Moduli Spaces, Cortona, August 2015
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5. Reider's Theorem. We start with an observation of Mumford:

For an ample line bundle L (of first chern class $c_1(L) = H$) on a smooth projective (complex) surface S, we may deduce:

$$\mathrm{H}^1(S, K_S + L) = 0$$

from the two inequalities:

Hodge Index. For any divisor class D,

$$(H \cdot H)(D \cdot D) \leq (H \cdot D)^2$$

Bogomolov Inequality. For any semi-stable bundle E on S,

$$c_1^2(E) \geq 2\mathrm{ch}_0(E)\mathrm{ch}_2(E)$$

Proof. By Serre duality, we have:

$$\mathrm{H}^1(S, K_S + L) = \mathrm{Ext}^1(\mathcal{O}_S, K_S + L) = \mathrm{Ext}^1(L, \mathcal{O}_S)^*$$

so the desired vanishing is equivalent to showing that each sequence:

$$(*)\ 0 \to \mathcal{O}_S \to E \to L \to 0$$

splits, i.e. that the extension class ϵ for $(*)$ is zero.

Step 1. The vector bundle E in $(*)$ is unstable by the computation:

$$\mathrm{ch}_0(E) = 2,\ c_1(E) = H \text{ and } \mathrm{ch}_2(E) = \frac{H^2}{2}$$

from which it follows that:

$$c_1^2(E) = H^2 < 2H^2 = 2\mathrm{ch}_0(E)\mathrm{ch}_2(E)$$

violates the Bogomolov inequality. This forces the bundle E to fit into a destabilizing exact sequence:

$$0 \to A \to E \to B \to 0$$

with B a rank one torsion-free sheaf and A invertible.

Step 2. Either the induced map $f : A \to E \to L$ is an isomorphism, splitting the sequence, or else there is an effective curve $C \subset S$ such that:

$$A = L(-C), \text{ and } B = \mathcal{O}_S(C) \otimes \mathcal{I}_W$$

from which we deduce the inequalities:

(a) $1 \leq H \cdot C < H^2/2$ (because C is effective and B destabilizes E).

(b) $\mathrm{ch}_2(E) = \mathrm{ch}_2(A) + \mathrm{ch}_2(B) \leq (H - C)^2/2 + C^2/2$, so $H \cdot C \leq C^2$.

which we multiply together to get:

$$(H \cdot C)^2 < \frac{H^2 C^2}{2}$$

violating the Hodge inequality. $\square$

Reider's idea is to apply the same strategy to:

$$\mathrm{H}^1(S, K_S \otimes L \otimes \mathcal{I}_Z)$$

where $Z \subset S$ has length d, the vanishing of which (for all Z) is the d-very-ample condition for the line bundle $K_S + L$.

Once again, we use Serre duality:

$$\mathrm{H}^1(S, K_S \otimes L \otimes \mathcal{I}_Z) = \mathrm{Ext}^1(L \otimes \mathcal{I}_Z, \mathcal{O}_S)^*$$

so we desire to prove that the sequences:

$$(*)\ 0 \to \mathcal{O}_S \to E \to L \otimes \mathcal{I}_Z \to 0$$

all split. In fact, by induction (if we assume that all sequences $(*)$ split for all schemes Z' of length $d' < d$), we may assume that E is a vector bundle. Otherwise, we'd have a similar sequence for E^{**}:

$$(**)\ 0 \to \mathcal{O}_S \to E^{**} \to L \otimes \mathcal{I}_{Z'} \to 0$$

that would split, inducing a splitting of $(*)$.

Step 1. E is unstable if the Bogomolov inequality is violated:

$$c_1^2(E) = H^2 < 2\mathrm{ch}_0(E)\mathrm{ch}_2(E) = 4\left(\frac{H^2}{2} - d\right)$$

i.e. if:

$$H^2 > 4d$$

so we assume first that this is the case, and:

$$0 \to A \to E \to B \to 0$$

is the destabilizing exact sequence.

