Lecture 9. Complex Affine Varieties

PCMI Summer 2015 Undergraduate Lectures on Flag Varieties

Lecture 9. When we think of the Lie groups $\text{GL}(n, \mathbb{C}), \text{SL}(n, \mathbb{C}), \text{SO}(n, \mathbb{C}), \text{Sp}(2n, \mathbb{C})$ over the complex numbers as algebraic groups, we may forget the groups entirely and replace them with their $\mathbb{C}$-algebras of regular functions. Within these regular functions the irreducible representations of the Lie groups will eventually be found.

Motivating Example. Because the circle $U(1)$ is abelian, all of its irreducible representations are one-dimensional characters, and it is easy to see that the continuous characters of $U(1)$ are the powers:

$$\chi_n(e^{i\theta}) \cdot v = e^{in\theta} \cdot v$$

On the other hand, consider $\text{GL}(1, \mathbb{C}) = \mathbb{C}^*$. The regular (algebraic) functions on $\mathbb{C}^*$ are Laurent polynomials in the variable $z$, which are organized by degree:

$$\mathbb{C}[z, z^{-1}] = \bigoplus_{n=-\infty}^{\infty} \mathbb{C} \cdot z^n$$

But when we act on these by $\mathbb{C}^*$ (with the variable $w$) by pulling back under the left multiplication, we get:

$$w \cdot z^n = (w^{-1}z)^n = w^{-n}z^n$$

which is just the one-dimensional representation $\chi_{-n}$ extended to $\mathbb{C}^*$.

In other words, the algebraic functions on $\mathbb{C}^*$ break up precisely into a complete set of characters of $\mathbb{C}^*$, which agree with the continuous characters of the circle $U(1)$. Our goal is to see how this idea extends to the non-abelian Lie groups, using some algebraic geometry.

**Definition 9.1.** (a) A $\mathbb{C}$-algebra $A$ is a commutative ring with $1 \in A$ together with an inclusion of $\mathbb{C} \subset A$ as a field of “scalars.”

(b) A ring homomorphism $f : A \to B$ of $\mathbb{C}$-algebras that is also the identity map on scalars is a $\mathbb{C}$-algebra homomorphism. When the context is clear, we will just say that $f$ is a homomorphism.

**Example 9.1** The polynomial rings $A = \mathbb{C}[x_1, ..., x_n]$ are $\mathbb{C}$-algebras.

**Definition 9.2:** A (proper) **ideal** is a subset $0 \in I \subset A$ that is:

(a) Closed under (internal) addition: $b_1, b_2 \in I \Rightarrow b_1 + b_2 \in I$.

(b) Closed under (external) multiplication: $a \in A, b \in I \Rightarrow ab \in I$.

(c) Satisfies $I \cap \mathbb{C} = 0$ (otherwise $I = A$ is improper).

**Remark.** An ideal in $A$ is **not** a subring of $I$, because $1 \notin I$.
Proposition 9.1. The kernel of a homomorphism \( f : A \to B \):
\[
\ker(f) = \{ a \in A \mid f(a) = 0 \}
\]
is an ideal and conversely, if \( I \subset A \) is a (proper) ideal, then \( A/I \) is a \( \mathbb{C} \)-algebra and \( q : A \to A/I \) is a \( \mathbb{C} \)-homomorphism with kernel \( I \).

Proof. Checking that the kernel is an ideal is straightforward. In the other direction, the interesting operation is coset multiplication:
\[
(a_1 + I)(a_2 + I) = (a_1 a_2 + a_1 I + a_2 I + I^2) = a_1 a_2 + I
\]
because of the closedness properties of an ideal. \( \square \)

There is a bijection of ideals under \( q : A \to A/I \)
\[
q^{-1} : \{ \text{ideals in } A/I \} \to \{ \text{ideals in } A \text{ that contain } I \}
\]
An ideal \( m \subset A \) is maximal if it is contained in no (proper) ideal.

Exercise 9.1. A \( \mathbb{C} \)-algebra with no proper ideal other than 0 is a field. The quotient of a \( \mathbb{C} \)-algebra by a maximal ideal is a field.

