Course Notes for Math 737 (Algebraic Geometry)

11. Varieties. The Categories of Affine Varieties. The Zariski topology is defined for affine varieties (and projective varieties in the exercises). The regular and rational categories of affine varieties over k are defined, as well as some natural contravariant functors from these categories to categories of integral domains and fields. The product of affine varieties always exists and is the same in either category.

Observation: Suppose $V \subset \mathbf{A}_k^n$ is an affine variety. If $I \subset k[V]$ is an ideal, define the algebraic subset:

$$V(I) := \{p \in V \mid f(p) = 0 \text{ for all } f \in I\}.$$

Let J be the ideal corresponding to I under the natural correspondence between ideals in $k[V] = k[x_1, ..., x_n]/I(V)$ and ideals in $k[x_1, ..., x_n]$ which contain $I(V)$. Then $V(J) \subset V$ and it is the same as the subset $V(I)$.

Conversely, if $W \subset V \subset \mathbf{A}_k^n$ is an algebraic subset, then $I(V) \subseteq I(W)$, so $W = V(I)$ for the ideal $I \subset k[V]$ corresponding to $I(W)$. Thus, algebraic subsets of V are the same as algebraic subsets of $\mathbf{A}_k^n$ contained in V.

The proof of Lemma 3.1 shows that if $I = \langle f_1, ..., f_r\rangle \subset k[V]$, then:

$$V(I) = V(f_1, ..., f_r) := \{p \in V \mid f_i(p) = 0 \text{ for all } i = 1, ..., r\}$$

Proposition 11.1: Let V be an affine variety. Declare:

$$Z \subset V \text{ is closed iff } Z \text{ is an algebraic subset.}$$

This defines a topology on V, called the **Zariski topology**.

Proof: We verify the four axioms for a topology:

(i) $\emptyset = V(k[V]) = V(1)$ is closed.

(ii) $V = V(0)$ is closed.

(iii) If $Z = V(I)$ and $W = V(J)$, then $Z \cup W = V(IJ)$ is closed.

(iv) If $Z_s = V(I_s)$ for $s \in S$, then $\cap_{s\in S} Z_s = V(\sum_{s\in S} I_s)$ is closed.

(iii) and (iv) follow more or less immediately from HW # 1, §3.

Remark: If $W \subset V \subset \mathbf{A}_k^n$ are varieties, then the Zariski topology on W agrees with the induced topology from the Zariski topology on V.

Exercise: Redo the previous page for projective varieties, replacing "ideal" by "homogeneous ideal". (It's not very interesting unless k is infinite!)

Examples: (a) If $V = \mathbf{A}_k^1$ or $\mathbf{P}_k^1$, then the Zariski topology is the "cofinite" topology (i.e. the open sets are $\emptyset$ and the complements of finite sets). This is clearly not Hausdorff if k is infinite!

(b) The product topology on the Cartesian product $\mathbf{A}_k^1 \times \mathbf{A}_k^1$ is the "cogrid" topology (i.e. the open sets are $\emptyset$, products of complements of finite sets, that is, complements of "grids", and intersections of these with complements of finite sets).

(c) The Zariski topology on $\mathbf{A}_k^2$ is **not** the "cogrid" topology. For instance, the complement of the hyperbola $V(xy-1)$ is an open set.

Here are some basic properties of this strange topology on V:

Proposition 11.2: (a) The complements of closed sets defined by principal ideals: $U_f := V - V(f)$ are a basis for the Zariski topology. In fact, every open set is a **finite** union of U_f's.

(b) If V is infinite (i.e. V is not a point), then any two open nonempty subsets of V have a nonempty intersection. Thus V is very non-Hausdorff!

(c) Any union $U = \cup_{s \in S} U_s$ of open subsets of V has a finite subset which also covers U. This is much stronger than compactness(!)

