Mathematics 7800– Quantum Kitchen Sink – Spring 2002

4. Quotients via GIT. Most interesting moduli spaces arise as quotients of schemes by group actions. We will first analyze such quotients with geometric invariant theory, then later consider "stack-theoretic" quotients.

Affine GIT: When a linearly reductive (affine!) linear group G acts on an affine scheme X of finite type over $\mathbf{C}$, then the sub-algebra of G-invariant regular functions:

$$\mathbf{C}[X]^G \subset \mathbf{C}[X]$$

is finitely generated, and the resulting morphism of affine schemes:

$$f : X = \mathrm{Spec}(\mathbf{C}[X]) \to \ \mathrm{Spec}(\mathbf{C}[X]^G) := X^G$$

has the following properties:

(a) f is surjective.

(b) $f(x) = f(x')$ if and only if the closures of their orbits intersect.

(c) There is a unique closed orbit $O(x)$ in each fiber $f^{-1}(y)$.

A closed point $x \in X$ is **stable** if its orbit is closed and its stabilizer is finite.

(d) The subset $X_S \subset X$ of stable points is open, its image $X_S^G \subset X^G$ is open, and $f_S : X_S \to X_S^G$ has the following "nice" properties:

(i) Each fiber of f_S is a single orbit.

(ii) Each G-invariant open $U \subset X_S$ is $f_S^{-1}(V)$ for an open $V \subset X_S^G$.

(iii) For U, V as in (ii), $\Gamma(V, \mathcal{O}_{X^G}) = \Gamma(U, \mathcal{O}_X)^G \subset \Gamma(U, \mathcal{O}_X)$

Remark: Any surjective map $f : X \to Y$ that is constant on orbits and satisfies (i)-(iii) above is called a **geometric quotient**.

Examples: (a) The (left) action of $\mathrm{SL}(W)$ on $\mathbf{A}^{mn} = \mathrm{Hom}(W, V)$ leaves invariant the subring

$$\mathbf{C}[\det(x_{i,j_l})] \subset \mathbf{C}[x_{ij}]$$

for multi-indices $J = (j_1, ..., j_m)$ as in §2. The injective homomorphisms have closed orbits with trivial stabilizers and all others have 0 in their orbit closure. The quotient is the affine cone over the Grassmannian of §2.

(b) The (conjugation) action of $\mathrm{SL}(W)$ on $\mathbf{A}^{m^2} = \mathrm{Hom}(W, W)$ has

$$\mathbf{C}[\mathrm{tr} = f_{m-1}, ..., \det = f_0] \subset \mathbf{C}[x_{ij}]$$

(the polynomial ring in the coefficients of the characteristic polynomial) as its ring of invariants. There are no stable points, because all matrices have positive-dimensional stabilizers, but the orbits of diagonalizable matrices are closed. Any orbit of a matrix with off-diagonal entries in Jordan form is not closed. The semi-simple matrices (distinct eigenvalues) have the smallest stabilizers. Question: Is $f : \{\text{semisimple matrices}\} \to \mathbf{A}^m - \{\mathrm{disc} = 0\}$ nevertheless a geometric quotient?

(c) Does the action of $\mathrm{SL}(W)$ on $\mathrm{Sym}^d(W^*)$ ($d > 2$) have stable points? If so, which homogeneous polynomials are stable? What about the case $m = 2$?

If $\sigma : G \times X \to X$ is the action and $x \in X$, let $\sigma_x = \sigma(*, x) : G \to X$.

Lemma 4.1: x is stable if and only if $\sigma_x : G \to X$ is proper.

Proof: If σ_x is proper, then $O(x)$ is closed, and the stabilizer $G_x = \sigma_x^{-1}(x)$ is complete and affine (G is affine!), hence finite. On the other hand, if $O(x)$ is closed, then to prove properness, it suffices to show that $\sigma_x : G \to O(x)$ is a finite morphism. Since the fibers are finite, the restriction of σ_x to $\sigma_x^{-1}(U)$ for some non-empty open subset $U \subset O(x)$ is finite (Hartshorne exercise!). Translation of U by elements of G then shows that σ_x is finite everywhere.