There are dueling notions of finite-generatedness for algebras and ideals:

Definition 9.3. (a) A \( \mathbb{C} \)-algebra \( A \) is finitely generated over \( \mathbb{C} \) if there is a surjective \( \mathbb{C} \)-algebra homomorphism from a polynomial ring:
\[
\phi : \mathbb{C}[x_1, ..., x_n] \to A
\]
in which case the images \( y_i = \phi(x_i) \) are generators of \( A \) (over \( \mathbb{C} \)).

(b) An ideal \( I \subset A \) is (finitely) generated by \( b_1, ..., b_n \in I \) if:
\[
I = \langle b_1, ..., b_n \rangle = \{ b_1 a_1 + \cdots + b_n a_n \mid a_1, ..., a_n \in A \}
\]

Let’s quickly relate this to algebraic groups:

Definition 9.4. The algebraic functions on the algebraic groups are:
(a) \( \mathbb{C}[\text{SL}(n, \mathbb{C})] = \mathbb{C}[x_{ij}]/\langle \det(X) - 1 \rangle \).
(b) \( \mathbb{C}[\text{GL}(n, \mathbb{C})] = \mathbb{C}[x_{ij}, y]/\langle y \det(X) - 1 \rangle \)
(c) \( \mathbb{C}[\text{O}(n, \mathbb{C})] = \mathbb{C}[x_{ij}]/\langle XX^T - \text{id} \rangle \)
(d) \( \mathbb{C}[\text{SO}(n, \mathbb{C})] = \mathbb{C}[x_{ij}]/\langle XX^T - \text{id}, \det(X) - 1 \rangle \).
where \( X = (x_{ij}) \) and \( XX^T - \text{id} \) is a system of \( n^2 \) polynomials.

Example 9.2. The algebraic functions on \( \text{SO}(2, \mathbb{C}) \) are:
\[
\mathbb{C}[x, y, z, w]/\langle x^2 + y^2 - 1, xz - yw, z^2 + w^2 - 1, xw - yz - 1 \rangle
\]
Two theorems of Hilbert are crucial:

Basis Theorem: Every ideal in \( \mathbb{C}[x_1, ..., x_n] \) is finitely generated.
Nullstellensatz. The maximal ideals in $\mathbb{C}[x_1, ..., x_n]$ are exactly:

$$m_p = \ker(e_p) = \langle x_1 - p_1, ..., x_n - p_n \rangle$$

for $p = (p_1, ..., p_n) \in \mathbb{C}^n$ where $e_p : \mathbb{C}[x_1, ..., x_n] \to \mathbb{C}$ is “evaluation at the point $p$.”

**Definition 9.5.** Let $A$ be a finitely generated $\mathbb{C}$-algebra. Then:

$$\text{mspec} A = \{ \text{maximal ideals } m \subset A \}$$

is the maximum spectrum of $A$. This is intrinsic to $A$, but if generators:

$$\phi : \mathbb{C}[x_1, ..., x_n] \to A$$

for the $\mathbb{C}$-algebra are chosen, as well as generators:

$$\langle f_1, ..., f_m \rangle = \ker(\phi)$$

for the kernel ideal, then we obtain a bijection:

$$\text{mspec}(A) \leftrightarrow \{ \text{maximal ideals } m_p \in \mathbb{C}[x_1, ..., x_m] \text{ containing } \langle f_1, ..., f_m \rangle \}$$

and the latter is in a natural bijection with the set of points:

$$V(f_1, ..., f_n) = \{ p \in \mathbb{C}^n \mid f_1(p) = f_2(p) = \cdots = f_m(p) = 0 \} \subset \mathbb{C}^n$$

i.e. with the set of solutions to the system of polynomial equations.