Proof: (a) By the Basis Theorem, the polynomial rings are Noetherian, and a quotient of a Noetherian ring is Noetherian (easy exercise) so that each $k[V]$ is Noetherian. But we've already asserted on the previous page that each $V(I) = V(f_1) \cap ... \cap V(f_r)$ if $I = \langle f_1, ..., f_r \rangle$, which is (a).

(b) It suffices by (a) to prove that any intersection of the form $U_f \cap U_g$ is nonempty. But $U_f \cap U_g = U_{fg}$, so since $I(V)$ is prime, it follows that $U_f, U_g \neq \emptyset$ implies $f, g \notin I(V)$, implies $fg \notin I(V)$ implies $U_{fg} \neq \emptyset$.

(c) Each U_s is a finite union of $U_{f_{s,i}}$'s by (a). If we let $Z = V - U$, then it follows that $Z = V(I)$ where I is the ideal generated by the $f_{s,i}$'s. Corollary 3.3 is readily modified to show that I is generated by finitely many of the $f_{s,i}$'s and it follows that U is a finite union of U_s's.

Recall the definition of irreducibility from §1, which we rewrite as:

Definition: A closed subset $Z \subset V$ is **irreducible** if Z is not the union of two smaller closed subsets.

Exercise: Points are the only irreducible closed subsets of a Hausdorff space.

Suppose that V an affine variety.

Proposition 11.3: (a) A closed subset $Z \subset V$ is irreducible if and only if the ideal $I(Z) := \{f \in k[V] \mid f(p) = 0 \text{ for all } p \in Z\}$ is prime.

(b) Any chain $V \supseteq W_1 \supseteq W_2 \supseteq W_3 \supseteq ...$ of closed subsets of V must stabilize (i.e. there is an r such that $W_r = W_{r+1} = ...$).

(c) Every closed subset $W \subset V$ is a finite union of irreducible subsets. The union can be written uniquely up to permutation of the factors.

Proof: (a) We've already proved this in Lemma 1.1 and §3 HW in case $V = \mathbf{A}^n_k$. But since prime ideals in $k[x_1, ..., x_n]$ containing $I(V)$ correspond to prime ideals in $k[V]$, (a) follows in general.

(b) Given a chain of closed subsets as above, there is a chain of ideals $I(W_1) \subseteq I(W_2) \subseteq ...$ which must stabilize since $k[V]$ is Noetherian. By HW §3, each $W_r = V(I(W_r))$, so the chain of closed sets must also stabilize.

(c) Suppose W is not a union of finitely many irreducible subsets. Then W is of course not irreducible, so $W = W_1 \cup Z_1$, and we may assume (exchanging W_1 and Z_1 if necessary) that $W_1 \subset W$ is a proper subset, which is also not a union of finitely many irreducible subsets. Reasoning in this way, we will get a chain $V \supseteq W \supseteq W_1 \supseteq ...$ violating (b), and a contradiction. Suppose $Z_1 \cup ... \cup Z_r = Z'_1 \cup ... \cup Z'_{r'}$ where the Z_i are irreducible, no Z_i is contained in a Z_j, and likewise for the $Z'_{i'}$. Then $Z_1 \subseteq Z'_{i'}$ for some i', otherwise $Z_1 = (Z_1 \cap Z'_1) \cup ...$ would be reducible. Similarly, by the irreducibility of $Z'_{i'}$, it follows that $Z_1 = Z'_{i'}$. Applying the same reasoning to the remaining factors gives uniqueness up to permutation of the factors.

Remark: If we require that the chain in (b) consist of distinct **irreducible** closed subsets of V, then we will discover that there is a uniform (sharp) upper bound for its length, which will be the dimension of V(!)

Definition: Suppose $W \subset \mathbf{A}^n_k$ and $V \subset \mathbf{A}^m_k$ are affine varieties. A **regular map** from W to V is by definition a map:

$$\Phi = (\phi_1, ..., \phi_m) : W \to V$$

where the $\phi_i \in k[W]$.

