim(X) - \dim(Y)).$$

Proof: We prove (b), of which (a) is a special case.

Let $V \subset Y$ be open and affine satisfying $V \cap W \neq \emptyset$, and observe that because $\Phi|_Z$ is dominant, we have $Z \cap \Phi^{-1}(V) \neq \emptyset$. So it suffices to assume $Y = V$ is affine, replacing X by $\Phi^{-1}(V)$, W by $W \cap V$ and Z by $Z \cap \Phi^{-1}(V)$.

Let r be the codimension of W in Y. Then by Proposition 15.4, there are regular functions $f_1, ..., f_r \in k[Y]$ such that W is a component of $V(f_1, ..., f_r)$. This implies that $Z \subseteq V(\Phi^*(f_1), ..., \Phi^*(f_r))$. I claim that Z is a component.

If $Z \subseteq Z' \subseteq V(\Phi^*(f_1), ..., \Phi^*(f_r))$ and Z' is closed and irreducible, then since $W = \overline{\Phi(Z)}$, it follows that $W \subseteq \overline{\Phi(Z')} \subseteq V(f_1, ..., f_r)$. Since $\overline{\Phi(Z')}$ is irreducible and W is a component of $V(f_1, ..., f_r)$, it follows that $W = \overline{\Phi(Z')}$, so $Z' \subseteq \Phi^{-1}(W)$. Since Z is a component of $\Phi^{-1}(W)$, it therefore follows that $Z = Z'$, as desired.

But now by the note following Krull's theorem, we see that the dimension of Z is at least $\dim(X) - r = \dim(X) - (\dim(Y) - \dim(W))$.

So every component of every fiber of a dominant map has at least the "expected" dimension. Keep in mind that this is false over non-algebraically closed fields. For example, the map $\Phi : \mathbf{R}^n \to \mathbf{R}$ given by $\Phi(x_1, ..., x_n) = x_1^2 + ... + x_n^2$ has a single point as the preimage of the origin. On the other hand, the "general" fiber is a sphere, which does have the expected dimension. The next Proposition says that over an algebraically closed field, the "general" fibers all have the expected dimension.

Theorem 15.11: If $\Phi : X \to Y$ is a dominant map of varieties over k, then there is a nonempty open $U \subset Y$ contained in the image of Φ such that:

(a) For all $p \in U$, the components of the fibers $\Phi^{-1}(p)$ all have the expected dimension $\dim(X) - \dim(Y)$, and more generally,

(b) If $W \subseteq Y$ is a closed irreducible set such that $W \cap U \neq \emptyset$, and if $Z \subseteq \Phi^{-1}(W)$ is a component dominating W, then

$$\dim(Z) = \dim(W) + (\dim(X) - \dim(Y)).$$

Proof: Once again, (a) is a special case of (b). As with Proposition 15.10, we may assume Y is affine. But this time we may assume X is affine as well. Cover $X = \cup_{i=1}^n V_i$ by affine open subsets V_i. If we assume the proposition for V_i, and let $U_i \subset \Phi(V_i)$ be the open subsets of Y satisfying (b) for each $\Phi|_{V_i}$, then $U := \cap_{i=1}^n U_i \subset \Phi(X)$ is an open subset of Y satisfying (b) for Φ.

Let $A = k[X]$ and $B = k[Y]$, and regard $\Phi^* : B \hookrightarrow A$ as an inclusion of rings. Let $r = \dim(X) - \dim(Y) = \ \mathrm{trd}_k(k(A)) - \ \mathrm{trd}_k(k(B))$, where trd_k denotes the transcendence degree over k. I claim that there are regular functions $f \in B$ and $a_1, ..., a_r \in A$ such that:

(i) The map $B_f[x_1, ..., x_r] \to A_f$ defined by $x_i \mapsto a_i$ is injective, and

(ii) A_f is integrally dependent upon $B_f[x_1, ..., x_r]$.