Proof of Affine GIT:

Step 1: $\mathbf{C}[X]^G$ is a finitely generated algebra.

Consider the action of G on the vector space $\mathbf{C}[X]$. If:

$$\sigma^* : \mathbf{C}[X] \to \mathbf{C}[G] \otimes \mathbf{C}[X]$$

is the $\mathbf{C}$-algebra homomorphism corresponding to σ, then the induced action on functions: $\overline{\sigma} : G \times \mathbf{C}[X] \to \mathbf{C}[X]$ is given by:

$$\overline{\sigma}(g, r) = \sum_{i=1}^{n} s_i(g) r_i \quad \text{where} \quad \sigma^*(r) = \sum_{i=1}^{n} s_i \otimes r_i$$

In particular, this is a **finite** sum, so each orbit $O(r) = Gr \subset \langle r_1, ..., r_n \rangle$ is contained in a finite-dimension subspace of $\mathbf{C}[X]$, and then the linear span of $O(r)$ is finite-dimensional, and evidently G-invariant. The actions of G on infinite-dimensional vector spaces with this property are called **rational**.

Linearly reductive means that for each finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$ and each invariant subspace $W \subset V$, there is a (unique) complementary invariant subspace $W' \subset V$ with $V = W \oplus W'$. We apply this to the rational action of G on $\mathbf{C}[X]$ to define the **Reynolds operator**

$$E : \mathbf{C}[X] \to \mathbf{C}[X]^G$$

which is the linear projection onto the subspace invariant functions.

Definition of E: If V is any vector space on which G acts rationally, and if $v \in V$ is not invariant, let W be a finite-dimensional invariant subspace containing v (by rationality) and then decompose $W = W^G \oplus W_G$ as the sum of the subspace of invariant vectors and the complementary invariant subspace W_G (using the linear reductivity of G). Then $v \notin W^G$, so W_G is nonempty, invariant and $W_G \cap V^G = 0$. Apply Zorn's lemma to the set of invariant subspaces $T \subset V$ with $T \cap V^G = 0$. Let V_G be a maximal such. If $V^G + V_G \neq V$, choose v in the complement, let W be finite dimensional an invariant containing v, and let W' be the invariant complement of the invariant subspace $W \cap (V_G + V^G) \subset W$. Then the span of V_G and W' would give a larger invariant subspace T in our set, violating maximality. Similarly, one shows that V_G contains every T in the set, hence is uniquely determined. The Reynolds operator E is now uniquely defined by the property that it has V_G as its kernel and projects onto V^G.

Properties of E:

(a) The Reynolds operators commute with G-linear maps $u : V \to V'$ of vector spaces on which G acts rationally.

(b) If $u : V \to V'$ in (a) is surjective, then $u^G : V^G \to V'^G$ is surjective.

(c) The Reynolds operator for $\mathbf{C}[X]$ satisfies the "Reynolds identity":

$$E(xy) = xE(y)$$

for all $x \in \mathbf{C}[X]^G$ and $y \in \mathbf{C}[X]$

(d) If I_λ is a family of invariant ideals in $\mathbf{C}[X]$, then

$$(\sum_\lambda I_\lambda) \cap \mathbf{C}[X]^G = \sum_\lambda (I_\lambda \cap \mathbf{C}[X]^G)$$

Proofs: Let E' be the Reynolds operator for V'. To show that $E' \circ u = u \circ E$, it suffices to show that $u(V^G) \subset (V')^G$ and $u(V_G) \subset (V')_G$. The first inclusion is obvious. For the second, suppose that $v \in V_G$, and let W be a finite-dimensional invariant subspace of V_G containing v. Then $W \cap \ker(u) \subset W$ is invariant, so we can take its invariant complement $W' \subset W$. But then u maps W' isomorphically onto $u(W') = u(W)$, hence $u(W)$ is invariant and $u(W) \cap (V')^G = 0$, so $u(W) \subset (V')_G$, so $u(v) \in (V')_G$.

