New Stabilities for Graded Modules

I-70 Algebraic Geometry

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Hilbert Polynomials

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$, let

$$M_\bullet = \bigoplus \Gamma(\mathbb{P}^n, \mathcal{F}(d))$$

be the associated module over $S = \mathbb{C}[x_0, \ldots, x_n]$ and let:

$$\chi_{\mathcal{F}}(d) = \chi(\mathbb{P}^n, \mathcal{F}(d))$$

be the Hilbert polynomial of $\mathcal{F}$. This is discrete invariant: constant on flat families over a connected base. Moreover:

$$\deg(\chi_{\mathcal{F}})$$

is the dimension of the support of $\mathcal{F}$ and $\mathcal{F}$ has pure dimension $m$ if $\deg(\chi_{\mathcal{E}}) = m$ for all $\mathcal{E} \subseteq \mathcal{F}$. 
Gieseker Slope

\( \chi \) is computed by the Hirzebruch-Riemann-Roch Theorem:

\[
\chi_{\mathcal{F}}(t) = \deg(ch(\mathcal{F}) \cdot td(\mathbb{P}^n) \cdot e^{tH})
\]

where \( ch(\mathcal{F}) \), \( td(\mathbb{P}^n) \) and \( H \) are cohomology classes on \( \mathbb{P}^n \). Thus:

\[
\chi_{\mathcal{F}}(t) = \text{rk}(\mathcal{F}) \cdot \frac{t^n}{n!} + \text{lower order}
\]

The Gieseker slope of \( \mathcal{F} \) is:

\[
\mu_{\mathcal{F}}(t) = \frac{\chi_{\mathcal{F}}(t)}{\text{leading coefficient}}
\]
Definition. (a) $\mathcal{F}$ is Gieseker/Simpson stable if:

(i) $\mathcal{F}$ is pure-dimensional and (ii) For all proper subsheaves $\mathcal{E} \subset \mathcal{F}$,

$$\mu_{\mathcal{E}}(t) < \mu_{\mathcal{F}}(t)$$

as polynomials in $t$

(b) $\mathcal{F}$ is semi-stable if pure-dimensional and there is a filtration:

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_N = \mathcal{F}$$

with each $\mathcal{F}_{i+1}/\mathcal{F}_i$ stable of the same slope.

Remark. (Semi)-stability are open conditions on flat families.

Theorem (Gieseker/Simpson). For fixed Hilbert polynomial $\chi$, there is a projective moduli space $\mathcal{M}_{\mathbb{P}^n}(\chi)$ parametrizing equivalence classes of semi-stable sheaves of Hilbert polynomial $\chi$. 
Examples.

\( n = 1 \). The only stable sheaves are line bundles \( \mathcal{O}_{\mathbb{P}^1}(d) \) and skyscraper sheaves \( \mathbb{C}_p \).

**Remark.** Minimal rank pure-dimensional sheaves are stable.

\( n = 2 \) Any Hilbert scheme of ideal sheaves.

**Note.** Stable points of the moduli spaces \( \mathcal{M}_{\mathbb{P}^2}(\chi) \) are smooth. This is because stable sheaves are simple, and:

\[
\text{Hom}(\mathcal{F}, \mathcal{F}) = \mathbb{C} \cdot \text{id}
\]

and Serre duality give the vanishing of obstruction spaces.

\( n \geq 3 \) “Pathological” moduli spaces abound. (Murphy’s Law).
The Gieseker slope

\[ \mu_{\mathcal{F}}(t) = \frac{\chi_{\mathcal{F}}(t))}{\text{leading coefficient}} \]

is not a good slope when evaluated at any \( t \). However:

**Theorem.** (Altavilla, B, Mu, Petkovic) The rational function:

\[ \nu_{\mathcal{F}}(t) = \frac{\chi_{\mathcal{F}}(t)}{\chi'_{\mathcal{F}}(t)} \]

defines a one-parameter family of Bridgeland stability conditions on the derived category \( \mathcal{D}^b(\mathbb{P}^n) \) of coherent sheaves on \( \mathbb{P}^n \).

