Course Notes for Math 737 (Algebraic Geometry)

0.1 Pre-Introduction. Beginning Category Theory. Here are some really fundamental (and at first really confusing!) concepts.

Categories: A **category** $\mathcal{C}$ consists of:

(i) A collection of objects, denoted $\mathrm{Ob}(\mathcal{C})$, together with

(ii) A collection of arrows $f : A \to B$ pointing from one object of $\mathcal{C}$ to another. The arrows are called morphisms, and the collection of arrows is $\mathrm{Mor}(\mathcal{C})$. $\mathrm{Mor}(\mathcal{C})$ is required to be closed under composition, in the sense that each triple of objects $A, B, C \in \mathrm{Ob}(\mathcal{C})$ and pair of morphisms $f : A \to B$, $g : B \to C$ determines a unique "composition" morphism $g \circ f : A \to C$.

There are two additional rules:

(Associativity) Given four objects $A, B, C, D \in \mathrm{Ob}(\mathcal{C})$ and morphisms $f : A \to B$, $g : B \to C$ and $h : C \to D$, the composition morphisms $(h \circ g) \circ f$ and $h \circ (g \circ f) : A \to D$ are the same morphism, and therefore the composition: $h \circ g \circ f$ is uniquely determined. (From this it follows that the composition of any finite number of morphisms is uniquely determined.)

(Identity) Each $A \in \mathrm{Ob}(\mathcal{C})$ admits an identity morphism id: $A \to A$ with the property that if $f : A \to B$ and $g : B \to A$, then $f \circ \mathrm{id} = f$ and $\mathrm{id} \circ g = g$. This property implies that the identity morphism is unique.

Using the identity morphisms, an **isomorphism** is defined to be a morphism f with a two-sided inverse (i.e. a g such that $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$.)

Examples: (a) Fix a set S. Let $\mathcal{P}(S)$ be the category whose objects are the subsets of S, and "inclusions of subsets" are the morphisms. Thus "equalities of subsets" are the identity morphisms, and they are the **only** isomorphisms.

(b) Let $\mathcal{S}$ be the category whose objects are all the sets(!), and "mapping of sets" are the morphisms. The identity mappings are the identity morphisms. The bijections are the isomorphisms.

(c) Let $\mathcal{T}$ be the category whose objects are all the topological spaces(!) Let "continuous mappings" be the morphisms, and note that the identity is continuous. The homeomorphisms are the isomorphisms.

(d) Let $\mathcal{R}$ (or $\mathcal{A}$ or $\mathcal{G}$ or $\mathcal{V}$) be the category whose objects are all rings (or abelian groups or groups, or vector spaces over a fixed field), and let "homomorphisms" be the morphisms, noting that the identity is always a homomorphism. The isomorphisms are the isomorphisms (hmmm...).

(e) Given a category $\mathcal{C}$, let $\overline{\mathcal{C}}$ be defined to be the category whose objects are the **isomorphism classes** of objects in $\mathcal{C}$ and whose morphisms are "morphisms modulo isomorphisms". The equivalence relation on morphisms is the following: if $f : A \to B$ and $f' : A' \to B'$ are morphisms in $\mathcal{C}$ and $\overline{A} = \overline{A}'$ and $\overline{B} = \overline{B}'$ as objects in $\overline{\mathcal{C}}$, then $f \sim f'$ if there are isomorphisms $\alpha : A \to A'$ and $\beta' : B' \to B$ such that $f = \beta' \circ f' \circ \alpha$.

There are two types of "maps" from one category to another.

Functors: A **covariant functor** $F : \mathcal{C} \to \mathcal{C}'$ between two categories is:

(i) A mapping $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$

(ii) A mapping $F : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C}')$ taking each morphism $f : A \to B$ to $F(f : A \to B) : F(A) \to F(B)$ (which we'll often shorten to $F(f)$).

Two rules: F is required to be compatible with composition and identity, i.e.