Proposition 11.4: (a) A regular map is continuous with respect to the Zariski topologies on W and V.

(b) A composition of regular maps is regular. The identity is regular.

(c) If Φ is a regular map as above, then the pull-back of functions is a homomorphism of k-algebras:

$$\Phi^* : k[V] \to k[W]; \ f \mapsto f(\phi_1, ..., \phi_m).$$

with the property that $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$.

Proof: First of all, observe that if $\Phi : (\phi_1, ..., \phi_m) : W \to V$ is a regular map, and if $f \in I(V)$, then $f(\phi_1, ..., \phi_m)(p) = 0$ for all $p \in W$, so that $f(\phi_1, ..., \phi_m) = 0$ as an element of $k[W] = k[x_1, ..., x_n]/I(W)$.

Thus the pull-back $\Phi^* : k[x_1, ..., x_m] \to k[W]$ given by $f \mapsto f(\phi_1, ..., \phi_m)$ is well-defined as a map $\Phi^* : k[V] \to k[W]$. It is a k-homomorphism because constants pull back to constants! To prove Φ is continuous, it suffices to show that $\Phi^{-1}(U_f)$ is open for all $f \in k[V]$ by 11.2 (a). But $\Phi^{-1}(U_f) = U_{\Phi^*(f)}$ ... compatibility! If we're additionally given $\Psi = (\psi_1, ..., \psi_l) : V \to X$, then choose lifts $\widetilde{\psi}_i$ of the ψ_i to elements of $k[x_1, ..., x_m]$ and the composition:

$$\Psi \circ \Phi = (\widetilde{\psi}_1(\phi_1, ..., \phi_m), ..., \widetilde{\psi}_l(\phi_1, ..., \phi_m))$$

shows that $\Psi \circ \Phi$ is a regular map. Obviously the identity is regular.

Finally, $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$ by the contravariantness of the pullback.

Definition: (a) Suppose A is a finitely generated k-algebra. Then a **framing of** A is a collection $a_1, ..., a_n \in A$ such that the map:

$$k[x_1, ..., x_n] \to A; \ x_i \mapsto a_i \text{ is surjective}$$

We'll let $(A; a_1, ..., a_n)$ stand for a framed k-algebra.

(b) Given framed k-algebras $(A; a_1, ..., a_n)$ and $(B; b_1, ..., b_n)$, we will say that a k-algebra homomorphism $f : A \to B$ **respects the framings** if $f(a_i) = b_i$ for all $i = 1, ..., n$.

Definition: The category of **framed k-algebras** is defined as follows:

The objects are the equivalence classes of framed k-algebras. Two framed k-algebras are equivalent of there is an isomorphism between them which respects the framings.

The morphisms are the homomorphisms of k-algebras modulo automorphisms which respect the framings. (See §0.1)

Proposition 11.5: The assignment:

$$(V \subset \mathbf{A}_k^n) \mapsto (k[V]; \overline{x}_1, ..., \overline{x}_n)$$

$$(\Phi : W \to V) \mapsto (\Phi^* : k[V] \to k[W])$$

is a fully faithful (contravariant) functor from the category of affine varieties and regular maps to the category of framed k-algebras.

Proof: Proposition 11.4 proves this is a functor. We only need to show:

Fully Faithful: Suppose $f : k[V] \to k[W]$ is a k-algebra homomorphism in the image of the functor, where $(k[V]; a_1, ..., a_n)$ and $(k[W]; b_1, ..., b_m)$ are in the image of the functor. If we let $I \subset k[x_1, ..., x_n]$ be the kernel of the evaluation map $x_i \mapsto a_i$, then $I = I(V)$, and we recover V because $V = V(I) \subset \mathbf{A}_k^n$ (by §3 HW). Similarly, we recover W, and finally we recover Φ because $\Phi = (f(a_1), ..., f(a_n)) : V \to W$. This construction produces the same ideals in the polynomials rings, hence the **same** $\Phi : V \to W$ if $f, k[V], k[W]$ are replaced by other representatives of their equivalence classes! Thus the functor is faithful. We can similarly construct Φ for an arbitrary f as above which satisfies $\Phi^* = f$, so the functor is fully faithful.

