

## Chapter 2

# Polynomials

A **polynomial** in the variable  $x$  looks like:

$$a_d x^d + a_{d-1} x^{d-1} + \dots + a_0$$

where  $a_0, \dots, a_d$  are the coefficients, which are usually elements of a field, and  $d$  is the degree (assuming the leading coefficient  $a_d$  is nonzero). The arithmetic of polynomials has a lot of the same features as the arithmetic of integers: both admit division with remainders, Euclid's algorithm and have infinitely many primes, and rational functions (polynomial fractions) play the role of rational numbers (integer fractions). This resemblance is formalized in the definition of a **Euclidean domain**, which includes both the integers and all polynomials with coefficients in a field, and where there is unique factorization into primes.

The **roots** tell us everything there is to know about a polynomial. Polynomials with rational number coefficients will be of particular interest to us. The rational roots test finds all the rational number roots of such polynomials, but there are often irrational roots. The quadratic and cubic formulas give the (possibly complex) roots of polynomials of degree 2 or 3, but things are considerably more complicated in higher degree. Even deciding whether or not a polynomial is prime can be difficult, and Eisenstein's criterion uses the finite fields produced by **clock arithmetic** to give one useful way of detecting prime polynomials. We know one thing for sure. All the roots are complex numbers. This is the **fundamental theorem of algebra** which we will prove using a few simple ideas from analysis.

**6-6** A **power series** in the variable  $x$  is a (usually infinite) sum:

$$f(x) = a_d x^d + a_{d+1} x^{d+1} + a_{d+2} x^{d+2} + \dots \quad (a_d \neq 0, d \geq 0)$$

where the coefficients all belong to a field  $F$ . Power series are added and multiplied as polynomials are added and multiplied, and they are easily seen to satisfy properties (a)-(d) of the beginning of this section.

The set of power series is denoted by  $F[[x]]$ . In  $\mathbb{Q}[[x]]$ :

- (a) Find the multiplicative inverse of  $1 + (a/b)x$ .
- (b) Find the multiplicative inverse of  $1 + 2x + 3x^3 + 4x^4 + \dots$ .

Hint: This power series is the derivative of  $1 + x + x^2 + \dots = 1/(1-x)$ .

(c) The units in  $F[[x]]$  are exactly the power series satisfying  $d = 0$ . Assuming this fact (which I could ask you to prove, but I won't!) show that the function:

$$\deg(a_d x^d + a_{d+1} x^{d+1} + a_{d+2} x^{d+2} + \dots) = d$$

is a degree function for the power series.

- (d) Prove that  $x$  has no multiplicative inverse in any  $F[[x]]$ .

(e) Prove that  $F[[x]]$  satisfies property (e) at the beginning of this section, so it is a domain.

Finally,  $F[[x]]$  satisfies a strong form of division with remainders. Namely, if  $\deg(f(x)) \leq \deg(g(x))$ , then:

$$f(x) \text{ divides } g(x)$$

(I am telling you this. If you want to prove it, go for it!)

In other words, this is division with remainders without remainders! So  $F[[x]]$  is yet another example of a Euclidean domain.