First, let's see how the Theorem follows from (i) and (ii). If we let $U_f = Y - V(f)$, then $\Phi^{-1}(U_f) = X - V(f)$, so both U_f and $\Phi^{-1}(U_f)$ are affine with $k[U_f] \cong B_f$ and $k[\Phi^{-1}(U_f)] \cong A_f$. The regular map $\Phi_f : \Phi^{-1}(U_f) \to U_f$ obtained by restricting Φ corresponds to

$$\Phi_f^* : B_f \to B_f[x_1, ..., x_r] \hookrightarrow A_f$$

which tells us that Φ_f factors as:

$$\Phi_f : \Phi^{-1}(U_f) \to U_f \times \mathbf{A}_k^r \to U_f$$

where the first map is finite and dominant and the second is the projection.

In particular, Φ_f is surjective, so $U_f \subset \Phi(X)$. Now suppose $W \subset Y$ is irreducible and closed, and $Z \subset X$ is a component of $\Phi^{-1}(W)$ dominating W. Then Φ_f restricts to a map from $Z \cap \Phi^{-1}(U_f)$ to $\{W \cap U_f\} \times \mathbf{A}_k^r$ with finite fibers followed by the projection. Putting this together with Proposition 15.10, we see that $\dim(Z) = \ \dim(W) + r$ as desired.

Let's see how to achieve (i) and (ii). Let $K = k(Y) = k(B)$, and consider the localization $R := A_{B-\{0\}}$ as a K-algebra. Notice that $k(X) = k(A)$ is isomorphic to the field of fractions $k(R)$ so by the transitivity of transcendence degree, we have:

$$\mathrm{trd}_K(k(R)) = \ \mathrm{trd}_k k(A) - \ \mathrm{trd}_k K = r$$

Thus by Noether normalization, there exist $\alpha_1, ..., \alpha_r \in R$ which are algebraically independent, and such that R is integrally dependent upon the subring $K[\alpha_1, ..., \alpha_r]$. In order to achieve (i) and (ii), we want the same result after replacing K by B_f for some $f \in B$, and R by A_f.

To get this, observe first that we may assume each $\alpha_i \in A$. After all, each α_i may be written as a_i/b_i for $a_i \in A, b_i \in B$, so replacing α_i by $b_i\alpha_i$ is a change of variables with coefficients in K(!). Thus, we have an inclusion:

$$B[x_1, ..., x_r] \hookrightarrow A; \quad x_i \mapsto a_i = \alpha_i b_i$$

If A is integrally dependent upon $B[a_1, ..., a_r]$, then we're done with $f = 1$. If not, then for each $a \in A$, we know that a is integral over $K[a_1, ..., a_r]$, by the discussion above. Thus for each $a \in A$, we may let f_a be the product of all denominators appearing in a monic equation for a over $K[a_1, ..., a_r]$, making a integral over $B_{f_a}[a_1, ..., a_r]$. But A is a finitely generated k-algebra, so if we let $g_1, ..., g_m$ be generators, and let $f = \prod_{i=1}^m f_{g_i}$, then *every* $a \in A$ is integral over $B_f[x_1, ..., x_r]$, and it follows that every $a/f^n \in A_f$ is integral over $B_f[x_1, ..., x_r]$, as well, giving (i) and (ii).

Definition: If X is a topological space, then $\phi : X \to \mathbf{R}$ is **upper-semicontinuous** if $\{x \in X \mid \phi(x) \geq \alpha\} \subset X$ is closed for each $\alpha \in \mathbf{R}$.

Corollary 15.12: If $\Phi : X \to Y$ is a regular map, then the (integer-valued) function $e : X \to \mathbf{R}$ defined by:

$$e(x) := \max\{\dim(Z) \mid x \in Z \text{ and } Z \text{ is a component of } \Phi^{-1}(\Phi(x))\}$$

is upper-semicontinuous.