Definitions: (a) The isomorphism classes of affine varieties together with regular maps modulo automorphisms is called the **regular category of affine varieties**.

(b) The isomorphism classes of (unframed) finitely generated k-algebras together with k-algebra homomorphisms modulo k-algebra automorphisms is called the **category of finitely generated k-algebras**.

Corollary: The assignments:

$$(V) \mapsto (k[V]) \text{ and } (\Phi) \mapsto (\Phi^*)$$

determine a fully faithful functor from the regular category of affine varieties to the category of finitely generated k-algebras.

Proof: Abstract nonsense.

Remark: The framing has disappeared in the corollary because k-algebras modulo k-algebra isomorphisms are the same as framed k-algebras modulo k-algebra isomorphisms!

Thus by the corollary, the regular category of affine varieties is equivalent to a full subcategory of the category of finitely generated k-algebras. It is the subcategory defined by the following:

A k-algebra A represents an element of this subcategory if and only if:

(1) A is an integral domain, and

(2) There is a surjective k-algebra homomorphism $k[x_1, ..., x_n] \to A$ whose kernel I is a geometric ideal.

Proposition 11.6: Products exist in the regular category of affine varieties.

Moreover, as sets, the product is the Cartesian product.

Proof: More precisely, we prove that the subcategory of finitely generated k-algebras satisfying (1) and (2) is closed under tensor products. Since the tensor product is a coproduct in this category, it follows that the corresponding affine variety is a product in the category of affine varieties.

Given affine varieties $V \subset \mathbf{A}_k^n$ and $W \subset \mathbf{A}_k^m$, consider the surjective k-algebra homomorphism ρ and kernel:

$$K \subset k[x_1, ..., x_n, y_1, ..., y_m] \stackrel{\rho}{\to} k[V] \otimes_k k[W];\ x_i \mapsto \overline{x}_i \otimes 1, y_j \mapsto 1 \otimes \overline{y}_j.$$

The universal property for tensor products implies that K is the ideal generated by the polynomials $f \in I(V)$ and $g \in I(W)$, when these are thought of as elements of $k[x_1, ..., x_n, y_1, ..., y_m]$. We think of $V(K) \subset \mathbf{A}_k^{n+m} = k^{n+m} = \mathbf{A}_k^n \times \mathbf{A}_k^m$, that is, the points of $V(K)$ can naturally be thought of as a belonging to the Cartesian product of $\mathbf{A}_k^n$ and $\mathbf{A}_k^m$.

Then the Proposition follows from:

(i) $V(K) = V \times W \subset \mathbf{A}_k^{n+m}$,

(ii) $V \times W$ is an irreducible algebraic subset (hence an affine variety), and

(iii) $K = I(V \times W)$, hence a geometric ideal.

Suppose $(p,q) = (p_1, ..., p_n, q_1, ..., q_m) \in V \times W$. Then $f(p) = 0$ for all $f \in I(V)$ and $g(q) = 0$ for all $g \in I(W)$, so that if $h = \sum f_i a_i + \sum b_j g_j \in K$, then $h(p,q) = 0 + 0 = 0$. So $V \times W \subseteq V(K)$.

If $(p,q) \notin V \times W$, then either $p \notin V$, so there is an $f \in I(V) \subset K$ such that $f(p,q) \neq 0$, or else $q \notin W$, in which case there is a $g \in I(W) \subset K$ such that $g(p,q) \neq 0$. Thus we have (i).

