

Summer High School 2009

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3. Induction and Related Stuff.

Let's think for a bit about the following two familiar equations:

Triangle Number Equation

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Square Number Equation:

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

What do they mean? n is clearly meant to represent a natural number, and the equations are clearly intended to be true *for all values of* n . In other words, they are each not one, but an infinite number of equations. We can easily check as many of them as we like by hand, but evidently we cannot check an infinite number of equations by hand!

How do you prove them? Actually, there are several ways to do this.

Geometric Proofs:

(a) Triangle Number Equation: $1 + 2 + 3 + \cdots + n$ counts points arranged in a right triangle with n rows, which can be viewed in either one of the following ways:

$$\begin{array}{cccc} * & & & * \cdots * * \\ * & * & & \vdots \\ * & * & * & \vdots \\ & \vdots & & * * * \\ * & * & \cdots & * \end{array} \quad \text{or} \quad \begin{array}{cccc} * & \cdots & * & * \\ & & \vdots & \\ & & \vdots & \\ & * & * & * \\ & & * & * \\ & & & * \end{array}$$

(depending upon whether you start counting at the top or the bottom).

The two triangles fit together to form an $(n+1) \times n$ rectangle, which has $n(n+1)$ points. So each triangle has $n(n+1)/2$ points.

A way of doing this (attributed to Gauss at a very young age) without pictures is to write the sum twice, reversing the order the second time:

$$\begin{array}{ccccccccc} 1 & + & 2 & + & 3 & + & \cdots & + & n \\ n & + & (n-1) & + & (n-2) & + & \cdots & + & 1 \end{array}$$

When you add them first vertically then horizontally, you get $n(n+1)$. When you add them first horizontally, then vertically, you get 2 times the triangle number.

(b) Square Number Equation: $1 + 3 + 5 + \cdots + (2n - 1)$ counts the points of an isosceles triangle:

$$\begin{array}{ccccccc}
 & & & & * & & \\
 & & & * & * & * & \\
 & & * & * & * & * & * \\
 & & & & \vdots & & \\
 * & * & * & * & \cdots & * & * & * & *
 \end{array}$$

We can slice the triangle vertically to get two right triangles:

$$\begin{array}{ccccccc}
 & & & & * & & \\
 & & & * & * & & * \\
 & & * & * & * & \text{and} & * & * \\
 & & & & \vdots & & \vdots & \\
 * & * & \cdots & * & * & & * & * & \cdots & *
 \end{array}$$

The two triangles fit together to give an $n \times n$ square!

Alternatively, the number of points of the two right triangles are two consecutive triangle numbers (for n and $n - 1$). So they have:

$$\frac{n(n+1)}{2} + \frac{(n-1)(n-1+1)}{2} = \frac{1}{2} (n^2 + n + n^2 - n) = n^2$$

total points.

The no-picture version. Break apart each odd number as the sum of consecutive natural numbers and regroup:

$$\begin{aligned}
 1 + 3 + 5 + \cdots + (2n - 1) &= \\
 (1 + 0) + (2 + 1) + (3 + 2) + \cdots + (n + (n - 1)) &= \\
 (1 + 2 + 3 + \cdots + n) + (1 + 2 + \cdots + (n - 1)) &
 \end{aligned}$$

to get the same result.

Geometric proofs are nice, since they somehow explain “why” the equations are true, which is something proofs don’t always accomplish! However, they are not always available. Proofs by induction, on the other hand, are useful for a wide variety of problems.

Definition 3.1. A sequence of infinitely many equations (statements) $E(n)$ are all simultaneously *proved by induction* once:

- (i) The initial ($n = 1$) equation (statement) is proved, and
- (ii) The following implication is proved for an arbitrary n :

Assuming $E(n)$ is true, then $E(n + 1)$ is also true

Why is this a proof? A proof by induction proves that the set of natural numbers n such that $E(n)$ is *false* can have no minimal element because (i) says $E(1)$ is true, and (ii) says that if $E(n)$ were false, then $E(n-1)$ would also be false. So $E(n)$ is **never false** by well-ordering.

Examples:

(a) Proof by Induction of the Triangle Number Equations.

$$E(n) \text{ is the equation } 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} = \frac{n^2 + n}{2}$$

(i) $E(1)$ is the equation $1 = (1^2 + 1)/2$, which is true.

(ii) Assuming that $E(n)$ is true, we have:

$$1 + 2 + \cdots + n = \frac{n^2 + n}{2} \text{ and, adding } n + 1 \text{ to both sides,}$$

$$(1 + 2 + \cdots + n) + (n + 1) = \frac{n^2 + n}{2} + (n + 1), \text{ and}$$

$$\frac{n^2 + n}{2} + (n + 1) = \frac{n^2 + n + (2n + 2)}{2} = \frac{(n + 2)(n + 1)}{2}$$

so that $E(n + 1)$ is also true. This completes the proof by induction!

(b) Proof by Induction of the Square Number Equations.

$$E(n) \text{ is the equation } 1 + 3 + \cdots + (2n - 1) = n^2$$

(i) $E(1)$ is the equation: $1 = 1^2$, which is true.