Looking back at the algebraic groups, we get natural bijections:

(a) $\text{mspec}(\mathbb{C}[\text{SL}(n, \mathbb{C})]) \leftrightarrow \text{SL}(n, \mathbb{C})$

(c) $\text{mspec}(\mathbb{C}[\text{O}(n, \mathbb{C})]) \leftrightarrow \text{O}(n, \mathbb{C})$

(d) $\text{mspec}(\mathbb{C}[\text{SO}(n, \mathbb{C})]) \leftrightarrow \text{SO}(n, \mathbb{C})$

and finally, (b),

$$\text{mspec}(\mathbb{C}[\text{GL}(n, \mathbb{C})]) \leftrightarrow \{(A, y) \mid \det(A)y = 1\} \subset \mathbb{C}^{n^2+1}$$

which projects bijectively onto the set $\text{GL}(n, \mathbb{C}) = \{ A \mid \det(A) \neq 0 \}$.

**The Zariski Topology** on $\text{mspec}(A)$ is also intrinsic. In this topology, ideals determine subsets of $\text{mspec}(A)$ via:

$$Z(I) = \{ m \mid I \subset m \} \subset \text{mspec}(A)$$

which are, by definition, the **closed sets** in the Zariski topology.

**Exercise 9.2.** Check that this defines a topology on $\text{mspec}(A)$, if we throw in the empty set as a closed set.

**Example 9.3.** This is a strange topology! For example, if $A = \mathbb{C}[x]$, then **every** ideal is principal, generated by $f(x) \in \mathbb{C}[x]$, so:

$$I = \langle f(x) \rangle = \langle c(x - p_1) \cdots (x - p_m) \rangle \Rightarrow Z(I) = \{ p_1, ..., p_m \}$$

and the closed sets are the finite sets. This is very non-Hausdorff.
Exercise 9.3. The complements of hypersurfaces:

$$U_f := \text{mspec}(A) - Z(f)$$ for \( f \in A \)

are a basis for the Zariski topology in the strong sense that every nonempty open set is a finite union of basis open sets.

From the basis theorem, it follows that any decreasing chain:

$$Z(I_1) \supseteq Z(I_2) \supseteq \cdots$$

does not stabilize at:

$$Z(I_n) = Z(I_{n+1}) = \cdots$$

A topological space with this property is called Noetherian.

We want to consider the sheaf of regular functions on \( X = \text{mspec}(A) \). For this it is useful to make the following:

**Restriction.** Suppose \( A \) is a domain (and finitely generated \( \mathbb{C} \)-algebra). Let \( X = \text{mspec}(A) \) with the Zariski topology. Then:

(a) \( X \) is an irreducible Noetherian topological space: if \( X = Z_1 \cup Z_2 \) is a union of closed sets, then either \( Z_1 = X \) or \( Z_2 = X \).

(b) The fraction field \( \mathbb{C}(A) \) of \( A \) defines the rational functions on \( X \).

(c) An element \( f \in A \) is evaluated at a point \( x \in X \) by:

$$f(x) := f \ (\text{mod } m_x) \in \mathbb{C}$$

where \( m_x \) is the maximal ideal corresponding to \( x \).

(d) Every element \( \phi \in \mathbb{C}(A) \) has an open domain of definition

$$U_\phi = \{ x \in X \mid \phi = \frac{f}{g} \text{ with } g(x) \neq 0 \}$$

(e) A sheaf of regular functions \( \mathcal{O}_X \) is defined by setting:

$$\mathcal{O}_X(U) = \{ \phi \in \mathbb{C}(X) \mid U \subseteq U_\phi \}$$

**Proposition 9.2.** The global regular functions \( \mathcal{O}_X(X) \) are all in \( A \).

**Proof.** Clearly \( A \subset \mathcal{O}_X(X) \). Conversely, if \( \phi \in \mathcal{O}_X(X) \subset \mathbb{C}(X) \), consider the ideal \( I = \{ g \mid g\phi \in A \} \) of denominators of \( \phi \). For all \( x \in X \), there is a \( g \in I \) such that \( g \notin m_x \). Thus, \( I \) is not in any maximal ideal of \( A \) (Nullstellensatz!), so \( 1 \in I \) and \( \phi \in A \).

