Course Notes for Math 738 (Algebraic Geometry)

16. Varieties. Differentials and Tangents. Kähler differentials are introduced and used to get an intrinsic definition of the nonsingular locus of a variety, which happens to always be an open subset. The Zariski tangent space is introduced, as well as the tangent cone which is an intrinsic affine subvariety. Finally, we define the blow-up of a variety at a point, and show how it replaces the point with its projectivized tangent cone.

Motivation: Suppose $k = \mathbf{R}$ or $\mathbf{C}$ and $f_1, ..., f_r \in k[x_1, ..., x_n]$ are given, together with a point $p \in V(f_1, ..., f_r)$. Then recall that the implicit function theorem tells us $V(f_1, ..., f_r)$ (which one should think of as the fiber $\Phi^{-1}(0)$ of the map $\Phi : k^n \to k^r$ given by $\Phi(p) = (f_1(p), ..., f_r(p)))$ is a submanifold of k^n in a neighborhood of p if the following Jacobian matrix has full rank:

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1}(p) & \frac{\partial f_1}{\partial x_2}(p) & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \\ \frac{\partial f_2}{\partial x_1}(p) & \frac{\partial f_2}{\partial x_2}(p) & \cdots & \frac{\partial f_2}{\partial x_n}(p) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_r}{\partial x_1}(p) & \frac{\partial f_r}{\partial x_2}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{vmatrix}$$

A variety X will be nonsingular at p if "X can defined by equations with a Jacobian matrix of the right rank at p". To get an intrinsic definition for this, we introduce some notions from commutative algebra:

Suppose A is a k-algebra and M is a module over A.

Definition: A **derivation** (or **k-derivation**) $d : A \to M$ is a homomorphism of the abelian groups under addition, which additionally satisfies:

$$d(aa') = a(da') + a'(da)$$

(i.e. Leibniz' rule) and also:

$$dc = 0 \text{ for all } c \in k.$$

Key Observation: If $d : A \to M$ is a derivation and $\alpha : M \to N$ is an A-module homomorphism, then $\delta := \alpha \circ d : A \to N$ is also a derivation. In fact, precomposing with α induces an A-module homomorphism:

$$h_{d,N} : \mathrm{Hom}_A(M, N) \to \{\text{derivations } \delta : A \to N\}$$

Definition: A derivation $d : A \to \Omega_A$ is **universal** if $h_{d,M}$ is an isomorphism for every A-module M. In other words, every derivation $\delta : A \to M$ is the composition $\alpha \circ d$ for a **unique** homomorphism $\alpha : \Omega_A \to M$ of A-modules.

Remark: By the usual general nonsense, if $d : A \to \Omega_A$ and $d' : A \to \Omega'_A$ are universal, then $\Omega_A \cong \Omega'_A$ by a uniquely determined isomorphism. We reserve the notation Ω_A for "the" A-module associated to a universal derivation. This A-module is called the **module of Kähler differentials on** A.

Proposition 16.1: (a) Universal derivations always exist.

(b) If $A \cong k[x_1, ..., x_n]/I$, then Ω_A is finitely generated as an A-module. Precisely, if $I = \langle f_1, ..., f_m \rangle$, then Ω_A is isomorphic to the quotient of the free module $\oplus_{i=1}^n A(dx_i)$ by the submodule generated by the $df_j := \sum_{i=1}^n \frac{\partial f_j}{\partial x_i} dx_i$.

Proof: (a) Given A, let F be the free A-module generated by the symbols $\{da \mid a \in A\}$ and let $N \subseteq F$ be the submodule generated by the relations:

(i) $d(a + a') - da - da'$ for all $a, a' \in A$,

(ii) $d(aa') - a(da') - a'(da)$ for all $a, a' \in A$ and

(iii) dc for all $c \in k$.

Then $\Omega_A := F/N$ with natural derivation $d : A \to \Omega_A$ sending $a \mapsto da$ is universal. If $A \cong k[x_1, ..., x_n]$, then $F/N \cong \oplus_{i=1}^n A(dx_i)$, and the universal $d : k[x_1, ..., x_n] \to \oplus_{i=1}^n k[x_1, ..., x_n]dx_i$ is given by $d(g) = \sum_{i=1}^n \frac{\partial g}{\partial x_i} dx_i$.

If $A \cong k[x_1, ..., x_n]/\langle f_1, ..., f_m \rangle$, let $K \subseteq \oplus_{i=1}^n A(dx_i)$ be the submodule generated by the images of the df_j. If $g = \sum_{j=1}^m h_j f_j \in I$, then

$$\sum_{i=1}^n \frac{\partial g}{\partial x_i} dx_i = \sum_{j=1}^m (\sum_{i=1}^n \frac{\partial h_j}{\partial x_i} dx_i) f_j + \sum_{j=1}^m h_j (\sum_{i=1}^n \frac{\partial f_j}{\partial x_i} dx_i)$$

so $\delta : A \to D/K$ given by $\delta(a) = \sum_{i=1}^n \frac{\partial g}{\partial x_i} dx_i$ (for any $g \in k[x_1, ..., x_n]$ which maps to $a \in k[x_1, ..., x_n]/I$) is a well-defined universal derivation.

Example: If $A = k[x, y]/\langle xy \rangle$, then $\Omega_A = Adx \oplus Ady/\langle xdy + ydx \rangle$. Note that the nonzero element $\omega := xdy = -ydx$ has the property that $x\omega = y\omega = 0$. Thus $A\omega = k\omega \subset \Omega_A$. The quotient module $\Omega_A/k\omega \cong k[x]dx \oplus k[y]dy$ looks like the sum of the modules of differentials on the x and y axes respectively.