(a) $F(g \circ f) = F(g) \circ F(f)$ for all $f : A \to B$ and $g : B \to C$, and

(b) $F(\mathrm{id}) = \mathrm{id}$ for all identity morphisms id: $A \to A$.

A **contravariant functor** F is defined similarly, except that in (ii), the arrows are reversed, i.e. $F(f) : F(B) \to F(A)$, and correspondingly, in rule (a), the order of composition must be reversed, i.e. $F(g \circ f) = F(f) \circ F(g)$.

More Examples: (a) The "reverse" functor. There is in some sense only one contravariant functor. Namely, if $\mathcal{C}$ is a category, let $\mathcal{C}^*$ be the "dual" category where the objects are the same, and the morphisms are also the same, except that they point in the opposite direction(!) Then there is a "reversing" functor $REV : \mathcal{C} \to \mathcal{C}^*$ which is the identity on the set of objects and reverses the direction of all arrows. A contravariant functor can be made covariant by composing with REV (though one rarely does this!).

(b) There is a canonical covariant functor $F : \mathcal{C} \to \overline{\mathcal{C}}$ that sends an object to its isomorphism class and a morphism to its equivalence class modulo isomorphisms (see above).

(c) Suppose that R is a ring (or A is an abelian group, etc.) and that $\mathcal{S}$ is the category of all sets. Then there is a contravariant functor $F : \mathcal{S} \to \mathcal{R}$ defined as follows:

$F(S)$ is the ring(!) of functions $\phi : S \to R$.

$F(f : S \to S')$ is the pull-back homomorphism $f^* : F(S') \to F(S)$ on functions defined by: $f^*(\phi')(p) = \phi'(f(p))$ for all functions $\phi' : S' \to R$.

(d) Suppose that R is a **continuous** ring (i.e. a topological space in which $*$ and $+$ are continuous functions) . For instance, $R = \mathbf{R}$ (the reals) or $\mathbf{C}$ (the complex numbers) or any ring with the discrete topology. Then in the previous example we can replace $\mathcal{S}$ with $\mathcal{T}$, the category of topological spaces, and let

$F(T)$ be the ring of continuous functions $\phi : T \to R$, and

$F(f : T \to T') = f^*$, as before. (Since f is a continuous mapping, the pull-back f^* on functions takes continuous functions on T' to continuous functions on T(!))

Now for some new examples:

Two Fundamental Functors: (i) **A presheaf**. Suppose that T is a topological space. Let $\mathcal{U}(T)$ be the category whose objects are the **open** subsets $U \subset T$ (as opposed to all subsets in the very first example.) Let inclusions be the morphisms, as before. Then:

A contravariant functor $F : \mathcal{U}(T) \to \mathcal{R}$ (or $\mathcal{A}$ or $\mathcal{G}$, etc.) is called a **presheaf of rings on** T (or abelian groups or groups, etc.)

Given F and an inclusion $V \subset U$ of open subsets of T, then the ring homomorphism $F(V \subset U) : F(U) \to F(V)$ is called the **restriction** homomorphism. It is denoted by $\rho^F_{U,V}$, or $\rho_{U,V}$ when the presheaf is understood.

As one example (but by no means the only one!), we could fix a continuous ring R and let $\mathcal{C}_{T,R}$ be the presheaf defined by setting:

$\mathcal{C}_{T,R}(U)$ to be the ring of continuous functions $\phi : U \to R$, where U has, of course, the induced topology, and

$\rho_{U,V}$ to be the honest restriction of functions from U to V.

This example has some other nice properties which we will later require in the definition of a **sheaf** (as opposed to a presheaf).