(ii) Assuming that $E(n)$ is true, we have:

$$1 + 3 + \cdots + (2n - 1) = n^2 \text{ and, adding } 2(n + 1) - 1 = 2n + 1,$$

$$(1 + 3 + \cdots + (2n - 1)) + (2(n + 1) - 1) = n^2 + (2n + 1) = (n + 1)^2$$

so that $E(n + 1)$ is also true. This completes the proof by induction.

We next want to get to the binomial theorem. But first, consider:

Definition 3.2 The product of the numbers from 1 to n is written:

$$n! := 1 \cdot 2 \cdot 3 \cdots n$$

and it is called n *factorial*.

Definition/Proposition 3.3. If $k < n$ are natural numbers, then:

$$\binom{n}{k} := \frac{n!}{k!(n - k)!}$$

is the number of ways of choosing k different objects from n objects. We call this “ n choose k ”, and notice that it is equal to n choose $n - k$.

Proof (of the counting statement): The first object you choose can be any one of the n , the second can be any of the $n - 1$ remaining, etc. up to the k th, so there are:

$$n(n - 1) \cdots (n - k + 1)$$

ways of choosing k objects *in a particular order*. On the other hand, $k!$ is the number of ways of ordering k objects, so the total number of ways of choosing k objects out of n (when order doesn't matter) is:

$$\frac{n(n - 1) \cdots (n - k + 1)}{k!} = \frac{n!}{k!(n - k)!}$$

since $n(n - 1) \cdots (n - k + 1) = n!/(n - k)!$

A Minor Addition: By convention, we set $0! = 1$, and in particular:

$$\binom{n}{0} = 1 = \binom{n}{n}$$

makes sense. There is one way to choose 0 objects (or n) from n .

Pascal's Identity: For all natural numbers $k < n$:

$$\binom{n}{k - 1} + \binom{n}{k} = \binom{n + 1}{k}$$

Remark: This is an infinite number of equations in “two directions” since there is one equation for each value of n and each value of $k < n$. One could write $E(k, n)$ to stand for each equation, and try to prove the whole mess by induction (it can be done!). Fortunately, there is a really “elegant” proof available:

Proof: Start with n red objects and one green object.

- There are $\binom{n+1}{k}$ ways of choosing k objects in all. But:
- The k objects may all be red (in $\binom{n}{k}$ ways), or else
- One of them is green, and $k - 1$ are red (in $\binom{n}{k-1}$ ways).

QED!

Pascal's Volcano: Start with a shell of 1s in the following shape:

$$\begin{array}{ccccccc}
 & & & & 1 & & 1 \\
 & & & & & & \\
 & & & 1 & & & 1 \\
 & & & & & & \\
 & & 1 & & & & 1 \\
 & & & & & & \\
 & 1 & & & & & 1 \\
 & & & & & & \\
 1 & & & & & & 1
 \end{array}$$

Fill in the volcano by adding pairs of adjacent numbers to produce the number beneath (and between) them:

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & 1 & & 2 & & 1 \\
 & & & & & & \\
 & & & & 1 & & 2 & & 1 \\
 & & & & 1 & & 3 & & 3 & & 1 \\
 & & & & & & & & & & \\
 & & & & & & & & 1 & & 1 \\
 & & & & & & & & 1 & & 2 & & 1 \\
 & & & & & & & & 1 & & 3 & & 3 & & 1 \\
 & & & & & & & & 1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & & & & & & & & & & & & & \\
 & & & & & & & & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1
 \end{array}$$

etc.

Pascal's Theorem: The n th row of Pascal's Volcano is:

$$1 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + \binom{n}{k-1} - \binom{n}{k} + \cdots + \binom{n}{n} = 1$$

Proof by Induction:

(i) The first row is $1 = \binom{1}{0} - \binom{1}{1} = 1$.

(ii) Assume that the n th row of the volcano is:

$$1 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + \binom{n}{n} = 1$$

Then the $n + 1$ st row (adding adjacent numbers in the n th row) is:

$$1 - \binom{n}{0} + \binom{n}{1} - \binom{n}{1} + \binom{n}{2} - \cdots + \binom{n}{k-1} + \binom{n}{k} - \cdots + 1$$

(the 1s on the left and right come from the shell). By Pascal's Identity, this is exactly the $n + 1$ st row that we want. QED.

Finally, we have the

Binomial Theorem:

$$(x + y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + y^n$$

That is, the coefficients of $(x + y)^n$ are the n th row of Pascal's volcano.

Proof: Also, by induction.

(i) The first equation is: $(x + y)^1 = x + y$. Duh!

(ii) Assume the coefficients of $(x + y)^n$ are the n th row of Pascal's volcano (this is the n th equation). Then:

$$\begin{aligned} (x + y)^{n+1} &= (x + y)^n(x + y) = x(x + y)^n + y(x + y)^n = \\ x^{n+1} &+ \binom{n}{1}x^ny + \binom{n}{2}x^{n-1}y^2 + \cdots + xy^n + \\ &x^ny + \binom{n}{1}x^{n-1}y^2 + \cdots + \binom{n}{n-1}xy^n + y^{n+1} \end{aligned}$$

and combining coefficients vertically, we see that as in Pascal's volcano, the adjacent coefficients of $(x + y)^n$ are added to get the coefficients of $(x + y)^{n+1}$. Thus the coefficients match with Pascal's volcano. $\square$

We may now read off the first few examples from Pascal:

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$