**Remark.** As a variation on Proposition 9.2, one can show:

$$\mathcal{O}_X(U_f) = A[f^{-1}]$$

if \( f \in A \) is any non-zero element of \( A \).
Example 9.4. Consider the example of \( A = \mathbb{C}[x, y] \). Then:
\[
X = \mathbb{C}^2, \quad Z((f)) = \{ p \in \mathbb{C}^2 \mid f(p) = 0 \}
\]
as plane curves, the complements of which satisfy:
\[
\mathcal{O}_{\mathbb{C}^2}(U_f) = \mathbb{C}[x, y, f^{-1}]
\]
but one can easily show that \( \mathcal{O}_{\mathbb{C}^2}(\mathbb{C}^2 - \{0\}) = \mathbb{C}[x, y] \), so that somehow the regular functions on \( \mathbb{C}^2 - \{0\} \) do not “see” the missing point.

Definition 9.6. The data consisting of:

(a) An irreducible Noetherian topological space \( X = \text{mspec}(A) \)

(b) The sheaf of regular functions \( \mathcal{O}_X(U) \) with \( \mathcal{O}_X(X) = A \)

is the **affine variety** (of finite type over \( \mathbb{C} \)) associated to \( A \).

Definition 9.7. A morphism of affine varieties (over \( \mathbb{C} \)) is:

(i) A continuous map \( f : X \to Y \) (Zariski topologies!) such that:

(ii) For all \( U \subset Y \), the function pull-back \( f^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U)) \)
maps regular functions to regular functions.

Remark. The global pull-back \( f^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X) \) is a \( \mathbb{C} \)-algebra homomorphism \( h : B \to A \) of the underlying domains, and conversely, a homomorphism \( h : B \to A \) of \( \mathbb{C} \)-algebras determines \( f : X \to Y \) defined by \( f(m_x) = h^{-1}(m_x) \), as well as the map on sheaves.

Moment of Zen: The category of affine varieties \( (X, \mathcal{O}_X) \) of finite type over \( \mathbb{C} \) is equivalent (contravariantly, i.e. with arrows reversed) to the category of finitely generated \( \mathbb{C} \)-algebra domains \( A \).

Given this equivalence, it is natural to ask of affine varieties:

Q1. What morphisms are associated to surjective maps of domains?

Q2. What morphisms are associated to injective maps of domains?

The first question is rather easy, because:

A surjective map of domains factors through the quotient:
\[
h : B \to B/\ker(h) = A
\]
where \( P = \ker(h) \) is the prime ideal kernel of the homomorphism \( h \).

When we reverse arrows, this is an isomorphism of affine varieties between \( \text{mspec}(A) = X \) and the closed subvariety \( \text{mspec}(B/P) \subset Y \) which is “an affine variety” structure on the irreducible closed set \( Z(P) \). Such a map of affine varieties is called a **closed embedding**.
As an example of the second question, consider the inclusion:

\[ B \subset A = B[g^{-1}] \text{ for some non-unit } g \in A \]

Then the arrow-reversed map of affine varieties:

\[ f : \text{mspec}(B[g^{-1}]) \to \text{mspec}(B) = Y \]

is an isomorphism from \( X \) to the basic open set \( U_g \subset Y \). This map is called an **open embedding**.

But for a *general* inclusion of rings, the answer is more interesting!

**Example 9.4.** (a) Projections. Let \( B \subset A = B[x_1, \ldots, x_n] \). Then:

\[ f : \text{mspec}(A) \to \text{mspec}(B) = Y \]

is the **projection** \( p : Y \times \mathbb{C}^n \to Y \).

(b) Blow-downs. Let \( h : \mathbb{C}[x, xy] \subset \mathbb{C}[x, y] \) be the inclusion of one polynomial ring in another. Then:

\[ f : \mathbb{C}^2 \to \mathbb{C}^2 \text{ is the map } f(a,b) = (a, ab) \]

whose image is neither open nor closed. It is the complement \( U_y \subset \mathbb{C}^2 \) of the \( y \)-axis together with the origin. Notice that \( f^{-1}(0,0) = \{(0,b)\} \) while \( f^{-1}(a,c) = \{(a/c, c)\} \) is a single point otherwise. For this reason, we say that \( f \) **blows down** the \( y \)-axis onto the origin.