(ii) The **Spec** (or spectrum) functor is the contravariant functor from the category $\mathcal{C}$ of commutative rings with 1 (and homomorphisms preserving 1) to the category $\mathcal{T}$ of topological spaces (and continuous maps) defined as follows:

$$Spec(R) \text{ is the set of prime ideals properly contained in } R$$

$Spec(R)$ is a topological space as follows. If R_f is the localization of R at an element $f \in R$, then there is a natural inclusion $Spec(R_f) \subset Spec(R)$ as the set of prime ideals which do not contain f. Since $Spec(R_f) \cap Spec(R_g) = Spec(R_{fg})$ (the ideals are prime!), these subsets generate a topology on $Spec(R)$. Notice that unions of these basis elements are not, in general, other basis elements. Of course, $\emptyset = Spec(R_0)$ and $Spec(R) = Spec(R_1)$ are open.

If $\phi : R \to R'$ is a ring homomorphism, then ϕ^{-1} takes prime ideals to prime ideals (why?). Thus it makes sense to define

$$Spec(\phi) : Spec(R') \to Spec(R) \text{ by setting } Spec(\phi)(\mathcal{P}') = \phi^{-1}(\mathcal{P}).$$

Suppose that $f \in R$. Then some quiet thought will convince you that:

$$Spec(\phi)^{-1}(Spec(R_f)) = \{\mathcal{P}' \subset R' \mid f \notin \phi^{-1}(\mathcal{P}')\} = Spec(R'_{\phi(f)}).$$

From which it follows that $Spec(\phi)$ is continuous.

We'll investigate these two functors at great length later. In particular, we will find a "natural" presheaf of rings (denoted by $\mathcal{O}$) on each $Spec(R)$, and more generally, each R-module M will determine a "natural" presheaf of abelian groups (actually, of $\mathcal{O}$-modules...to be defined).

More on Functors:

(a) If $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{C}''$ are functors, then $G \circ F$ is the functor from $\mathcal{C}$ to $\mathcal{C}''$ defined by composing the maps on objects and arrows.

(b) The **identity** functor $\mathrm{id} : \mathcal{C} \to \mathcal{C}$ is the identity map on objects and arrows.

Does this make all categories and functors into a "monster" category?

(c) A functor F from a $\mathcal{C}$ to $\mathcal{C}'$ is an **equivalence of categories** if there is a two-sided inverse functor G from $\mathcal{C}'$ to $\mathcal{C}$.

Example: $REV : \mathcal{C} \to \mathcal{C}^*$ is an equivalence of categories. $REV^{-1} = REV$.

(d) A functor $F : \mathcal{C} \to \mathcal{C}'$ is **faithful** if it is injective as a map from $Mor(\mathcal{C})$ to $Mor(\mathcal{C}')$. Notice that this forces F to be injective as a map from $Ob(\mathcal{C})$ to $Ob(\mathcal{C}')$, too. (Why?)

Examples: (i) The "forgetful" functors from the categories: $\mathcal{T}$ or $\mathcal{R}$ or $\mathcal{G}$ of topological spaces, rings or groups to the category $\mathcal{S}$ of sets, are defined by forgetting the extra structure (i.e. each object is taken to its underlying set, and each morphism to the underlying mapping of sets). These are all obviously faithful...two homomorphisms, for example, are the same if they are the same as maps of sets!

(ii) The functor $F : \mathcal{C} \to \overline{\mathcal{C}}$ defined earlier is faithful if and only if the only isomorphisms in $\mathcal{C}$ are the identity morphisms, in which case it is an equivalence of categories(!)

(iii) The spectrum functor is **not** faithful. For example, consider the ring $R = k[x]/\langle x^2 \rangle$ (k is a field). Then $Spec(R)$ is a point(!) The two homomorphisms $\phi =$ "identity" and $\psi : R \to R$, which is the identity on k but takes x to 0, are obviously different homomorphisms, but $Spec(\phi) = Spec(\psi)$. (They don't have much chance to be different, do they?) We will need to worry about this later.