As for (b), just apply (a)(!)

$$(V')^G = E'(V') = E'(u(V)) = u(E(V)) = u(V^G)$$

For (c), apply (a) to the G-linear endomorphism $y \mapsto xy$ of $\mathbf{C}[X]$.

Finally, apply (a) to the G-linear inclusions $I_\lambda \subset \mathbf{C}[X]$ of spaces with rational G-actions. Then $E(f) \in I_\lambda \cap \mathbf{C}[X]^G$ for each $f \in I_\lambda$, so if $f = \sum f_\lambda \in (\sum I_\lambda) \cap \mathbf{C}[X]^G$ (a finite sum!), then $f = E(f) = \sum_\lambda E(f_\lambda) \in \sum_\lambda (I_\lambda \cap \mathbf{C}[X]^G)$.

Proof of Step 1: Let $f_1, ..., f_r$ be generators of $\mathbf{C}[X]$, and let V be a finite-dimensional invariant subspace containing them (by rationality!). Then for the induced action of G on $\mathbf{C}[x_1, ..., x_n] = \mathrm{Sym}^*(V)$, the surjective map

$$u : \mathbf{C}[x_1, ..., x_n] \to \mathbf{C}[X]$$

is G-linear, so by (b) the induced map $u^G : \mathbf{C}[x_1, ..., x_n]^G \to \mathbf{C}[X]^G$ is also surjective. Thus it suffices to show that $\mathbf{C}[x_1, ..., x_n]^G$ is finitely generated.

Since the action of G on $\mathbf{C}[x_1, ..., x_n]$ preserves degree, $\mathbf{C}[x_1, ..., x_n]^G$ is graded, too. Let $I \subset \mathbf{C}[x_1, ..., x_n]$ be the ideal generated by all positive-degree invariant polynomials. Then I is a finitely generated homogeneous ideal and generators $F_1, ..., F_m \in I$ may be chosen homogeneous and invariant.

We claim that $1, F_1, ..., F_m$ generate $\mathbf{C}[x_1, ..., x_n]^G$ as an algebra. Indeed, by induction, we may assume that $1, F_1, ..., F_m$ generate $\mathbf{C}[x_1, ..., x_n]^G$ in degrees less than d. If P is homogeneous of degree d and invariant, then $P \in I$, so we can write $P = \sum Q_i F_i$ for $Q_i \in \mathbf{C}[x_1, ..., x_n]$, and by (c), we then have

$$P = E(P) = \sum E(Q_i) F_i$$

Since the degrees of the $E(Q_i)$ are all smaller than d, they are in the algebra generated by $1, F_1, ..., F_m$, and we are done.

Step 2: The rest of the proof of Affine GIT. We start with a geometric version of property (d) of the Reynolds operator. If $(Z_\lambda) \subset X$ is a family of closed, invariant subsets, then:

$$\overline{f(\cap Z_\lambda)} = \cap \overline{f(Z_\lambda)}.$$

Now suppose $Z \subset X$ is closed and invariant and $y \in \overline{f(Z)}$ is a closed point. Then since $f^{-1}(y)$ is also closed and invariant, it follows that

$$\overline{f(Z \cap f^{-1}(y))} = \overline{f(Z)} \cap \{y\} = \{y\}$$

so $Z \cap f^{-1}(y) \neq \emptyset$ and $y \in f(Z)$. Thus f maps invariant closed sets to closed sets. In particular, the map f itself, which is dominant, is thus surjective.