Here is a more natural construction of Ω_A. Given a k-algebra A, let $\mu : A \otimes_k A \to A$ be the multiplication homomorphism $\mu(\sum a \otimes a') = \sum aa'$, and consider the ideal $J := \ker(\mu) \subset A \otimes_k A$. Since J is an $A \otimes_k A$-module, it follows that J/J^2 is an $A = A \otimes_k A/J$-module.

Proposition 16.2: The map $d : A \to J/J^2$ defined by setting $d(a)$ to be the image of $1 \otimes a - a \otimes 1$ is a universal derivation.

Proof: I'm going to pass freely between elements of J/J^2 and their representatives in J. For instance, if $m \in J/J^2$ and $a \in A$, then we can compute am by taking a lift of m to J, multiplying by $a \otimes 1$ or $1 \otimes a$ and passing back down to J/J^2 (why?).

From the identity: $a \otimes a' + a' \otimes a = 1 \otimes aa' + aa' \otimes 1$ (modulo J^2) it follows that $d(aa') = 1 \otimes aa' - aa' \otimes 1 = (a \otimes 1)(1 \otimes a' - a' \otimes 1) + (a' \otimes 1)(1 \otimes a - a \otimes 1) = a(da') + a'(da)$. Of course, $dc = 1 \otimes c - c \otimes 1 = 0$ for $c \in k$, so d is a derivation.

To see that $d : A \to J/J^2$ is universal, suppose we are given a derivation $\delta : A \to M$. We obtain a ring $R = A \oplus M$ by setting:

$$(a, m) \cdot (a', m') = (aa', am' + a'm).$$

The map $f : A \times A \to R$ given by $f(a, a') = (aa', a(\delta a'))$ is bilinear and induces a ring homomorphism $f : A \otimes_k A \to R$ because δ is a derivation. Moreover, $f(J) \subset M$, by definition, and it follows that $f(J^2) \equiv 0$, so we obtain an A-module homomorphism $\overline{f} : J/J^2 \to M$. Since $\overline{f}(1 \otimes a - a \otimes 1) = \delta a$, (check this!) we see that $\delta = \overline{f} \circ d$. Exercise: Why is $\overline{f}$ the unique A-module homomorphism with this property?

We need to extend our notion of localizing to modules:

Definition: If M is an A-module and $S \subset A$ is a multiplicative set, then:

$$M_S := \{\frac{m}{s} \mid m \in M, s \in S\}/\sim$$

where $\frac{m}{s} \sim \frac{m'}{s'}$ iff $a(ms' - m's) = 0$ for some $a \in S$. Observe that M_S is naturally an A_S-module.

Universal derivations localize to universal derivations:

Proposition 16.3: If $d : A \to \Omega_A$ is a universal derivation and $S \subset A$ is a multiplicative set, then $d_S : A_S \to (\Omega_A)_S$ defined by the quotient rule:

$$d_S(\frac{a}{s}) := \frac{s(da) - a(ds)}{s^2}$$

is a universal derivation. So $(\Omega_A)_S \cong \Omega_{A_S}$ by a canonical isomorphism.

Proof: Exercise.

We will need the following standard results from field theory:

Existence of a primitive element: If $K \subset L$ is a separable field extension, then there is an element $\alpha \in L$ such that $L = K[\alpha]$.

Existence of a separating basis: If $k \subset K$ is a field extension of finite type (see §4) and k is algebraically closed (or more generally "perfect"), then there is a trancendence basis $\alpha_1, ..., \alpha_n$ for K such that the field extension $k(\alpha_1, ..., \alpha_n) \subset K$ is separable. Such a basis is referred to as separating.

We assume k is algebraically closed for the rest of this section.

Let X be a variety over k. To each point $p \in X$, we have the module of Kähler differentials $\Omega_{\mathcal{O}_{X,p}}$ over the local ring $\mathcal{O}_{X,p}$ (with maximal ideal m_p). This is finitely generated by Propositions 16.1(b), 16.3 and HW #1(a). Thus

$$\Omega_X(p) := \Omega_{\mathcal{O}_{X,p}} / m_p \Omega_{\mathcal{O}_{X,p}}$$

is a finite dimensional vector space over $k = \mathcal{O}_{X,p}/m_p\mathcal{O}_{X,p}$.

Theorem 16.4: (a) For each $p \in X$, there is a canonical isomorphism:

$$\mathrm{Hom}_k(\Omega(p), k) \cong \mathrm{Hom}_k(m_p/m_p^2, k)$$

(this k-vector space is called the **Zariski tangent space to X at p**.)

(b) The function $e : X \to \mathbf{R}$ defined by:

$$e(p) := \dim_k(m_p/m_p^2)$$

is upper-semi continuous. Moreover, $e(p) \geq \dim(X)$ for all $p \in X$ and $e(p) = \dim(X)$ on a nonempty open subset of X.

Proof: Let $A = \mathcal{O}_{X,p}$, and $k = A/m_p$ as an A-module. We start with the canonical isomorphism of A-modules:

$$\mathrm{Hom}_A(\Omega_A, k) \cong \{\text{derivations } \delta : A \to k\}$$

On the left side, $\mathrm{Hom}_A(\Omega_A, k) = \mathrm{Hom}_k(\Omega(p), k)$ (HW # 1(e)).

On the right side, a derivation $\delta : A \to k$ has the property that $\delta(m_p^2) \equiv 0$, so it certainly determines a linear map $\overline{\delta} : m_p/m_p^2 \to k$. On the other hand, given $l \in \mathrm{Hom}_k(m_p/m_p^2, k)$, define $\delta_l(a) := l(a - a(p))$. Then

$$\delta_l(ab) = l(ab - a(p)b(p)) = a(p)l(b - b(p)) + b(p)l(a - a(p))$$

as you can easily check, so δ_l is a derivation, and (a) follows.

