Why does pi keep popping up?

Undergraduate Colloquium, October 2007

I. Definitions and Archimedes

II. Digits and some silliness (and Ramanujan)

III. Antidote: pi is irrational.

IV. Pi popping up in factorials.

A. Gamma function: "$(1/2)! = \sqrt{\pi}$"

B. Stirling's formula.

C. Sums of inverse even powers.

V. Why?

I. Definitions and some geometry.

Definition: π is the ratio of circumference to diameter of a circle:

$$\pi := \frac{c}{d}$$

which does not depend upon the circle (it's a "dimensionless constant").

Observation: The ratio of the area to the radius squared:

$$\Pi := \frac{A}{r^2}$$

also does not depend upon the circle. In fact:

$$\Pi = \pi$$

Proof: Thinking of A, c and d as functions of the radius:

$$\frac{dA}{dr} = c(r) \quad \text{(basic calculus)}$$

Thus the derivative with respect to r:

$$c(r) = A'(r) = (\Pi r^2)' = \Pi(2r) = \Pi d(r)$$

and dividing both sides by $d(r)$ gives the result.

Exercise: If S is a sphere in $\mathbb{R}^3$, show that:

$$\frac{V}{r^3} = \frac{4}{3}\pi \quad \text{and} \quad \frac{A}{r^2} = 4\pi$$

where V, A and r are the volume, surface area and radius.

Challenge: Find a formula for the ratios:

$$\frac{V}{r^n} \quad \text{and} \quad \frac{A}{r^{n-1}}$$

for the sphere in $\mathbb{R}^n$:

$$S = \{(x_1, \ldots, x_n) \mid x_1^2 + \cdots x_n^2 = r^2\}$$

Approximating π. The Direct Method:

$$3 < \pi < 4$$

Proof: Two pictures—inscribed hexagon and circumscribed square.

Consider a regular n-gon with sides of length s. The radius R of the *circumscribed* circle satisfies:

$$\sin\left(\frac{\pi}{n}\right) = \frac{(s/2)}{R} = \frac{s}{2R}$$

and the radius of the *inscribed* circle is:

$$\tan\left(\frac{\pi}{n}\right) = \frac{(s/2)}{r} = \frac{s}{2r}$$

and thus if we let $P = ns$ be the perimeter of the n-gon, then:

$$n \sin\left(\frac{\pi}{n}\right) = \frac{P}{2R} < \pi < \frac{P}{2r} = n \tan\left(\frac{\pi}{n}\right)$$

Example: Let $n = 6$ (the hexagon).

From $\sin(\pi/6) = 1/2$ and $\cos(\pi/6) = \sqrt{3}/2$, we get:

$$3 = 6 \sin\left(\frac{\pi}{6}\right) < \pi < 6 \tan\left(\frac{\pi}{6}\right) = 2\sqrt{3} \approx 3.5$$

Using the trigonometric half-angle formulas:

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos(\theta)}{2}},\ \cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos(\theta)}{2}},\ \tan\left(\frac{\theta}{2}\right) = \frac{1-\cos(\theta)}{\sin(\theta)}$$

one could "in principle" get better inequalities by hand. For example:

$$\pi < 12 \tan\left(\frac{\pi}{12}\right) = 12(2-\sqrt{3}) \approx 3.2$$

Archimedes carried this out to $n = 96$ to obtain:

$$3.141 \approx 3\frac{10}{71} < \pi < 3\frac{1}{7} = \frac{22}{7} \approx 3.143$$

Question: How did he do this? (Continued fractions!)

II. Digits and silliness. Rational approximations to π range from:

$$\pi \approx 3 \quad \text{(Babylonians)}$$

$$\pi \approx \frac{256}{81} \approx 3.16 \quad \text{(Egyptians)}$$

$$\pi \approx \frac{22}{7} \approx 3.143 \quad \text{or} \quad \pi \approx 3.14 \quad \text{(unhappy kids)}$$

to the remarkably good approximation:

$$\pi \approx \frac{335}{113} \approx 3.1415929$$

Mathematical urban legends abound of state legislatures in the USA trying to legislate the value of π to be precisely 3, or 3.14, or 22/7. Such foolishness is demonstrated in the widely distributed famous crank book of Carl Theodore Heisel, published in 1931, which rediscovered the Egyptian approximation and claimed it was an exact value for π(!) Many interesting mathematical corollaries would follow, including the "fact" that π is the area of a $16/9 \times 16/9$ square ("squaring the circle"). It is an interesting sociological question to ask whether the emergence of the internet will give crackpots a wide audience for such ridiculous theories. It is our responsibility as trained experts to be vigilant.