In particular, \( \nu(t) \) defines a good slope with quasi-projective moduli for coherent sheaves of Castelnuovo-Mumford regularity

\[ k = \lfloor t \rfloor \]
Regularity

**Definition.** \( \mathcal{F} \) is \( k \)-regular if:

\[
H^i(\mathbb{P}^n, \mathcal{F}(k - i)) = 0 \text{ for all } i > 0
\]

**Basic Properties.** (i) If \( \mathcal{F} \) is \( k \)-regular, then it is \( k + 1 \)-regular.

(ii) \( \mathcal{F} \) is \( k \)-regular if and only if \( \mathcal{F}(k) \) is generated by global sections with linear syzygies, i.e. \( \mathcal{F} \) has a free resolution:

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-k - n)^{a_n} \rightarrow \cdots \rightarrow \mathcal{O}_{\mathbb{P}^n}(-k)^{a_0} \rightarrow \mathcal{F} \rightarrow 0
\]

or, equivalently,

\[
\mathcal{F} = [\mathcal{O}_{\mathbb{P}^n}(-k - n)^{a_n} \rightarrow \cdots \rightarrow \mathcal{O}_{\mathbb{P}^n}(-k)^{a_0}] \in D^b(\mathbb{P}^n)
\]
Consider the example of the ideal sheaf of three points $Z \subset \mathbb{P}^2$.

(a) If the points are not collinear, then $I_Z$ is 2-regular, and:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3)^2 \to \mathcal{O}_{\mathbb{P}^2}(-2)^3 \to I_Z \to 0$$

(b) If the points are collinear, then $I_Z$ is not 2-regular and:

$$\mathcal{O}_{\mathbb{P}^2}(-4) \to \mathcal{O}_{\mathbb{P}^2}(-3)^3 \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^3 \oplus \mathcal{O}_{\mathbb{P}^2}(-4) \oplus \mathcal{O}_{\mathbb{P}^2}(-3)$$

is the resolution. But both are 3-regular with resolution:

$$\mathcal{O}_{\mathbb{P}^2}(-5)^3 \to \mathcal{O}_{\mathbb{P}^2}(-4)^9 \to \mathcal{O}_{\mathbb{P}^2}(-3)^7$$
The following theorem was a precursor to stability conditions:

**Theorem** (King ’91) Let

\[ A_k = \{ F^\bullet = [\mathcal{O}_{\mathbb{P}^n}(-k - n)^{a_n} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}(-k)^{a_0}] \} \]

be the (abelian) category of complexes. Then any assignment:

\[ z_i = z(\mathcal{O}_{\mathbb{P}^n}(-k - i)[i]) \in \mathbb{H} \]

defines a GIT quotient space for the action of \( G = \prod GL(a_i) \) on complexes with dimension vector \( a = (a_n, \ldots, a_0) \) in which:

\[ F^\bullet \] has a GIT-stable orbit iff \( \arg(\sum z_i b_i) < \arg(\sum z_i a_i) \)

for each dimension vector \( b \) of a subcomplex \( E^\bullet \subset F^\bullet \).
Consider the resolution of 3 non-collinear points in $\mathbb{P}^2$

$$\mathcal{O}_{\mathbb{P}^3}(-3)^2 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2)^3 \in \mathcal{A}_2$$

According to the Theorem of King, we assign two complex vectors:

$$z_1 = z(\mathcal{O}_{\mathbb{P}^3}(-3)[1]) \text{ and } z_0 = z(\mathcal{O}_{\mathbb{P}^3}(-2)) \in \mathcal{H}$$

then we have a GIT quotient of the space of complexes. We need:

$$\arg(z_1) > \arg(z_0),$$

and then $F^\bullet$ is stable if it has no subcomplexes with any of the dimension vectors: $(1, 0), (2, 0), (1, 1), (2, 1), (2, 2)$

Ideal sheaves are then stable, as are the sheaves:

$$\epsilon : 0 \rightarrow \mathcal{O}_l(-3) \rightarrow \mathcal{F} \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow 0$$

for lines $l \subset \mathbb{P}^2$ and non-trivial extension class.
Our Theorem

The more precise version of our Theorem is:

**Theorem.** The assignment:

$$z(F^\bullet) = \chi_{F^\bullet}'(t) + i\chi_{F^\bullet}(t) \in \mathbb{C}$$

on complexes maps objects of $\mathcal{A}_{[t]}$ to $\mathbb{H}i$.