And if $f(x) = f(x') = y$, then

$$f(\overline{O(x)} \cap \overline{O(x')}) = f(\overline{O(x)}) \cap f(\overline{O(x')}) = \{y\},$$

so $\overline{O(x)} \cap \overline{O(x')} \neq \emptyset$. If $O(x)$ is an orbit of minimal dimension in $f^{-1}(y)$, then $\overline{O(x)} = O(x)$, otherwise $Z := \overline{O(x)} - O(x)$ would be invariant, of smaller dimension, containing smaller dimensional orbits. So $O(x)$ is closed. But uniqueness follows since any two of these intersect!

Finally, consider $\Psi = (\sigma, \mathrm{id}) : G \times X \to X \times X$. If $(x, x) \in \Delta$ is a closed point, then $\Psi^{-1}(x, x)$ is isomorphic to the stabilizer G_x. Moreover, there is a section of Ψ over the diagonal given by $(x, x) \mapsto (1, x)$. Thus we can apply uppersemicontinuity at the section, and because the fibers are groups:

$$X^{\mathrm{reg}} := \{x \in X | G_x \text{ is of minimal dimension}\}$$

is invariant and open in X. If the minimal dimension is positive, there is nothing to prove. Otherwise, $X - X^{\mathrm{reg}}$ is closed and invariant, so $f(X - X^{\mathrm{reg}})$ is closed, and $X^G_S = X^G - f(X - X^{\mathrm{reg}})$ and $X_S = f^{-1}(X^G_S)$ are both open.

Finally, we need to see why $f_S : X_S \to X^G_S$ is a geometric quotient. Property (i) is done already. For (ii), suppose $U \subset X_S$ is open and invariant. Then $Z = X - U$ is closed and invariant, so $f(Z)$ is closed, and one checks that $f^{-1}(X^G_S - f(Z)) = U$. This gives (ii). Finally, suppose that $V = D(f)$ is the open affine subset of X^G defined by the nonvanishing of $f \in \mathbf{C}[X]^G$. Then $\Gamma(V, \mathcal{O}_{X^G}) = \mathbf{C}[X]^G_f$, and $\Gamma(f^{-1}(V), \mathcal{O}_X) = \mathbf{C}[X]_f$. But it is a simple consequence of the Reynolds identity (Property (c)) that $\mathbf{C}[X]^G_f$ is the ring of invariants of $\mathbf{C}[X]_f$ and (iii) follows.

Now suppose (X, L) is a projective scheme with an ample line bundle. An action of G on X is **linearized** (with respect to L) when the action is extended to an action on the geometric line bundle L such that:

$$\sigma_g(a+b) = \sigma_g(a) + \sigma_g(a) \text{ and } \sigma_g(ka) = k\sigma_g(a)$$

for a, b over a point $x \in X$. This gives an action on the local sections of L:

$$\sigma_g(s)(p) = \sigma_g(\sigma_g^*(s)(\sigma_g^{-1}(p)))$$

Suppose in addition that L is very ample, and that the complete linear series embedding: $f : X \hookrightarrow \mathbf{P}^n$ with $L = f^*\mathcal{O}_{\mathbf{P}^n}(1)$ is projectively normal, i.e. that the map:

$$\mathbf{C}[x_0, ..., x_n] \to \oplus\Gamma(X, L^{\otimes d})$$

is surjective, and an isomorphism in degree 1. (This can always be arranged by replacing L with a large enough tensor power $L^{\otimes m}$). Then the linearized action of G is an action of G on the affine cone $C(X) \subset \mathbf{A}^{n+1}$ over X, which preserves the grading of the (affine) coordinate ring:

$$\mathbf{C}[x_0, ..., x_n]/I(X) = \mathbf{C}[C(X)]$$

and, of course, $C(X) = \mathrm{Spec}(\mathbf{C}[X])$ and $X = \mathrm{Proj}(X)$.