(e) Suppose that F and G are covariant functors from $\mathcal{C}$ and $\mathcal{C}'$. A **transformation** t from F to G is a map:

$$t : Ob(\mathcal{C}) \to Mor(\mathcal{C}')$$

with the property that $t(X) : F(X) \to G(X)$ for all $X \in Ob(\mathcal{C})$ and the compositions $G(f) \circ t(X) = t(Y) \circ F(f)$ for all morphisms $f : X \to Y$ in $Mor(\mathcal{C})$. (The definition is adjusted by requiring $t(X) \circ F(f) = G(f) \circ t(Y)$ in case F and G are contravariant.) The collection of covariant (resp. contravariant) functors from $\mathcal{C}$ to $\mathcal{C}'$ becomes a category with transformations as the morphisms (why? and what is the identity transformation?). In particular, two functors F and G are **isomorphic** if there is a transformation t from F to G with a two-sided inverse.

Example: Suppose that $\mathcal{F}$ and $\mathcal{G}$ are presheaves of rings on a topological space T. Then a morphism (or transformation of functors) from $\mathcal{F}$ to $\mathcal{G}$ is a collection of ring homomorphisms $t(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ which "commute with restriction". You want a concrete example? Suppose that $f : A \to R$ is a ring homomorphism. If we let $\mathcal{F}$ be the presheaf of functions from T to A and $\mathcal{G}$ be the presheaf of functions from T to R, then we get a transformation of functors $t : \mathcal{F} \to \mathcal{G}$ by letting $t(s) = f \circ s : U \to R$ for each function $s : U \to A$.

Notation: We will let $Cov(\mathcal{C}, \mathcal{C}')$ and $Cont(\mathcal{C}, \mathcal{C}')$ be the categories, respectively, of covariant and contravariant functors from $\mathcal{C}$ to $\mathcal{C}'$.

One more category definition: Suppose $\mathcal{C}$ is a category and $\mathcal{B} \subset \mathcal{C}$ is a subcategory. (i.e. $\mathcal{B}$ is a category, $Ob(\mathcal{B}) \subset Ob(\mathcal{C})$ and $Mor(\mathcal{B}) \subset Mor(\mathcal{C})$). Then we say that $\mathcal{B}$ is a **full subcategory** of $\mathcal{C}$ if the only morphisms in $\mathcal{C}$ between objects of $\mathcal{B}$ are the morphisms in $\mathcal{B}$.

Example: Any faithful functor from $\mathcal{C}$ to $\mathcal{C}'$ is an equivalence of categories between $\mathcal{C}$ and its image, which is a subcategory of $\mathcal{C}'$ (why?). The image of the forgetful functor $F : \mathcal{A} \to \mathcal{S}$ from the category of abelian groups to sets is **not** a **full** subcategory. For instance, if $\mathbf{Z}_2$ is the group with two elements, then we can think of $F(\mathbf{Z}_2) = \{0, 1\}$, and the map of sets from $\{0, 1\}$ to itself taking $0 \to 1$ and $1 \to 0$ is not in the image of F because it is not a homomorphism of abelian groups(!)

Definition: Any faithful functor whose image is a full subcategory is called a **fully faithful functor** (try saying that three times fast!).

A very important example:

Grothendieck's Functor of Points: Suppose that $\mathcal{C}$ is a category and that $X \in Ob(\mathcal{C})$. We define the functor of points $h_X : \mathcal{C} \to \mathcal{S}$ to the category $\mathcal{S}$ of sets by letting:

(i) $h_X(Z) = \{\text{morphisms } f : Z \to X\}$ and

(ii) $h_X(g : W \to Z) : h_X(Z) \to h_X(W)$. is "pre-composition with g". That is, $h_X(g)(f) = f \circ g : W \to X$.

Then h_X is a contravariant functor.

We call it the functor of points because in many examples there is an object $Z \in \mathcal{C}$ with the property that $h_X(Z)$ is "the set of points of X".

And even (or especially) in examples where this is **not** true, the functor of points plays a very important role.