Suppose $V \times W = Z' \cup Z''$. Then each $V \times \{q\}$ is either contained in Z' or in Z'' because V is irreducible. But as in the proof of the irreducibility of the quadric Q in §2, the sets:

$$W' = \{q \in W \mid V \times \{q\} \subset Z'\} \text{ and } W'' = \{q \in W \mid V \times \{q\} \subset Z''\}$$

are algebraic subsets of W, which is irreducible, so either $W' = W$, in which case $Z' = V \times W$, or else $W'' = W$, and $Z'' = V \times W$. Thus we have (ii).

Finally, suppose $h \in I(V \times W)$. Let $\rho(h) = \sum_{i=1}^k h_i \otimes h_i' \in k[V] \otimes_k k[W]$. If $\rho(h) \neq 0$, then $h_i \otimes h_i' \neq 0$ for some i, hence we can find $p \in V$ and $q \in W$ such that $h_i(p) \neq 0$ and $h_i'(q) \neq 0$. But then letting $c_j = h_j(p)$, we have $\sum_{i=1}^k c_j h_j'(q) = 0$ for all points $q \in W$, so $\sum_{i=1}^k c_j h_j' = 0$ in $k[W]$, and we can solve for h_i' in terms of the other h_j', which allows us to express $\rho(h)$ as a sum of $k-1$ tensors. Continuing, we eventually get $\rho(h) = 0$, proving (iii).

Example: $\mathbf{A}_k^2$ is a product of $\mathbf{A}_k^1$ with itself because $k[x] \otimes_k k[y] \cong k[x,y]$. Remember that we have already established that the topology on $\mathbf{A}_k^2$ is not the product topology!

Notice that the Proposition does NOT tell us that a tensor product of integral domains is an integral domain (but see §12):

Exercise: $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$ is not an integral domain!

(Why does this not contradict the Proposition?)

Definition: (a) Suppose $W \subset \mathbf{A}_k^n$ and $V \subset \mathbf{A}_k^m$ are affine varieties. A **rational map** from W to V is by definition a "map":

$$\Phi = (\phi_1, ..., \phi_m) : W - - > V$$

where the $\phi_i \in k(W)$.

(b) Φ is called a **dominant** rational map if the image is dense in V.

Examples: We have already encountered dominant rational maps in §4 in the definition of a parametrization $\Phi : \mathbf{A}_k^n - - > V$. In addition, a regular map is always a rational map, but not vice versa.

Remark: A rational map is not a map in the usual sense, because it is not defined at all points of W. It would be absurd to call something a map if it were not defined anywhere(!) but fortunately, we have the following:

Proposition 11.7: (a) The domain of a rational function $\phi \in k(W)$ is Zariski open and nonempty.

(b) The domain of a rational map Φ is Zariski open and nonempty.

Proof: Suppose $f, 0 \neq g \in k[W]$ and $\frac{f}{g}$ is a representative for ϕ. Then:

$$U := \{p \in W \mid g'(p) \neq 0 \text{ and } \frac{f'}{g'} \sim \frac{f}{g} \text{ for some } f', g' \in k[W]\}$$

is the domain of ϕ, because the value $\phi(p) = \frac{f'(p)}{g'(p)}$ is independent of the expression of ϕ as a fraction, provided that the denominator does not vanish at p. But this means that $U = \cup U_{g'}$ for g' as above, so U is Zariski open. It is nonempty because each $U_{g'}$ is nonempty(!) Part (b) follows from (a) because the domain of $\Phi = (\phi_1, ..., \phi_n)$ is the intersection of the domains of the ϕ_i, which is Zariski open, and nonempty by Proposition 11.2 (b).

Bad Example: Let $\Phi = (t, \frac{1}{t}) : \mathbf{A}_k^1 - - > \mathbf{A}_k^2$ and $\Psi = (\frac{1}{xy-1}) : \mathbf{A}_k^2 - - > \mathbf{A}_k^1$. Then $\Psi \circ \Phi$ is undefined! The problem here is that the image of Φ is contained in the complement of the domain of Ψ.