Next, we use (a) to prove (b) since now $e(p) = \dim_k(\Omega_X(p))$. Choose an affine neighborhood V of p and suppose $k[V] \cong k[x_1, ..., x_n]/\langle f_1, ..., f_m\rangle$. There are two important observations to make. First, by Proposition 16.3, we have $(\Omega_{k[V]})_{m_p} \cong \Omega_{\mathcal{O}_X,p}$, and it follows from HW #1(b) that:

$$\Omega_X(p) \cong (\Omega_{k[V]})_{m_p}/m_p(\Omega_{k[V]})_{m_p} \cong \Omega_{k[V]}/m_p\Omega_{k[V]}$$

(we sloppily let m_p stand for both the maximal ideal in $k[V]$ and in $\mathcal{O}_{X,p}$.) Second, using 16.1(b) and HW # 1(c), we have:

$$\Omega_{k[V]}/m_p\Omega_{k[V]} \cong \oplus_{i=1}^n k(dx_i)/\langle \sum_{i=1}^n \frac{\partial f_1}{\partial x_i}(p)dx_i, ..., \sum_{i=1}^n \frac{\partial f_m}{\partial x_i}(p)dx_i\rangle$$

From this we conclude that $\dim_k(\Omega_X(p)) = n - \mathrm{rank}(|\frac{\partial f_j}{\partial x_i}(p)|)$ where $|\frac{\partial f_j}{\partial x_i}(p)|$ is the Jacobian matrix associated to the f_j. By the same reasoning, this is true for all $q \in V$. The set of $q \in V$ such that $\mathrm{rank}(|\frac{\partial f_j}{\partial x_i}(q)|) \leq n - \alpha$ is closed in V since it is given by the vanishing of the appropriate minors of a matrix of polynomials, hence $e : X \to \mathbf{R}$ is upper-semi continuous on each affine open V and it follows that e is upper-semi continuous.

To prove the last assertions, it suffices by upper-semi continuity to prove that $e(p) = n := \dim(X)$ for all p in an open (hence dense) subset of X. Let $V \subset X$ be an open affine subset. The algebra results listed above imply that $k(V) \cong k(x_1, ..., x_n)[x_{n+1}]/f(x_{n+1})$, and we may as well assume, clearing denominators, that $f \in k[x_1, ..., x_{n+1}]$ is irreducible. It follows that $k(V(f)) \cong k(V)$ for the hypersurface $V(f) \subset \mathbf{A}_k^{n+1}$.

Thus there is a birational isomorphism $\Phi : V - - > V(f) \subset \mathbf{A}_k^{n+1}$, hence a birational **regular map** $\Phi : U \to V(f)$ from the domain of Φ. By Proposition 15.13, such a map is an isomorphism from some open subset of U to an open subset of $V(f)$, so it suffices to prove that $\dim_k(\Omega_{V(f)})(q) = n$ for all q in an open subset of $V(f)$. Here the Jacobian matrix is the "gradient" $\nabla f := (\frac{\partial f}{\partial x_1}, ... \frac{\partial f}{\partial x_{n+1}})$ so $e(q) = n$ iff some $\frac{\partial f}{\partial x_i}(q) \neq 0$ ($e(q) = n+1$ otherwise). Suppose f has degree N. Then each $\frac{\partial f}{\partial x_i}$, if nonzero, is a polynomial of degree $N-1$, and by the irreducibility of f, it is impossible that $V(f) \subseteq V(\frac{\partial f}{\partial x_i})$. In characteristic zero, the gradient is never identically zero. But in characteristic $p > 0$, this happens exactly when every variable appears in every term of f as a pth power(!) But this would imply $f = g^p$ for some $g \in k[x_1, ..., x_{n+1}]$, violating the irreducibility of f. Thus the open subset of $V(f)$ where the gradient of f fails to vanish is not empty and the proof is complete.

Remarks: Some important results are embedded in this proof(!)

(i) Every variety over k has a nonempty open subset which is isomorphic to an open subset of an affine hypersurface.

(ii) If $p \in V \subset X$ is an affine neighborhood of p, then we can compute $\dim(m_p/m_p^2)$ as the corank in k^n of the Jacobian matrix $|\frac{\partial f_j}{\partial x_i}(p)|$ for any set of generators $f_1, ..., f_m$ for the ideal of $V \subset \mathbf{A}_k^n$.

Definition: A point $p \in X$ is **nonsingular** if $\dim_k(m_p/m_p^2) = \dim(X)$. The variety X is nonsingular if every point is nonsingular.

By the theorem, the set of nonsingular points of a variety is always open.

Examples: (a) $\mathbf{A}_k^n$ and $\mathbf{P}_k^n$ are nonsingular.

(b) If $f \in k[x_1, ..., x_n]$ is irreducible then $V(f, \nabla f) := V(f, \frac{\partial f}{\partial x_1}, ...) = \emptyset$ if and only if $V(f) \subset \mathbf{A}_k^n$ is nonsingular.

(c) (Exercise) If $F \in k[x_0, ..., x_n]_d$ is homogeneous and irreducible, then $V(\nabla F) = \emptyset$ if and only if $V(F) \subset \mathbf{P}_k^n$ is nonsingular. For example, the "Fermat" hypersurfaces $V(x_0^m + ... + x_n^m) \subset \mathbf{P}_k^n$ are nonsingular. If $\mathrm{char}(k)$ divides m, then what happens?