Proof: We may as well assume Φ is dominant, since nothing in the definition of e changes if we replace Y by $\overline{\Phi(X)}$. Also, it suffices to show that for all **integers** a, the sets $X_a := \{x \in X \mid e(x) \geq a\}$ are closed. And by Proposition 15.10, $X_a = X$ for $a \leq r := \dim(X) - \dim(Y)$.

Let $U \subset Y$ satisfy the conditions of the theorem. If $x \in \Phi^{-1}(U)$, then by the theorem, $e(x) = r$, so if $a > r$, it follows that $X_a \subset X - \Phi^{-1}(U)$. If we let $X - \Phi^{-1}(U) = Z_1 \cup ... \cup Z_N$ be the decomposition into irreducibles, then by induction on the dimension of X, we may assume that each $(Z_i)_a := \{x \in Z_i \mid e(x) \geq a\} \subset Z_i$ is closed. If $x \in X_a$, then there is a component $Z \subset \Phi^{-1}(\Phi(x))$ of dimension at least a, hence this component is contained in $X - \Phi^{-1}(U)$, so it must lie entirely in some Z_i by irreducibility. Thus $X_a = \cup_{i=1}^N (Z_i)_a$ is a finite union of closed sets, so it is closed.

Examples: (a) If $\Phi : \mathbf{A}_k^2 \to \mathbf{A}_k^2$ is the map given by $\Phi(x, y) = (x, xy)$, then $e(x, y) = 0$ if $x \neq 0$ and $e(x, y) = 1$ if $x = 0$.

(b) If $p : X \to \mathbf{P}_k^{n^2-1}$ is the projection from the variety of kernels, then

$$e(x) \geq r \Leftrightarrow x \in \Phi^{-1}(D_{r+1})$$

Birational maps are an important special case of dominant maps:

Definition: A regular, dominant map $\Phi : X \to Y$ is **birational** if the induced map of fields $\Phi^* : k(Y) \to k(X)$ is an isomorphism.

Birational maps are not usually isomorphisms. They may not be surjective, for instance, as in the case of an open immersion. They may be finite-to-one over closed subsets of Y, as is the case with the normalization of a (not normal) variety. They may even have fibers of positive dimension over closed subsets of Y. But the following Proposition says that at least "generically" a birational map is indeed an isomorphism.

Proposition 15.13: If $\Phi : X \to Y$ is a birational map of varieties over k, then there are open sets $U \subset X$ and $V \subset Y$ such that $\Phi(U) = V$ and the restricted map $\Phi|_U : U \to V$ is an isomorphism.

Exercise: Prove this.

Strategy: First show that Y may be assumed to be affine (this is easy) and X may also be assumed to be affine (this is a more involved!) Then prove the following algebraic fact: If $k[Y] \hookrightarrow k[X]$ is an inclusion, such that the induced map $k(Y) \hookrightarrow k(X)$ is an isomorphism, then there is an $f \in k[Y]$ such that the induced map $k[Y]_f \hookrightarrow k[X]_f$ is also an isomorphism. Show how the proposition follows from this algebraic fact.

Tenth Homework Assignment

1.- 4. Do the four exercises in the text.

5. Prove that if $X_1, ..., X_m \subset \mathbf{P}^n_k$ are projective varieties of codimension $s_1, ..., s_m$, and if $s_1 + ... + s_m \le n$, then $X_1 \cap ... \cap X_m \ne \emptyset$.

6. (a) Suppose $\Phi : X \to Y$ is a dominant, closed map of varieties over k, suppose $Z \subset X$ is a closed subset with the following property: $\Phi(Z)$ is irreducible, and for each $y \in \Phi(Z)$, the set $\Phi^{-1}(y) \cap Z$ is irreducible of dimension r (independent of y). Conclude that Z is irreducible. Find counterexamples to the same statement if either Φ is not a closed map or r is allowed to depend upon y.

(b) Prove that the variety of kernels is irreducible.

(c) Prove that $D_1 = \mathbf{P}^{n^2-1}_k -$ PGL(n, k) is an irreducible hypersurface.