Step 2. If the induced $f : A \to Z \otimes \mathcal{I}_Z$ is not an isomorphism, then:

$$A = L(-C) \text{ and } B = \mathcal{O}_S(C) \otimes \mathcal{I}_W$$

as before, **and** $Z \subset C$ is a subscheme of the curve C. In this case:

(a) $1 \le H \cdot C < H^2/2$ as before, but:

(b) $H \cdot C \le C^2 + d$, and:

(c) $H^2 C^2 \le (H \cdot C)^2$ by Hodge Index.

Thus, in this case we obtain:

$$1 - d \le C^2 \le d \text{ and } H \cdot C \le C^2 + d$$

The nonexistence of such curves plus $H^2 > 4d$ implies the vanishing.

Examples. ($d = 1$) $K_S + L$ is base point free if $H^2 > 4$ and there are no effective curves $C \subset S$ satisfying:

$$C^2 = 0 \text{ and } H \cdot C = 1 \quad \text{or} \quad C^2 = 1 \text{ and } H \cdot C = 2$$

In particular, if $L = nA$ and $n \geq 3$, then $K_S + L$ is base point free.

($d = 2$) $K_S + L$ is very ample if $H^2 > 8$ and there are no effective curves $C \subset S$ satisfying:

$$C^2 = -1, H \cdot C = 1 \quad \text{or} \quad \dots \text{ or} \quad C^2 = 2, H \cdot C = 4$$

but the last case is out, because it (together with $H^2 > 8$) violates (c). Thus, in particular, if $L = nA$ and $n \geq 4$, then $K + L$ is very ample, which is optimal (as is (a)), considering the example of $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$.

A related computation is relevant to stability conditions on S.

Let $\alpha = \alpha_0 + \alpha_1 + \alpha_2 \in \mathrm{H}^0(S, \mathbb{Q}) \oplus \mathrm{NS}(S)_{\mathbb{Q}} \oplus \mathrm{H}^4(S, \mathbb{Q})$ satisfy:

$$(\dagger) \quad \alpha_0 > 0 \text{ and } \alpha_1^2 > 2\alpha_0\alpha_2$$

(the strict Bogomolov inequality) and let $H \in \mathrm{NS}(S)$ be an ample class.

Then:

$$\deg(\mathrm{ch}(E) \cdot \alpha \cdot H) = 0 \Rightarrow \deg(\mathrm{ch}(E) \cdot \alpha) < 0$$

for all (Mumford) semi-stable torsion-free sheaves on S.

Proof: The former equality gives:

$$\alpha_0 c_1(E) \cdot H + \mathrm{ch}_0(E)\alpha_1 \cdot H = 0 \text{ and}$$
$$H \cdot (\alpha_0 c_1(E) + \mathrm{ch}_0(E)\alpha_1) = 0$$

so we may conclude from the Hodge index theorem that:

$$(\alpha_0 c_1(E) + \mathrm{ch}_0(E)\alpha_1)^2 \leq 0$$

On the other hand, the second expression is:

$$\alpha_0(E)\mathrm{ch}_2(E) + \alpha_1 \cdot c_1(E) + \mathrm{ch}_0(E)\alpha_2 <$$
$$\alpha_0 \left(\frac{c_1^2(E)}{2\mathrm{ch}_0(E)} \right) + \alpha_1 \cdot c_1(E) + \mathrm{ch}_0(E) \left(\frac{\alpha_1^2}{2\alpha_0} \right) =$$
$$\frac{1}{2\alpha_0 \mathrm{ch}_0(E)} (\alpha_0 c_1(E) + \mathrm{ch}_0(E)\alpha_1)^2 \leq 0$$

giving the desired inequality.

This is enough to show that the pair $(Z, \mathcal{A})$ consisting of:

(i) The "central charge"

$$Z(E) = \big(-\deg(\mathrm{ch}(E) \cdot \alpha) + \sqrt{-1}\deg(\mathrm{ch}(E) \cdot \alpha \cdot H)\big)$$

(ii) The “tilt” $\mathcal{A}$ of $Coh(S)$ with respect to the torsion pair:

- $\mathcal{T}$ generated by torsion and semi-stable E with $\deg(\mathrm{ch}(E)\alpha H) > 0$
- $\mathcal{F}$ generated by torsion-free semi-stable E with $\deg(\mathrm{ch}(E)\alpha H) \leq 0$

satisfies the upper-half plane condition for objects of $\mathcal{A}$.