Examples: (a) Let $\mathcal{S}$ be the category of sets. Let Z be any set with one element(!) Then $h_X(Z)$ is naturally bijective to the set X. (Send a morphism to the image of the element(!)) The same is obviously true for the category $\mathcal{T}$, since the map from one point to any topological space is continuous(!)

(b) Let $\mathcal{G}$ be the category of groups. Let $Z = \mathbf{Z}$, the integers, thought of as an abelian group with addition. Then $h_G(\mathbf{Z})$ is "the set G" for any group G, by sending $f : \mathbf{Z} \to G$ to $f(1) \in G$(!)

(c) Let $\mathcal{C}$ be the category of commutative rings with 1, with 1-preserving homomorphisms. Then $Z = \mathbf{Z}[x]$ does the trick, with f going to $f(x)$.

(d) Here's a new category: $\mathcal{CW}$ has, as its objects the CW-complexes, and as its morphisms, the homotopy classes of continuous maps. Then the one-point CW-complex Z yields $h_Z(X) = \pi_0(X)$ for all CW-complexes X. This is certainly not the "set of points of X", and it is pretty easy to see that there is no CW-complex Z such that $h_Z(X)$ is the "set of points of X". The problem here: we have identified continuous maps if they are homotopic.

(e) In another direction, in the category $\mathcal{P}(S)$ of subsets of S, there is no subset Z with the desired properties (provided S has more than one element). The problem here: there are not enough morphisms.

One more level of abstraction:

Let $\mathcal{C}$ be a category and $\mathcal{S}$ the category of sets. Consider the covariant functor:

$$F : \mathcal{C} \to Cont(\mathcal{C}, \mathcal{S})$$

defined as follows:

$F(X)$ is the "functor of points" h_X (or $h_X(-)$, if you prefer).

$F(\phi : X \to Y)$ is "post-composition with ϕ", i.e. the transformation $h_X \to h_Y$ of functors obtained by taking each $f : Z \to X$ to $\phi \circ f : Z \to Y$.

You check that this is really a covariant functor, and the important:

Yoneda Lemma: F (defined above) is a fully faithful functor(!)

Proof: A mind-bending exercise.

The main motivation for all this nonsense is:

Definition: Given a category $\mathcal{C}$ and a contravariant functor $G : \mathcal{C} \to \mathcal{S}$, we say G is **representable** if G is isomorphic to a functor $F(X) = h_X$ for some $X \in Ob(\mathcal{C})$, in which case we say that X represents the functor G.

You will have to take my word for it that when $\mathcal{C}$ is the category of projective schemes (to be defined), there are some very interesting functors (to be defined) which are representable, and others (to be defined) which are even more interesting, which are tantalizing close to being representable. Moreover, at this point you will have to take my word for it that it is a **big deal** if a functor in the category of projective schemes is representable!

Some Spec Examples

The topological spaces $Spec(R)$ take some getting used to. (By the way, this topology is called the Zariski topology.) Here are some examples.

If R is a discrete valuation ring, then R has two prime ideals: 0 and m, the maximal ideal. Thus $Spec(R) = \{x_0, x_m\}$ is a space with two points. But it does not have the discrete topology! In fact, $x_m \in \overline{\{x_0\}}$. You can check this directly or appeal to the useful:

Exercise: In general, if $x_{\mathcal{P}}, x_{\mathcal{Q}} \in Spec(R)$ are points corresponding to prime ideals $\mathcal{P}, \mathcal{Q} \subset R$, then:

$$x_{\mathcal{Q}} \in \overline{\{x_{\mathcal{P}}\}} \text{ if and only if } \mathcal{P} \subseteq \mathcal{Q}$$

so that, in particular, the points corresponding to maximal ideals are the only closed points, and the zero ideal (in an integral domain) corresponds to x_0, which is **dense** in $Spec(R)$. x_0 is called the **generic** point of $Spec(R)$.