(d) If $V = V(f_1, ..., f_m) \subset \mathbf{A}_k^n$ is a veriety of dimension $n - m$, then $p \in V$ is a nonsingular point if and only if the Jacobian matrix $|\frac{\partial f_j}{\partial x_i}|$ has rank m at p (hence also in an open neighborhood of p). Compare this with the motivation at the beginning of this section.

Corollary 16.5: The local ring $\mathcal{O}_{X,p}$ is a DVR if and only if $\dim(X) = 1$ and p is a nonsingular point of X.

Proof: If $\dim(X) > 1$ or p is a singular point, then $\dim_k(m_p/m_p^2) > 1$ by the theorem, hence it certainly requires more than one element to generate m_p (why?). On the other hand, if p is a nonsingular point and $\dim(X) = 1$, then $\dim_k(m_p/m_p^2) = 1$, and if $a \in m_p$ is an element with a nonzero image in m_p/m_p^2, then the map $\mathcal{O}_{X,p} \to m_p$ given by $b \mapsto ba$ must be surjective by Nakayama's lemma, since it is surjective after taking the quotient by m_p(!) So m_p is principal, and $\mathcal{O}_{X,p}$ is a DVR.

Here is another characterization of a nonsingular point:

Proposition 16.6: $p \in X$ is nonsingular iff $\Omega_{\mathcal{O}_{X,p}}$ is a free $\mathcal{O}_{X,p}$-module.

Proof: Suppose $\dim_k(m_p/m_p^2) = \dim_k(\Omega_X(p)) = r$. Then as in the corollary above, if we choose elements $a_1, ..., a_r \in \mathcal{O}_{X,p}$ such that the images of $da_1, ..., da_r$ are a basis for $\Omega_X(p)$, then Nakayama's lemma applied to:

$$\oplus_{i=1}^r \mathcal{O}_{X,p} \to \Omega_{\mathcal{O}_{X,p}} \text{ given by } (b_1, ..., b_r) \mapsto b_1 da_1 + ... + b_r da_r$$

shows that the da_i generate $\Omega_{\mathcal{O}_{X,p}}$. If V is an affine neighborhood of p and $\Omega_{k[V]}$ has generators $\alpha_1, ..., \alpha_n$, then we can solve $\alpha_j = \sum_{i=1}^r b_{i,j} da_i$ for $b_{i,j} \in \mathcal{O}_{X,p}$ and taking f to be the product of denominators of the $b_{i,j}$ and da_i (thought of as elements of $(\Omega_{k[V]})_{m_p}$), it follows that the da_i generate $\Omega_{k[U_f]}$.

Consider the kernel K of the surjective homomorphism of $k[U_f]$-modules:

$$\oplus_{i=1}^r k[U_f] \to \Omega_{k[U_f]} \text{ given by } (b_1, ..., b_r) \mapsto b_1 da_1 + ... + b_r da_r.$$

If $r = \dim(X)$, then since the da_i generate $\Omega_{\mathcal{O}_{X,q}}$ for every $q \in U_f$, it follows in particular that U_f is contained in the nonsingular locus of X. Moreover, if $\kappa = (g_1, ..., g_r) \in K$, then the functions $g_i \in k[U_f]$ must evaluate to zero on **every** $q \in U_f$, otherwise we would have $\dim(\Omega_X(q)) < \dim(X)$. But this implies $\kappa = 0$, hence $\Omega_{k[U_f]}$ is free, and so is every $\Omega_{\mathcal{O}_{X,q}}$ for $q \in U_f$.

If $r > \dim(X)$, then $\Omega_{\mathcal{O}_{X,p}}$, if free, would require r generators. As above, we would then be able to find an affine neighborhood $p \in U_f$ such that $\Omega_{k[U_f]}$ is generated by the same elements. Then $\Omega_{k[U_f]}$ would be free of rank r since any relation among the generators would persist after localizing at m_p. Thus $\dim_k(\Omega_X(q)) = r$ for all $q \in U_f$ which contradicts the fact that U_f intersects the nonsingular locus of X (which is dense in X).

Next, I want to take a more concrete point of view of the Zariski tangent space as a "linear" approximation to a variety X at a point $p \in X$. We will also introduce the tangent cone to X at p which turns out to be a better nonlinear approximation when p is a singular point. We initially work locally.

Suppose $p = (0, ..., 0) \in V \subset \mathbf{A}^n_k$ and let $I = I(V) \subset k[x_1, ..., x_n]$. Notice that if $f \in I$ and $f = f_0 + f_1 + ... + f_d$ is its decomposition as a sum of homogeneous polynomials, then $f_0 = 0$. If $f = f_e + ... + f_d$ and $f_e \neq 0$, then we will call f_e the first homogeneous term of f.

Definitions: (a) Let I_{lin} be the ideal generated by the linear terms of the elements of I. It follows that $V(I_{lin})$ is an sub-vector space of $k^n = \mathbf{A}^n_k$. Let

$$T_pV := V(I_{lin}) \subset \mathbf{A}^n_k.$$

be this "explicit tangent space" to V at p.

(b) Let I_{init} be the ideal generated by the first homogeneous terms of the elements of I. Then $V(I_{init})$ is a cone over a closed subset of $\mathbf{P}^{n-1}_k$. Let

$$C_pV := V(I_{init}) \subset \mathbf{A}^n_k$$

be this "explicit tangent cone" to V at p.

Observations: (a) Since $I_{lin} \subseteq I_{init}$, we have $C_pV \subseteq T_pV$.

(b) The linear terms in a collection of generators for I will also generate I_{lin}. However, the first nonvanishing homogeneous terms in a collection of generators for I need not generate I_{init}. If I is principal, though, then I_{init} is also principal, as you can check.