Thus a composition of rational maps need not be well-defined! However, we do have the following analogue of Proposition 11.4:

Proposition 11.8: (a) A rational map is continuous (where defined).

(b) A composition of **dominant** rational maps is well-defined, and is another dominant rational map.

(c) If Φ is a dominant rational map, then the pull-back of rational functions is an **injective** map of fields of finite type (see §4) over k:

$$\Phi^* : k(V) \hookrightarrow k(W);\ \phi \mapsto \phi(\phi_1, ..., \phi_m).$$

with the property that $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$.

Proof: If $\Phi = (\phi_1, ..., \phi_n) : W - - > V$ is rational and $f \in k[V]$, then as in Proposition 11.4, the pull-back $\Phi^*(f) := f(\phi_1, ..., \phi_n) \in k(W)$ is well-defined, and we have the compatibility $\Phi^{-1}(U_f) = U_{\Phi^*(f)} \subset \mathrm{domain}(\Phi)$, which is open in W. So Φ is continuous as a map from domain(Φ) to V. Notice that the domain of $\Phi^*(f)$ is included in the domain of Φ, so that this makes sense.

We would like to define $\Phi^* : k(V) \to k(W)$ similarly, but we run into the difficulty (as in the example above) that if $\phi = \frac{f}{g} \in k(V)$, then it may be the case that $\Phi^*(g) = 0$, in which case the pull-back of ϕ is undefined. But this occurs if and only if $\Phi^{-1}(U_g) = \emptyset$, which occurs if and only if the image of Φ lies in $V(g)$. If Φ is dominant, then this doesn't happen for any g **by definition**, so that indeed $\Phi^* : k(V) \to k(W)$ is well-defined, is not the zero map because $\Phi^*(1) = 1$, so it is injective.

The composition of dominant rational maps $\Psi = (\psi_1, ..., \psi_m) : V \to X$ and Φ is well-defined because if $\psi_i(\phi_1, ..., \phi_n) = \frac{f_i(\phi_1,...,\phi_n)}{g_i(\phi_1,...,\phi_n)}$ were not well-defined, then we would have $g_i(\phi_1, ..., \phi_n) = 0$ and the image of Φ would lie in $V(g_i)$. One checks that $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$ as in 11.4, and that the image of $\Psi \circ \Phi$ is dense, because otherwise one would have $(\Psi \circ \Phi)^{-1}(U_g) = \emptyset$ for some nonzero $g \in k[X]$, hence $\Phi^*(\Psi^*(g)) = 0$ by compatibility. But Φ^* and Ψ^* are injective, so this can't happen.

Definition: (a) Suppose K is a field of finite type over k. Then a **framing of** K is a collection $\alpha_1, ..., \alpha_n \in K$ such that every $\alpha \in K$ is a rational function of the α_i.

We'll let $(K; \alpha_1, ..., \alpha_n)$ stand for a framed k-algebra.

(b) Given framed fields $(K; \alpha_1, ..., \alpha_n)$ and $(L; \beta_1, ..., \beta_n)$, we will say that an injection $f : K \to L$ of fields over k **respects the framings** if $f(\alpha_i) = \beta_i$ for all $i = 1, ..., n$.

Remark: A framing of K is **not** equivalent to a surjection $k(x_1, ..., x_n) \to K$. Rather, it is equivalent to a surjection $k[x_1, ..., x_n] \to A \subset K$ to an integral domain A with the property that K is the field of fractions of A.

Proposition 11.9: The assignment:

$$(V \subset \mathbf{A}^n_k) \mapsto (k(V); \overline{x}_1, ..., \overline{x}_n)$$

$$(\Phi : W \to V) \mapsto (\Phi^* : k(V) \to k(W))$$

is a fully faithful (contravariant) functor from the category of affine varieties and dominant rational maps to the category of framed fields over k modulo isomorphisms which respect the framings.