If $x \in X$ is a closed point, let $\widetilde{x} \in C(X)$ denote any lift of x to $C(X) - \{0\}$. The following are easily seen to be well-defined, independent of $\widetilde{x}$:

Definition: With respect to a linearized action of G on (X, L):

(i) $x \in X$ is unstable if $0 \in \overline{O(\widetilde{x})}$.

(ii) $x \in X$ is semistable if $0 \notin \overline{O(\widetilde{x})}$.

(iii) $x \in X$ is stable if $O(\widetilde{x})$ is closed with finite stabilizer.

Definition: With respect to a linearized action of G on (X, L) :

(i) $X_U(L) := \{\text{unstable points of } X\}$.

(ii) $X_{SS}(L) := \{\text{semistable points of } X\} = X - X_U(L)$.

(iii) $X_S(L) := \{\text{stable points of } X\} \subset X_{SS}(L)$.

Definition: If G is linearly reductive, with linearized action on (X, L), then:

$$\overline{f} : X = \text{ Proj}(\mathbf{C}[C(X)]) - - > \text{ Proj}(\mathbf{C}[C(X)]^G) =: X^G$$

is the associated (projective) GIT quotient of X by G.

Projective GIT: The map $\overline{f}$ has the following properties:

(a) $\overline{f}$ is defined on $X_{SS}(L)$, and $\overline{f}_{SS} : X_{SS}(L) \to X^G$ is surjective.

(b) $X_S(L) \subset X$ is also open, its image $X_S^G \subset X^G$ is open, and

$$\overline{f}_S : X_S(L) \to X_S^G$$

is a geometric quotient.

(c) If $x, x' \in X_{SS}(L)$, then $\overline{f}(x) = \overline{f}(x')$ iff $\overline{O(x)} \cap \overline{O(x')} \cap X_{SS}(L) \neq \emptyset$.

(d) If $Z \subset X_{SS}(L)$ is closed and invariant, then $\overline{f}(Z) \subset X^G$ is closed.

Proof: By definition, the projective GIT quotient descends from the affine GIT quotient of the cones (minus their origins!):

$$\begin{array}{ccc} C(X) & \stackrel{f}{\to} & C(X)^G \\ \cup & & \cup \\ C(X) - \{0\} & & C(X)^G - \{0\} \\ \pi \downarrow & & \pi \downarrow \\ X & \stackrel{\overline{f}}{- - >} & X^G \end{array}$$

By affine GIT, $f^{-1}(0) = \{0\} \cup \widetilde{X}_U(L)$, the lifts of the unstable points, and it follows that $\widetilde{X}_{SS}(L)$ (and hence $X_{SS}(L)$) is open in $C(X)$, and that $\overline{f}$ is defined on $X_{SS}(L)$. This gives (a). Notice that $f^{-1}(0) = V(\mathbf{C}[C(X)]_d^G)$ is precisely the common zero locus of the homogeneous ideal generated by the invariant homogeneous polynomials of positive degree.

The homogeneous polynomials $h \in \mathbf{C}[C(X)]_d^G$ of positive degree thus give an open cover: $\cup D(h) = X_{SS}(L)$. Each such open set is invariant, and the restriction $\overline{f}_h : D(h) = \text{Spec}(\mathbf{C}[C(X)]_{(h)}) \to \text{Spec}(\mathbf{C}[C(X)]_{(h)}^G) = D(h)^G$ is precisely the affine GIT quotient (Reynolds identity again!). Part (b) follows from affine GIT, provided we can show that the two notions of stability coincide for points of $D(h)$. Suppose that $x \in D(h)$ and $\widetilde{x}$ is any lift. Then $G_{\widetilde{x}} \subset G_x$ and $G_x/G_{\widetilde{x}}$ is in bijection with $O(\widetilde{x}) \cap \pi^{-1}(x)$, which is finite since x is semi-stable. So one stablizer is finite iff the other one is, too. Once the stabilizers are finite, it follows that $O(x)$ is closed if and only if x is stable, if and only if for each $y \notin O(x)$ there is a $p \in \mathbf{C}[C(X)]_{(h)}^G$ so that $p(x) = 0$ and $p(y) \neq 0$. But if this is true, then $p = \frac{P}{h^d}$ for some invariant homogeneous P (and d) and then $P(\widetilde{y}) \neq 0$ while $P(\widetilde{x}) = 0$. Conversely, once a homogeneous P separates $\widetilde{y}$ from $O(\widetilde{x})$, then $p = \frac{P^k}{h^d}$ separates y from $O(x)$.