We want to define an abstract variety (of finite type over \( \mathbb{C} \)) to be an irreducible topological space with a sheaf of regular functions on it that is locally affine and globally Hausdorff, as we did for manifolds, but we have already mentioned a problem with the Hausdorff property for the Zariski topology. We need a categorical work-around.

**Products** exist in the category of affine varieties. The contravariant functor above tells us this is equivalent to the existence of **coproducts** in the category of \( \mathbb{C} \)-algebra domains. These are furnished by the tensor product of \( \mathbb{C} \)-algebras over \( \mathbb{C} \):

\[ A \otimes \mathbb{C} B \text{ together with } a : A \otimes 1 \to A \otimes B, b : 1 \otimes B \to A \otimes B \]

is a coproduct (the only hard thing is to prove that it is a domain!) Via our contravariant functor, this means that:

\[ \text{mspec}(A \otimes B), \text{ with the maps } a^* \text{ and } b^* \text{ is a product!} \]

but this product of varieties does **not** have the product topology! For example, \( \mathbb{C}^2 = \mathbb{C}^1 \times \mathbb{C}^1 \) via this construction, but its Zariski topology has many more closed sets in it (plane curves!) than does the product of the cofinite topologies.
Observation. The diagonal mapping:

\[ X \rightarrow X \times X \]

is the closed embedding corresponding to the surjective map:

\[ A \otimes A \rightarrow A; \; a_1 \otimes a_2 \mapsto a_1 \cdot a_2 \]

converting tensor product to ordinary product. The kernel is:

\[ \langle a_i \otimes 1 - 1 \otimes a_i \mid a_i \in A \text{ generate } A \text{ as an algebra!} \rangle \]

which in particular proves that the diagonal in \( X \times X \) is **closed**.

Looking back at the section on topology, we noticed (Exercise 6.2 (b)) that Hausdorff was equivalent to the diagonal being closed provided that products had the product topology. Here they don’t, so we go with the categorical definition instead.

**Definition 9.8.** An object \( X \) in a category of topological spaces with products is **separated** if the diagonal \( X \rightarrow X \times X \) is a homeomorphism onto a closed subset \( \Delta \subset X \times X \).

**Remark.** Here the product doesn’t have the product topology as a topological space. Amusingly, in Grothendieck’s theory of schemes, the product of schemes isn’t even the Cartesian product as a set!

By analogy with manifolds, we now try to make the following:

**Definition 9.9.** A **variety** is a Noetherian topological space \( X \) together with a sheaf of regular functions \( \mathcal{O}_X \) so that the pair \((X, \mathcal{O}_X)\) is locally affine and globally separated.

But there is a problem here! What is the product of \( X \) with itself? We did it for affine varieties by appealing to the tensor product, but we have enlarged our category to include pairs \((X, \mathcal{O}_X)\) that are locally affine. Well, it can be done, even in Grothendieck’s wild category of locally affine schemes. I refer you to Hartshorne’s book or you can think about it as an ambitious exercise.
Exercises.

9.1. (a) A $\mathbb{C}$-algebra with no proper ideal other than 0 is a field.
(b) The quotient of a $\mathbb{C}$-algebra by a maximal ideal is a field.

9.2. (a) Check that declaring $Z(I)$ to be closed if $I \subset A$ is an ideal defines a topology on $\text{mspec}(A)$ (after you throw in the empty set). This is the Zariski topology.
(b) Check that $I(Z(I))$ contains the radical of $I$, defined by:
$$\sqrt{I} = \{ f \in A \mid f^r \in I \text{ for some } r > 0 \}$$
and also check that $\sqrt{I}$ is, in fact, an ideal.

Remark. It is a consequence of the Nullstellensatz that $\sqrt{I} = I(Z(I))$.

9.3. Check that the complements of hypersurfaces:
$$U_f := \text{mspec}(A) - Z(f)$$
for $f \in A$
are a basis for the Zariski topology in the strong sense that every nonempty open set is a finite union of basis open sets.

9.4. (Ambitious) Create a category of locally affine spaces with sheaves and prove that products exist in your category, so that the notion of being separated makes sense!