Examples: (a) (Nodal plane cubic) If $V = V(y^2 - x^2(x+1)) \subset \mathbf{A}^2_k$. Then $T_pV = \mathbf{A}^2_k$, and $C_pV = V(x^2 - y^2) = V(x-y) \cup V(x+y)$ is a pair of lines intersecting at the origin.

(b) (Cuspidal plane cubic) Suppose $V = V(y^2 - x^3) \subset \mathbf{A}^2_k$. Then again $T_pV = \mathbf{A}^2_k$, but $C_pY = V(y^2)$. Here the definition of C_pV really is not right. Although the tangent cone looks linear, we will want to remember that the equation is a square. (Schemes would help us here!)

(c) Suppose $I = \langle F_1, ..., F_m \rangle$ where the F_i are homogeneous, that is, suppose V is the cone over a projective variety. Then it is easy to see that $I_{init} = I$ in this case, so $C_pV = V$.

Suppose now that $p \in V \subset X$ is an affine neighborhood in a variety over k, and V is identified with a closed, irreducible subset of $\mathbf{A}_k^n$ in such a way that p is identified with the origin. If $f \in I(V)$, then f_1 is transformed into $df(p)$ when we replace the x_i's by dx_i's. Remark (ii) following Theorem 16.4 then implies that T_pV is isomorphic to the Zariski tangent space. In particular, the dimension of T_pV does not depend upon the choice of an affine neighborhood of p. In the next proposition, we show that the tangent cone, too, is intrinsic. In other words we find an affine "cone" in the Zariski tangent space at p (thought of as affine space) which is identified with C_pV for every choice of an affine neighborhood of p.

Proposition 16.7: If $p \in V \subset X$ as above, then there are isomorphisms:

$$\tau : \mathrm{Sym}^*(m_p/m_p^2) \overset{\sim}{\to} k[x_1, ..., x_n]/I_{lin} \text{ and}$$

$$\gamma : \oplus_{d=0}^{\infty} m_p^d/m_p^{d+1} \overset{\sim}{\to} k[x_1, ..., x_n]/I_{init}$$

as graded k-algebras such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Sym}^*(m_p/m_p^2) & \overset{\rho}{\to} & \oplus_{d=0}^{\infty} m_p^d/m_p^{d+1} \\ \tau \downarrow & & \gamma \downarrow \\ k[x_1, ..., x_n]/I_{lin} & \overset{r}{\to} & k[x_1, ..., x_n]/I_{init} \end{array}$$

where $m_p \subset \mathcal{O}_{X,p}$ is the maximal ideal, and ρ and r are the natural quotients.

In particular, the ideal $J := I_{init}/I_{lin} \subseteq k[x_1, ..., x_n]/I_{lin}$ which has the property that $V(J) = C_pV \subseteq T_pV$, is mapped isomorphically by τ^{-1} to the ideal $\ker(\rho) \subset \mathrm{Sym}^*(m_p/m_p^2)$. Thus $V(\ker(\rho))$ defines an intrinsic tangent cone sitting in the Zariski tangent space to X at p.

Proof: Let's start with some notation:

$m = \langle x_1, ..., x_n \rangle \subset k[x_1, ..., x_n]$,

$\tilde{m}_p = m/I \subset k[V]$ is the maximal ideal of $p \in V$, and

m_p is the corresponding ideal in $k[V]_{m_p} \cong \mathcal{O}_{X,p}$

For each $d \geq 1$, we have natural isomorphisms:

$$m_p^d/m_p^{d+1} \cong \tilde{m}_p^d/\tilde{m}_p^{d+1} \cong m^d/(m^{d+1} + I \cap m^d) = (k[x_1, ..., x_n]/I_{init})_d.$$

The first isomorphism is the isomorphism of HW # 1(b), the second is a fundamental isomorphism for ideals once we observe that $\tilde{m}_p^d = m^d/I$, and the third equality follows from the definition of I_{init}. Notice also that $(k[x_1, ..., x_n]/I_{init})_1 = (k[x_1, ..., x_n]/I_{lin})_1$. One can check that these isomorphism carry products in $\oplus_{d=0}^{\infty} m_p^d/m_p^{d+1}$ to products in $k[x_1, ..., x_n]/I_{init}$, yielding the isomorphism γ. The isomorphism τ of graded k-algebras is uniquely determined by its degree one isomorphism $m_p/m_p^2 \cong (k[x_1, ..., x_n]/I_{lin})_1$ since $\mathrm{Sym}^*(m_p/m_p^2)$ is a polynomial ring and $k[x_1, ..., x_n]/I_{lin}$ is generated by its degree one part, and also a polynomial ring. This also explains why the two k-algebra homomorphisms from $\mathrm{Sym}^*(m_p/m_p^2)$ to $k[x_1, ..., x_n]/I_{init}$, which agree in degree one, must agree in all degrees.

Finally, notice that if V is a vector space over k of dimension n, then $\mathrm{Sym}^*(V)$ is naturally the coordinate ring of the dual space $\mathrm{Hom}_k(V, k)$ viewed as affine n-space. Thus $V(\ker(\rho))$ may be viewed as a closed cone in the Zariski tangent space $\mathrm{Hom}_k(m_p/m_p^2, k)$ when thought of as affine space.

Remark: In the proof, I have had to dance around the statement I would have liked to make, namely: "$V(\ker(\rho))$ is isomorphic to C_pV". If C_pV is reducible, we could finesse this by saying that the components are isomorphic. But even when C_pV is irreducible, I actually have more information than the mere isomorphism of affine varieties, since I know that the defining ideals are isomorphic (think of the cusp example above). With the theory of schemes, one can more simply define the tangent cone as the affine scheme associated to the ring $\oplus_{d=0}^{\infty} m_p^d/m_p^{d+1}$, which remembers the isomorphism of ideals.