And now that we have established that the GIT quotient $D(h) \to D(h)^G$ corresponds to the affine GIT quotient, properties (c),(d) follow easily.

Warning: This doesn't say that stable points in X have closed orbits in X! For example, consider the action of $\mathbf{C}^*$ on $\mathbf{C}[x, y]$ given by:

$$\sigma_\lambda(x) = \lambda x, \sigma_\lambda(y) = \frac{1}{\lambda} y$$

Then the GIT quotient is:

$$\overline{f} : \mathbf{P}^1 - - > \mathrm{Proj}(\mathbf{C}[xy]) = \mathrm{pt}$$

and all points of $\mathbf{P}^1$ other than $0, \infty$ are stable, while $0, \infty$ are unstable. When we consider the "cone" $\mathbf{C}^2$, we see that the orbits of $(a, b) = \widetilde{(a : b)}$ other than $(0, b)$ and $(a, 0)$ are closed (they are the hyperbolas $xy = \text{constant}$) but the orbit of $(a : b)$ down on $\mathbf{P}^1$ is not closed!!!

So there is a trade-off here. If we want a projective quotient, we take semi-stable points. If we want a geometric quotient, we take stable points (giving a quasi-projective quotient). We'd like the projective quotient to have some nice property, too. There is one:

Definition: A **categorical quotient** of the action of G on a scheme X is a morphism $f : X \to Y$ to a scheme Y satisfying:

(i) f is constant on the orbits of the closed points of X.

(ii) For any scheme T and morphism $\phi : X \to T$ which satisfies (i), there is a uniquely determined morphism $\psi : Y \to T$ such that $\phi = \psi \circ f$.

It follows from general nonsense that a categorical quotient is unique. The following criterion is useful for detecting them:

Lemma 4.2: If G acts on X, and $f : X \to Y$ is a morphism which is constant on orbits, then it is a categorical quotient if:

(i) For all open $V \subset Y$, $f^*(\Gamma(V, \mathcal{O}_Y)) = \Gamma(f^{-1}(V), \mathcal{O}_X)^G$, and

(ii) If $Z \subset X$ is invariant and closed, then $f(Z)$ is closed. If (Z_λ) is a system of invariant closed subsets of X, then

$$f(\cap Z_\lambda) = \cap f(Z_\lambda)$$

Proof: (i) implies f is dominant, and (ii) implies it is surjective. Suppose that $\phi : X \to T$ is constant on orbits. We need to construct $\psi : Y \to T$. Choose an affine open cover (W_λ) of T, and let $Z_\lambda = X - \phi^{-1}(W_\lambda)$. These are closed and invariant in X, so by (ii), $V_\lambda = Y - f(Z_\lambda)$ are open in Y, and $f^{-1}(V_\lambda) \subset \phi^{-1}(W_\lambda)$. Since (W_λ) is an open cover of T, it follows that $\cap Z_\lambda = \emptyset$, so by (ii), $\cap f(Z_\lambda) = \emptyset$, so (V_λ) is an open cover of Y.