**Proof.** Since $\chi(\mathcal{O}(t)) = (t + 1) \cdots (t + n)/n!$,

$$\chi(\mathcal{O}) = 1 \text{ and } \chi'(\mathcal{O}) > 0$$

$$\chi(\mathcal{O}(-i)[i]) = 0 \text{ and } \chi'(-i)[i]) < 0$$

for all $i = 1, \ldots, n$. Moreover, as $t \downarrow -1$, the values:

$$\chi(\mathcal{O}(-i)[i]) \text{ move clockwise, staying within } \mathbb{H}i$$
For $t \in (1, 2]$ the stable objects in $\mathcal{A}_2$ with class $\chi = \chi_{I_Z}(t)$ are:

(i) Ideal sheaves of 3 non-collinear points and

(ii) Sheaves $\mathcal{F}$ from the earlier slide

Where are the ideal sheaves of collinear points? They are 3-regular, and the sheaf inclusion:

$$\mathcal{O}_{\mathbb{P}^2}(-1) \subset I_Z$$

is an inclusion of complexes that is destabilizing when

$$\nu_{\mathcal{O}_{\mathbb{P}^2}(-1)}(t) \geq \nu_{I_Z}(t)$$

i.e. when $t \leq 2 + \sqrt{6}$. After that, it no longer destabilizes!
Twisted Cubics

The Hilbert scheme of twisted cubic curves contains:

- $I_C$ the ideal sheaf of a twisted cubic
- $I_{E \cup p}$ plane cubic and general point
- $I_{E \cup p^*}$ point in the same plane as $E$

The latter two have subsheaves:

- $I_p(-1) \subset I_{E \cup p}$ and $\mathcal{O}_{\mathbb{P}^3}(-1) \subset I_{E \cup p^*}$

that destabilize the respective sheaves up until:

- $t \approx 6.24$ and $t \approx 7.47$, respectively

Schmidt and Xia have shown that these are the only “walls” for $\chi$ (at which the stable objects change) and they used this to recover the description of the Hilbert scheme due to Kleiman and Piene.
Questions about $\mathbb{P}^n$

We can easily show that if $F^\bullet \in D^b(\mathbb{P}^n)$, then:

(i) If $F^\bullet$ is not a sheaf, then $F^\bullet \not\in \mathcal{A}_k$ for large $k$, so in particular, $F^\bullet$ is not stable for large $t$.

(ii) If $\mathcal{F}$ is not pure-dimensional, then it is unstable for large $t$.

(iii) If $\mathcal{F}$ is Gieseker unstable, then it is unstable for large $t$.

Question. If $\mathcal{F}$ is Gieseker stable, then is it stable for large $t$?
(True for $n = 2, 3$.)

Difficulty. It is hard to “see” the subcomplexes of a complex!
Are there other varieties $X$ for which:

$$z = \chi'(t) + i\chi(t)$$

define stability conditions on $D^b(X)$?

And if so, what are the analogues of the categories $A_k$?

**Examples.** All Riemann surfaces (“rotated” standard stability).
All algebraic surfaces of positive signature $K_S^2 > 8\chi(O_S)$.
Odd dimensional quadrics (with varying exceptional collections).

For surfaces of signature zero, we can get “close:”

$$z_\epsilon(t) = \chi'(t) + i\chi(t) - \epsilon\chi''(t)$$

define stability conditions for $0 < \epsilon << 1$ and it seems to be an interesting question to study the analogues of $A_k$ for, e.g. Hirzebruch surfaces.