The tangent cone allows us to define types of singularities. E.g.

Definition: A singular point $p \in X$ is called a **hypersurface singularity** if $\dim_k(m_p/m_p^2) = \dim(X) + 1$. A hypersurface singularity of a curve is called a **plane curve singularity**. A plane curve singularity is **ordinary** if $\ker(\rho)$ (which is a homogeneous ideal in $k[x, y]$) is principal, generated by a quadratic polynomial. It is an:

(a) **ordinary node** if the generator is a product of distinct linear factors.

(b) **ordinary cusp** if the generator is a square.

I want to prove that the tangent cone to X at p always has the same dimension as X (unlike the Zariski tangent space, which jumps in dimension). The most convenient way to do this is to introduce the notion of blowing up.

Blowing up a point in affine space: Consider the rational map:

$$\Phi : \mathbf{A}_k^n - - > \mathbf{P}_k^{n-1} \text{ given by } \Phi(p_1, ..., p_n) = (p_1 : ... : p_n)$$

in other words, Φ maps a point $p \in \mathbf{A}_k^n$ (other than the origin) to the point in projective space corresponding to the line through p and the origin. Let

$$\Gamma = \{((p_1, ..., p_n), (p_1 : ... : p_n)) \mid (p_1, ..., p_n) \neq (0, ..., 0)\} \subset \mathbf{A}_k^n \times \mathbf{P}_k^{n-1}$$

i.e. Γ is the graph of Φ, and let $\widetilde{\mathbf{A}}_k^n := \overline{\Gamma}$ be its closure in $\mathbf{A}_k^n \times \mathbf{P}_k^{n-1}$.

Consider the polynomial ring $k[x_1, ..., x_n, y_1, ..., y_n]$ graded by degree in the y-variables as in the proof of Theorem 14.11, so that in particular, the closed subsets of $\mathbf{A}_k^n \times \mathbf{P}_k^{n-1}$ are all of the form $V(I)$ for homogeneous ideals I by the claim in that theorem. Then:

Theorem 16.8: (a) The first projection restricts to a birational regular map:

$$\sigma : \widetilde{\mathbf{A}}_k^n \to \mathbf{A}_k^n$$

(σ is variously called a blow-down, σ-process or monoidal transformation) More precisely, $\sigma^{-1}(0, ..., 0) = \{(0, ..., 0)\} \times \mathbf{P}_k^{n-1}$ and σ is an isomorphism elsewhere.

(b) $\widetilde{\mathbf{A}}_k^n = V(\langle x_i y_j - x_j y_i \rangle_{i,j=1}^n)$

(c) The open sets $U_i := \widetilde{\mathbf{A}}_k^n - V(y_i)$ are isomorphic to $\mathbf{A}_k^n$ with natural isomorphisms $k[U_i] \cong k[\frac{y_1}{y_i}, ..., \frac{y_{i-1}}{y_i}, x_i, \frac{y_{i+1}}{y_i}, ..., \frac{y_n}{y_i}]$.

Proof: We start with the affines $U_{y_i} := \mathbf{A}_k^n \times \mathbf{P}_k^{n-1} - V(y_i) \cong \mathbf{A}_k^{n+(n-1)}$ with coordinate rings $k[x_1, ..., x_n, \frac{y_1}{y_i}, ..., \frac{y_n}{y_i}]$. The equations $x_i y_j = x_j y_i$ may be rewritten as $x_j = x_i(\frac{y_j}{y_i})$ on $k[U_{y_i}]$, so we obtain the isomorphism:

$$U_{y_i} \cap V(\langle x_i y_j - x_j y_i \rangle) \cong \mathbf{A}_k^n, \text{ with } k[\mathbf{A}_k^n] = k[\frac{y_1}{y_i}, ..., \frac{y_{i-1}}{y_i}, x_i, \frac{y_{i+1}}{y_i}, ..., \frac{y_n}{y_i}]$$

(in particular, check that the other equations $x_k y_l - x_l y_k$ are redundent.)

Let $W_i = U_{y_i} \cap V(\langle x_i y_j - x_j y_i \rangle_{i,j=1}^n)$ for the moment. Since the U_{y_i} cover $\mathbf{A}_k^n \times \mathbf{P}_k^{n-1}$, since each of the W_i is irreducible, and finally since each intersection $W_i \cap W_j$ is nonempty (as you may easily check), it follows that $V(\langle x_i y_j - x_j y_i \rangle) = \cup_{i=1}^n W_i$ is irreducible.

If $(p_1, ..., p_n) \neq (0, ..., 0)$, then $p_i q_j = p_j q_i$ for all i and j if and only if $(p_1 : ... : p_n) = (q_1 : ... : q_n)$, so $\Gamma = V(\langle x_i y_j - x_j y_i \rangle) - \{(0, ..., 0)\} \times \mathbf{P}_k^{n-1}$, hence $\Gamma \subset V(\langle x_i y_j - x_i y_i \rangle)$ is open, which allows us to conclude (b) and (c) where now $U_i = W_i$.