Proof: Exercise. Follow the model for Proposition 11.5.

Definitions: (a) The isomorphism classes of affine varieties together with dominant rational maps is called the **rational category of affine varieties**.

(b) The isomorphism classes of fields of finite type over k together with k-homomorphisms is called the **category of fields of finite type over** k.

Corollary: The assignment:

$$(V \subset \mathbf{A}^n_k) \mapsto (k(V)) \text{ and } (\Phi) \mapsto (\Phi^*)$$

determines a fully faithful functor from the rational category of affine varieties to the category of fields of finite type over k.

Proof: Abstract nonsense.

Proposition 11.10: Products in the rational category of affine varieties exist and agree with products in the regular category.

Proof: Exercise.

Definition: Affine varieties V and W are said to be **birational** if they are isomorphic in the rational category. They are said to be **isomorphic** if they are isomorphic in the regular category.

Sixth Homework Assignment

1. Suppose $S \subset V$ is any subset of an affine variety. Prove that $V(I(S)) = \overline{S}$, the closure of S in the Zariski topology on V.

2. Suppose $\Phi : V \to W$ is a regular map of affine varieties and $Z \subset V$ and $Y \subset W$ are algebraic subsets. Prove:

(a) $\Phi^{-1}(Y) = V(J(Y)) \subseteq V$, where $J(Y) \subseteq k[V]$ is the ideal generated by the set $\Phi^*(I(Y))$, and

(b) $\overline{\Phi(Z)} = V(J(Z)) \subseteq W$, where $J(Z) = \Phi^{*-1}(I(Z)) \subseteq k[W]$.

3. (a) Prove the analogue of Proposition 11.1 for projective varieties.

(b) If $V \subset \mathbf{A}_k^n$ is an affine variety, prove that $\overline{V} \subset \mathbf{P}_k^n$ (as defined in §4) is the closure of V in the Zariski topology on $\mathbf{P}_k^n$.

4. Prove the only irreducible closed subsets of a Hausdorff space are points.

5. Exercises on tensor products:

(a) Prove that $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$ is not an integral domain. Why doesn't this contradict Proposition 11.5?

(b) Prove that $\mathbf{C}(x) \otimes_{\mathbf{C}} \mathbf{C}(y)$ is not a field. Thus the product of affine varieties in the rational category does not correspond to the tensor product.

6. Suppose k is an infinite field of characteristic not equal to two. Let $V = V(x - y^2) \subset \mathbf{A}_k^2$, and let $\pi : V \to \mathbf{A}_k^1$ be defined by $\pi(x, y) = x$. Prove that the **fiber product** (of sets) $V \times_{\mathbf{A}_k^1} V \subset V \times V$ is an algebraic subset, but is not irreducible. Conclude from this that the analog of Proposition 11.6 cannot be true for fibre products. Find an example in characteristic two.

7. (a) Prove that affine varieties are separated (see §0.3).

(b) Prove that the only proper affine variety is a point (see §0.3).

Remark: Affine varieties are the analogues in algebraic geometry of open balls in $\mathbf{R}^n$. Thus # 7 is the analogue of the statements: open balls are Hausdorff and open balls are not compact. Notice that by Proposition 11.2, this is precisely the opposite of what the Zariski topology gives us!

8. (a) Prove that the hyperbola $V(xy-1) \subset \mathbf{A}_k^2$ is birational to $\mathbf{A}_k^1$ but is not isomorphic to it. (Assume k is infinite here and in (b)).

(b) Ditto for the cuspidal and nodal cubics of §4.

(c) Suppose k is an algebraically closed field of characteristic p. Prove that the regular map $f : \mathbf{A}_k^1 \to \mathbf{A}_k^1;\ x \mapsto x^p$ is a bijection, but it does not have a regular inverse.

9. Prove Proposition 11.9.

10. Prove Proposition 11.10.