Other Pet Peeves: Dan Brown (DaVinci Code) writes the golden mean as: $\phi = 1.618$ and makes like this is a **very** mysterious number.

Well, it isn't 1.618. It's

$$\frac{1+\sqrt{5}}{2}$$

which is a **lot less** mysterious than π. It solves a simple equation:

$$\frac{1}{x-1} = x$$

and **that's** why it is chosen so often by nature.

For another pet peeve, consider the singers on http://pi.ytmnd.com. My beef with them is that the song itself seems to imply some sort of regularity of the digits of π, which isn't there because π is irrational!

Question: Suppose you were stranded on a desert island with only paper and pencils (lots of them!). Could you prove that:

$$3.14159 < \pi < 3.1416$$

before you starved to death?

Calculus to the rescue. The Maclaurin series for $\arctan(x)$:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

immediately yields the Gregory-Leibniz formula (using $\arctan(1) = \frac{\pi}{4}$):

$$\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots)$$

but this will not get you off the island in time. However, the following identity due to Machin:

$$\arctan(1) = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$$

was widely used in the 19th century, together with Maclaurin series, to get really good approximations with paper and pencil.

Challenge: Can you prove the identity? Hint (thanks to Wikipedia):

$$(5+i)^4(-239+i) = -114244 - 114244i$$

Some Approximations using Machin's identity:

$\pi < 16 * (\frac{1}{5}) = 3.2$

$\pi > 16 * (\frac{1}{5} - \frac{1}{125*3}) - 4 * \frac{1}{239} \approx 3.1406$

$\pi < 16 * (\frac{1}{5} - \frac{1}{375} + \frac{1}{15625}) - 4 * (\frac{1}{239} - \frac{1}{239^3*3}) \approx 3.1416$

Ramanujan was another mathematician with extraordinary powers. He gave:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 (396)^{4k}}$$

which converges absurdly rapidly (at $k = 0$ already $\pi \approx 3.1415927$(!)).

III. Niven's Proof that π is irrational (take that, Mr. Heisel!)

Suppose:

$$\pi = \frac{a}{b} \quad \text{for integers } a, b \geq 1$$

Construct the polynomials:

$$f_n(x) = \frac{x^n b^n (\pi - x)^n}{n!} = \frac{x^n (a - bx)^n}{n!}$$

These polynomials do not have integer coefficients, but:

$$f_n(0) = f_n'(0) = f_n''(0) = \cdots = f_n^{(n-1)}(0) = 0 \text{ and } f_n^{(n)}(0), f_n^{(n+1)}(0), \cdots \in \mathbf{Z}$$

and since $f_n(\pi - x) = f_n(x)$, the same is true of the derivatives at $x = \pi$.

Notice that as $n \to \infty$, the functions $f_n(x)$ for $0 \leq x \leq \pi$ are all positive and go uniformly to 0. That's because:

$$0 < f_n(x) < \frac{(\pi^2 b)^n}{n!}$$

Finally, define new functions $F_n(x)$ by:

$$F_n(x) = f_n(x) - f_n''(x) + f_n^{(4)}(x) - f_n^{(6)}(x) + \cdots$$

(this is a finite sum since $f(x)$ is a polynomial). Then:

$$F_n(x) + F_n''(x) = f_n(x)$$

and $F(0)$ and $F(\pi)$ are integers. It follows that:

$$(-F_n(x)\cos(x) + F_n'(x)\sin(x))' = f_n(x)\sin(x)$$

and then:

$$\int_0^\pi f_n(x)\sin(x)dx = (-F_n(x)\cos(x) + F_n'(x)\sin(x))|_0^\pi = F_n(0) - F_n(\pi) \in \mathbf{Z}$$

But this integral must be positive, and for large enough n this is a contradiction, since:

$$\lim_{n\to\infty} \int_0^\pi f_n(x)\sin(x)dx = 0$$

but the integrand is positive on $(0, \pi)$, hence the integral **isn't** zero.