Exercises: (a) I said that, in general, $Spec(R_f) \cup Spec(R_g)$ is not another basis element. However, if R is a principal ideal domain, show that **every** open subset $U \subset Spec(R)$ is of the form $Spec(R_h)$ for some $h \in R$.

(b) Also, in a PID, all prime ideals except the zero ideal are maximal, so there is only one point, namely the generic point, which is not closed.

Some specific PID examples:

(1) $Spec(k[x]) = \{x_0, x_{\langle f \rangle}\}$ as f ranges over all the irreducible monic polynomials (corresponding to the prime ideals of $k[x]$). If k is algebraically closed, then all the irreducible polynomials are linear, so we can think of:

$$Spec(k[x]) = \{x_0\} \cup k$$

by identifying each $f = x - a$ with its root. In general, we get $Spec(k[x]) = \{x_0\} \cup (\overline{k}/\sim)$, where $\alpha \sim \beta$ when they are roots of the same minimal polynomial over k. If $k \subset K$ is the field extension obtained by adjoining a root of f, then $x_{\langle f \rangle}$ is called a "K-valued" point of $Spec(k[x])$. In the Zariski topology, the closed sets are the finite subsets of k.

(2) $Spec(\mathbf{Z}) = \{x_0, x_{\langle p \rangle}\}$ as p ranges over all the prime numbers. The closed sets are the finite sets of primes.

The following exercise is also very useful:

Exercise: Suppose that $I \subset R$ is an ideal. Then the map:

$$Spec(R/I) \to Spec(R) \text{ associated with the quotient } R \to R/I$$

is a homeomorphism onto the closed subset $\{x_{\mathcal{P}} \in Spec(R) \mid I \subset \mathcal{P}\}$.

Some more examples:

(a) $Spec(k[x_1, ..., x_n])$ is "affine space as a scheme". It contains $\mathbf{A}^n_k = k^n$ as a subset, if you identify each maximal ideal of the form $\langle x_1 - a_1, ..., x_n - a_n\rangle$ with the closed point $(a_1, ..., a_n) \in k^n$. If k is algebraically closed, then the Nullstellensatz tells us that these are the only closed points. In general, if k is algebraically closed, then there is a bijection between the points of $Spec(k[x_1, ..., x_n])$ and the affine varieties in $\mathbf{A}^n_k$, sending $x_{\mathcal{P}}$ to $V(\mathcal{P})$

(b) Every prime ideal in $\mathbf{Z}[x]$ is one of:

0, which gives the generic point.

$\langle p, f(x)\rangle$, where p is a prime and $\overline{f(x)} \in \mathbf{F}_p[x]$ reduces to an irreducible polynomial. These are the closed points.

$\langle p\rangle$ with p prime. The closure of such a point consists of itself and all the closed points $\langle p, f(x)\rangle$.

$\langle f(x)\rangle$ for an irreducible $f(x) \in \mathbf{Z}[x]$. Its closure consists of itself and all the closed points $\langle p, f(x)\rangle$.

Finally, some examples of the continuous maps:

(a) The inclusion $k[x] \hookrightarrow k[x, y]$ gives $\pi : Spec(k[x, y]) \to Spec(k[x])$, a surjective map. The (closed) fibers $\pi^{-1}(x_{\langle f\rangle})$ are each homeomorphic to $Spec(K[y])$, where $k \subset K$ is the field extension obtained by adjoining a root of f. This comes from the natural isomorphism $K[y] \cong k[x, y]/\langle f\rangle$ and the exercise above. The fiber over x_0 is not closed and is more mysterious (for now). Notice that $\text{char}(k) = \text{char}(K)$ for all these fields.

(b) The inclusion $\mathbf{Z} \hookrightarrow \mathbf{Z}[x]$ has a similar structure(!) In this case, the fibers are identified with the affine space schemes $Spec(\mathbf{F}_p[x])$ via the isomorphisms $\mathbf{Z}[x]/\langle p\rangle \cong \mathbf{F}_p[x]$. Here, of course, the characteristic of the field does vary(!)