One checks directly that $\sigma^{-1}(0, ..., 0) = (0, ..., 0) \times \mathbf{P}_k^{n-1}$ using (b). The first projection retricts on U_i to $\sigma : U_i \to \mathbf{A}_k^n$ corresponding to the inclusion:

$$\iota : k[x_1, ..., x_n] \hookrightarrow k[\frac{y_1}{y_i}, ..., \frac{y_{i-1}}{y_i}, x_i, \frac{y_{i+1}}{y_i}, ..., \frac{y_n}{y_i}] \text{ sending } x_j \mapsto x_i \left(\frac{y_j}{y_i}\right).$$

Notice that $\sigma^{-1}(0, ..., 0) \cap U_i = V(x_i)$. On the coordinate rings of the open complements of $V(x_i) \subset U_i$ and $V(x_i) \subset \mathbf{A}_k^n$ the map ι localizes to:

$$\iota_{x_i} : k[\frac{x_1}{x_i}, ..., \frac{x_n}{x_i}] \to k[\frac{y_1}{y_i}, ..., \frac{y_n}{y_i}] \text{ sending } \frac{x_j}{x_i} \mapsto \frac{y_j}{y_i},$$

and it follows that σ becomes an isomorphism from each $U_i - \sigma^{-1}(0, ..., 0) = U_i - V(x_i)$ to its image. You check now that (id, Φ) furnishes an inverse to $\sigma : \widetilde{\mathbf{A}}_k^n - \sigma^{-1}(0, ..., 0) \to \mathbf{A}_k^n - (0, ..., 0)$ which gives the rest of (a).

Example: We have already encountered a blow-down! If $n = 2$, then the restriction $\sigma : U_2 \to \mathbf{A}_k^2$ corresponds to the inclusion $k[x, y] \hookrightarrow k[x, \frac{x}{y}]$ which means that this restriction of σ is none other than the map $(x, y) \mapsto (x, xy)$ we have considered before.

Remark: Notice that $\widetilde{\mathbf{A}}_k^n$ is nonsingular, since it is covered by affine spaces(!) Notice also that the "exceptional" fiber $\sigma^{-1}(0, ..., 0)$ is locally given by one equation (namely the equations $x_i = 0$ on each U_i). Thus blowing up replaced the origin in $\mathbf{A}_k^n$ with a copy of $\mathbf{P}_k^{n-1}$ (which you should think of as the collection of lines through the origin), which is locally described by a single equation. This becomes quite important in the next theorem.

Suppose $V \subset \mathbf{A}_k^n$ is a closed, irreducible subset passing through the origin. Let $\widetilde{V} \subset \mathbf{A}_k^n \times \mathbf{P}_k^{n-1}$ be the closure of the graph of Φ restricted to V, that is, the closure of the graph $\Gamma_V := \{((p_1, ..., p_n), (p_1 : ... : p_n)) \mid (p_1, ..., p_n) \in V\}$.

Let $\mathbf{P}(C_p(V)) := V(I_{init}) \subset \mathbf{P}_k^{n-1}$ be the "projectivized tangent cone". Of course, it may have many components. But one consequence of the next theorem is that each component has the same dimension.

Let $I = I(V)$. For each $f \in I(V)$, let $\mathrm{mult}_p(f)$ be the degree of the first homogeneous term in f. Notice that if $\mathrm{mult}_p(f) = e$, then

$$\frac{f}{x_i^e} = f_e(\frac{x_1}{x_i}, ..., \frac{x_n}{x_i}) + x_i f_{e+1}(\frac{x_1}{x_i}, ..., \frac{x_n}{x_i}) + ... + x_i^{d-e} f_d(\frac{x_1}{x_i}, ..., \frac{x_n}{x_i})$$

Substituting $\frac{y_j}{y_i}$ for each $\frac{x_j}{x_i}$ gives an element of $k[U_i]$ from part (c) of the previous theorem, which we will be sloppy and denote by $\frac{f}{x_i^e}$. For each $i = 1, ..., n$ let $\tilde{I}_i \subset k[U_i]$ be the ideal generated by the $\frac{f}{x_i^e}$ for each $f \in I$.

Exercise: $g \in \tilde{I}_i$ iff $g = x_i^m(\frac{f}{x_i^e})$ for some $f \in I, m \geq 0$.

Theorem 16.9: (a) The first projection restricts to a birational regular map:

$$\sigma : \widetilde{V} \to V$$

such that $\sigma^{-1}(0, ..., 0) = \{(0, ..., 0)\} \times \mathbf{P}(C_p(V))$ and σ is an isomorphism elsewhere.

(b) Each $\tilde{I}_i \subset k[U_i]$ is a prime ideal, and $\widetilde{V} \cap U_i = V(\tilde{I}_i) \subset U_i$.

(c) Each intersection $\sigma^{-1}(0, ..., 0) \cap U_i \subset \widetilde{V} \cap U_i$ is the zero locus of the (non-zero!) image of x_i in $k[V(\tilde{I}_i)] = \mathcal{O}_{\widetilde{V}}(\widetilde{V} \cap U_i)$.

Proof: Our first goal is to prove that $\cup_{i=1}^n V(\tilde{I}_i)$ is a closed, irreducible subset of $\widetilde{\mathbf{A}}_k^n = \cup_{i=1}^n U_i$. To see that it is closed, it suffices to check that for each i, $\cup_{i=1}^n V(\tilde{I}_i) \cap U_i$ is closed. In fact, from the identities:

$$\frac{f}{x_i^e} = \frac{f}{x_j^e}\left(\frac{x_j}{x_i}\right)^e \text{ and } x_j = x_i\left(\frac{y_j}{y_i}\right).$$

it follows that each $V(\tilde{I}_j) \cap U_i \subseteq V(\tilde{I}_i)$, hence in fact $\cup_{i=1}^n V(\tilde{I}_i) \cap U_i = V(\tilde{I}_i)$.

To see irreducibility, observe that from the exercise above it follows that if $x_i h \in \tilde{I}_i$, then $h \in \tilde{I}_i$, so x_i is not a zero divisor in $k[U_i]/\tilde{I}_i$. Next, we have natural isomorphisms $(k[U_i]/\tilde{I}_i)_{x_i} \cong (k[V]/I(V))_{x_i}$, as you may easily check, so in particular, $(k[U_i]/\tilde{I}_i)_{x_i}$ is an integral domain. But these two facts together imply $k[U_i]/\tilde{I}_i$ is an integral domain, as well (why?). Thus $\tilde{I}_i$ is a prime ideal, and as in the previous theorem, the irreducibility of each $V(\tilde{I}_i)$ and nonempty intersections implies that the union is irreducible.