IV. Pi popping up.

A. The Gamma function is “the” analytic function such that:

$$(a)\ \Gamma(1) = 1 \quad \text{and} \quad (b)\ x\Gamma(x) = \Gamma(x+1) \text{ for all } x > 0$$

In particular,

$$\Gamma(2) = 1,\ \Gamma(3) = 2 \cdot 1,\ \Gamma(4) = 3 \cdot 2 \cdot 1,\ \Gamma(5) = 4!$$

and

$$\Gamma(n+1) = n!$$

so we wouldn't be too remiss in declaring that:

$$x! = \Gamma(x+1)$$

Definition: Candidate for the Γ function:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

Proof of (a):

$$\Gamma(1) = \int_0^\infty e^{-t}dt = -e^{-t}|_0^\infty = e^0 = 1$$

Proof of (b): *Integration by parts* (using $(t^x e^{-t})'$) gives:

$$x\Gamma(x) = t^x e^{-t}|_0^\infty + \Gamma(x+1)$$

and if $x > 0$, then the extra quantity is zero.

Finally, consider:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}dt$$

If we perform the *u-substitution* $t = u^2/2$, then $dt = udu$ and:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-u^2/2}}{u/\sqrt{2}}(udu) = \sqrt{2}\int_0^\infty e^{-\frac{u^2}{2}}du = \sqrt{\pi}$$

The last equality comes from integrating in *polar coordinates*!

$$\int_0^\infty e^{-\frac{u^2}{2}}du = \sqrt{\int_0^\infty \int_0^\infty e^{-\frac{u^2}{2}}e^{-\frac{v^2}{2}}dudv} = \sqrt{\int_0^\infty \int_0^{\frac{\pi}{2}} re^{-\frac{r^2}{2}}drd\theta} = \sqrt{\frac{\pi}{2}}$$

Conclusion:

$$\left(-\frac{1}{2}\right)! = \sqrt{\pi},\ \left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2},\ \left(\frac{3}{2}\right)! = \frac{3\sqrt{\pi}}{4}, \ldots$$

Claim: The constant for the sphere in $\mathbb{R}^n$ is:

$$\frac{V}{r^n} = \frac{\pi^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!}$$

B. The Basel Problem

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

Euler's "Proof:" Consider the Taylor series for:

$$f(x) := \frac{\sin(x)}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{5!} - \cdots$$

This function satisfies:

$$f(1) = 1 \text{ and } f(x) = 0 \Leftrightarrow x = \pm\pi, \pm 2\pi, \cdots$$

so, treating it as a polynomial, we can write:

$$f(x) = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\cdots$$

and combining the terms pairwise:

$$f(x) = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{(2\pi)^2}\right)\left(1 - \frac{x^2}{(3\pi)^2}\right)\cdots$$

Finally, multiplying the terms out, we get:

$$f(x) = 1 - \left(\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}\right) x^2 + \cdots$$

so that equating the coefficients of x^2 gives:

$$-\frac{1}{6} = -\frac{1}{\pi^2} \sum_{n=1\infty} \frac{1}{n^2}$$

and multiplying both sides by $-\pi^2$ gives the result.

More Generally: Let

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

Then each *even* value of the ζ function is of the form:

$$\zeta(2k) = \pi^{2k} \cdot \frac{a}{b} \text{ for some (explicit!) rational number } \frac{a}{b}$$

C. Stirling's Formula.

$$\lim_{n\to\infty} n! \left(\frac{e}{n}\right)^n \frac{1}{\sqrt{n}} \approx \sqrt{2\pi}$$

This time **both** e and π appear(!)

V. Why? As with the golden mean, π's appearance is explained by a very simple (this time differential) equation:

$$y'' = -y$$

whose solutions,

$$y = e^{ix} = \cos(x) + i\sin(x)$$

have *period* 2π (so maybe the constant should be 2π....see Bob Palais):

Euler's Equation:

$$e^{i\pi} = -1$$

The simplicity the equation "explains" the ubiquity of π (!?).

Periods: Kontsevich and Zagier have proposed a vast generalization of the notion of a period. Namely, for them, **any** number of the form:

$$\int_R \frac{P(x_1, \cdots, x_n)}{Q(x_1, \cdots, x_n)} dx_1 dx_2 \cdots dx_n$$

where $R \subset \mathbb{R}^n$ is a region bounded by polynomial equations in the x_1 with rational coefficients should be considered a period. Thus:

$$\pi = \int_{x^2+y^2 \leq 1} dxdy \quad \text{is a period}$$

but also:

$$\ln(k) = \int_1^k \frac{1}{x} dx \text{ is a period}$$

and so is every algebraic number. The study of periods (is e a period?) is their contribution to the volume on "21st century" mathematics.