If $\psi : Y \to T$ satisfies $\phi = \psi \circ f$, then we must have $\psi(V_\lambda) \subset W_\lambda$ for all λ, so the restriction of ψ to each V_λ is determined by a ring homomorphism $r_\lambda : \Gamma(W_\lambda, \mathcal{O}_T) \to \Gamma(V_\lambda, \mathcal{O}_Y)$. But by property (ii), $\Gamma(V_\lambda, \mathcal{O}_Y)$ injects as the invariant subring of $\Gamma(f^{-1}(V_\lambda), \mathcal{O}_X)$, which contains the invariant subring of $\Gamma(\phi^{-1}(W_\lambda), \mathcal{O}_X)$, so the r_λ are **uniquely determined** by the requirement that $\phi = \psi \circ f$. Thus the resulting maps from V_λ to W_λ are uniquely defined, and glue together to give ψ.

Corollary 4.3: (a) Any geometric quotient is categorical.

(b) Any affine GIT quotient $f : X \to X^G$ is categorical.

(c) Any projective GIT quotient $\overline{f}_{SS} : X_{SS}(L) \to X^G$ is categorical.

Proof: All three quotients satisfy the conditions of the Lemma...the first by definition, the second from the proof of Affine GIT, and the third from the local (on $X_{SS}(L)$) identification of the projective GIT quotient with the affine GIT quotient.

Assume that a linearized action of G on (X, L) is given (for L as above). We want some practical method for detecting the stability (or instability) of points $x \in X$. The method given here dates back to Hilbert, and is usually called the **Hilbert-Mumford numerical criterion.** Roughly speaking, the idea is that if a point $x \in X$ is unstable, then its lifts $\widetilde{x}$ can be run off to $0 \in C(X)$ using the group elements of a subgroup $\mathbf{C}^* \subset G$.

Definition: A **one-parameter subgroup** of G (abbreviated 1-PS) is a non-trivial homomorphism $\lambda : \mathbf{C}^* \to G$

If $\sigma_{\widetilde{x}} \circ \lambda : \mathbf{C}^* \to C(X)$ extends to a morphism $\tau : \mathbf{C} \to C(X)$ we will write

$$\lim_{t \to 0} \lambda(t)\widetilde{x} = \tau(0)$$

and say the limit exists. Otherwise, we will write $\lim_{t\to 0} \lambda(t)\widetilde{x} = \infty$ and say that the limit doesn't exist.

Definition: Given a 1-PS λ, then we say x is:

(i) **λ-stable** if $\lim_{t\to 0}\lambda(t)\widetilde{x} = \infty$.

(ii) **λ-semistable** if $\lim_{t\to 0}\lambda(t)\widetilde{x} \neq 0$ (i.e. if the limit exists, it is not 0)

(iii) **λ-unstable** if $\lim_{t\to 0}\lambda(t)\widetilde{x} = 0 \in C(X)$.

Note: Since X is proper, $\sigma_x \circ \lambda : \mathbf{C}^* \to X$ always extends uniquely to $\overline{\tau} : \mathbf{C} \to X$. The point x is λ-stable when $\overline{\tau}$ doesn't lift to $\tau : \mathbf{C} \to C(X)$.

Theorem (The Numerical Criterion): If $G =$ SL$(m, \mathbf{C})$, then:

(a) $x \in X_{SS}(L)$ if and only if x is λ-semistable for all 1-PS λ.

(b) $x \in X_S(L)$ if and only if x is λ-stable for all 1-PS λ.

Proof: One direction is easy. If x is not λ-semistable for some λ, then 0 is in the closure of the orbit of $\widetilde{x}$ under that action of $\mathbf{C}^*$, hence under the action of G. So $x \notin X_{SS}(L)$.