One checks as in the previous theorem that $\Gamma_V \subset \cup_{i=1}^n V(\tilde{I}_i)$ is the complement of $\sigma^{-1}(0, ..., 0)$, hence $\widetilde{V}$ is equal to this union. Also, as in the previous theorem, the isomorphisms $(k[U_i]/\tilde{I}_i)_{x_i} \cong (k[V]/I(V))_{x_i}$ correspond to isomorphisms $V(\tilde{I}_i) - \sigma^{-1}(0, ..., 0) = V(\tilde{I}_i) - V(x_i) \xrightarrow{\sim} V - V(x_i)$ which taken together imply that $(\mathrm{id}, \Phi) : V - \{(0, ..., 0)\} \to \widetilde{V} - \sigma^{-1}(0, ..., 0)$ is an isomorphism, with σ as its inverse. Notice also that as in the previous theorem, $\sigma^{-1}(0, ..., 0) \cap U_i \subset \widetilde{V} \cap U_i$ is given by one equation, namely $V(x_i)$.

It only remains to show that $\sigma^{-1}(0, ..., 0) = \{(0, ..., 0)\} \times \mathbf{P}(C_pV)$. But on each $V(\tilde{I}_i) = \widetilde{V} \cap U_i$, we have $\sigma^{-1}(0, ..., 0) \cap V(\tilde{I}_i) = V(x_i) \cap V(\tilde{I}_i)$, so just by examining the equations defining $V(\tilde{I}_i) \subset U_{y_i}$, we see that

$$\sigma^{-1}(0, ..., 0) \cap V(\tilde{I}_i) = V(x_1, ..., x_n, \{f_e(\frac{y_1}{y_i}, ..., \frac{y_n}{y_i}) \mid f \in I\})$$

which precisely gives $(0, ..., 0) \times V(I_{init}) - V(y_i) \subset (0, ..., 0) \times \mathbf{P}_k^{n-1} - V(y_i)$(!)

Corollary 16.10: The dimension of every component of tangent cone to X at p is equal to $\dim(X)$. The point $p \in X$ is nonsingular if and only if the tangent cone at p is equal to the Zariski tangent space.

Proof: By 16.9(a),(c), and Krull's theorem, we see that every component of the projectivized tangent cone has dimension equal to $\dim(X) - 1$. By the first exercise in §15, we get the dimension result. Since every component of the tangent cone has dimension equal to $\dim(X)$, and since every component is a closed, irreducible subset of the Zariski tangent space, it follows from Krull's theorem that the dimensions are the same if and only if the tangent cone equals the Zariski tangent space.

Combining several results of this section, we get the following:

Definition: The local ring $\mathcal{O}_{X,p}$ associated to a point in a variety over an algebraically closed field k is **regular** if, equivalently:

(a) $\dim_k(m_p/m_p^2) = \dim(X)$

(b) $\Omega_{\mathcal{O}_{X,p}}$ is a free $\mathcal{O}_{X,p}$-module.

(c) The tangent cone at p is equal to the Zariski tangent space.

Remark: There is a fourth characterization using completions, which allows one to conclude to conclude that a regular local ring is always a UFD. Since a UFD is integrally closed, this would tell us that nonsingular implies normal.

Eleventh Homework Assignment

1. Easy properties of modules. Let A be a commutative ring with 1:

(a) If M is generated by $m_1, ..., m_n$ as an A-module, show that M_S is generated by $\frac{m_1}{1}, ..., \frac{m_n}{1}$ as an A_S-module for any multiplicative set $S \subset A$.

(b) If $S \subset A$ is a multiplicative subset and $I \subset A$ is an ideal, let $I_S \subset A_S$ be the corresponding ideal. Then exhibit an isomorphism:

$$(A/IA)_S \cong A_S/I_S A_S$$

and more generally, given an A-module M, an isomorphism of A_S-modules:

$$(M/IM)_S \stackrel{\sim}{\to} M_S/I_S M_S.$$

If $S = A - m$ for a maximal ideal m, note that $A/mA \cong A_S/m_S A_S$ and more generally, prove that $M/mM \cong M_S/m_S M_S$. Finally, if $I, J \subset A$ are ideals, show that $(IJ)_S = I_S J_S$ and use this to prove that:

$$m^d/m^{d+1} \cong m_S^d/m_S^{d+1}$$

for any integer $d \geq 0$.

(c) If $N \subset M$ are modules over A and $I \subset A$ is an ideal, show that N/IN is a submodule of M/IM and $(M/N)/I(M/N) \cong (M/IM)/(N/IN)$

(d) Prove Proposition 16.3

(e) If $I \subset A$ is an ideal and M, N are A-modules, show that:

$$\mathrm{Hom}_A(M, N/IN) = \mathrm{Hom}_{A/I}(M/IM, N/IN)$$

2. (a) Prove Euler's formula: If $F \in k[x_0, ..., x_n]_e$, then:

$$eF = \frac{\partial F}{\partial x_0} x_0 + ... + \frac{\partial F}{\partial x_n} x_n$$

(b) Prove that a hypersurface $V(F) \subset \mathbf{P}_k^n$ is nonsingular iff $V(\nabla F) = \emptyset$.

3. (a) If $d : A \to \Omega_A$ is a universal derivation, show that Ω_A is generated as an A-module by the image of d.

(b) Given an example to show that d in (a) need not be surjective.