And if x is not λ-stable for some λ, then $\sigma_{\widetilde{x}} \circ \lambda : \mathbf{C}^* \to C(X)$ extends to a map from $\mathbf{C}$. On the other hand, every representation of $\mathbf{C}^*$ splits into one-dimensional invariant subspaces. So there is a basis for $\mathbf{C}^m$ under which the image of $\mathbf{C}^*$ in SL$(m, \mathbf{C})$ has the form:

$$\mathrm{diag}\{t^{r_1}, ..., t^{r_m}\} := \begin{pmatrix} t^{r_1} & & & & 0 \\ & t^{r_2} & & & \\ & & \ddots & & \\ & & & t^{r_{m-1}} & \\ 0 & & & & t^{r_m} \end{pmatrix}$$

with $\sum r_i = 0$. Since λ is non-trivial, it must be that some $r_i < 0$, and $\lambda : \mathbf{C}^* \to G$ therefore cannot be extended to a map from $\mathbf{C}$. But this means $\sigma_{\widetilde{x}} : G \to C(X)$ is not proper by the valuative criterion, so $x \notin X_S(L)$.

To get the converses, let $\mathcal{O} = \mathbf{C}[[t]]$ be the ring of formal-power series and $K = \mathbf{C}((t))$. Then by the (analytic) valuative criterion:

(a) $\sigma_{\widetilde{x}} : G \to C(X)$ is not proper iff there is a morphism $\alpha :$ Spec$(K) \to G$ such that $\sigma_{\widetilde{x}} \circ \alpha$ extends to a morphism $\tau :$ Spec$(\mathcal{O}) \to C(X)$, but α does not extend to a morphism Spec$(\mathcal{O}) \to G$.

(b) 0 is in the closure of $\sigma_x(G)$ iff there is a morphism α as above, where the closed point of Spec($\mathcal{O}$) is sent to 0 under τ.

So suppose x is not stable. Let α be the map from (a). Such a map is equivalent to an element of $\mathrm{SL}(n, K)$, that is, to a matrix $M(t)$ whose entries are each in some $t^{-k}\mathcal{O}$. By the theory of elementary divisors there are matrices $A(t)$ and $B(t)$ in $\mathrm{SL}(n, \mathcal{O})$ that diagonalize $M(t)$:

$$A(t)M(t)B(t) = \mathrm{diag}\{t^{r_1}, ..., t^{r_m}\}$$

where $\sum_{i=1}^m r_i = 0$ and because α did not extend, some r_i is negative. Define:

$$\lambda(t) = A(t)M(t)B(t)B(0)^{-1}$$

and I claim that $\lim_{t\to 0} \lambda(t)\widetilde{x} \neq \infty$.

The action of G on the affine cone $C(X) \subset \mathbf{C}^{n+1}$ is induced from a *linear* action of G on $\mathbf{C}^{n+1}$ by definition, so the action of λ on $\mathbf{C}^{n+1}$ diagonalizes. Let $e_0, ..., e_n$ be a basis with the property that $\lambda(t)e_i = t^{s_i}e_i$ with $s_0 \leq ... \leq s_n$. Let $\hat{b}_{i,j}(t)$ be the (power series) entries of the matrix $B(0)B(t)^{-1}$ acting on V with respect to this basis. In particular, $\hat{b}_{i,j}(0) = \delta_{i,j}$.

Then if we write $\widetilde{x} = \sum x_i e_i$, we have:

$$\begin{aligned} A(t)M(t)\widetilde{x} &= \textstyle\sum_i x_i A(t)M(t)e_i &&= \textstyle\sum_i x_i \lambda(t)B(0)B(t)^{-1}e_i \\ &= \textstyle\sum_i x_i \lambda(t)\left(\sum_j \hat{b}_{i,j}(t)e_j\right) &&= \textstyle\sum_j \left(\sum_i \hat{b}_{i,j}(t)x_i\right)t^{s_j}e_j \end{aligned}$$

By assumption, this has a finite limit, but since $\hat{b}_{i,j}(0) = \delta_{i,j}$, this implies that $x_j = 0$ when $s_j < 0$, and then $\lim_{t\to 0}\lambda(t)\widetilde{x} = \sum \lim_{t\to 0} t^{s_j}x_j e_j \neq \infty$ and $\widetilde{x}$ is not λ-stable(!)

If in addition $\lim_{t\to 0} M(t)\widetilde{x} = 0$, then in the previous paragraph, $x_j = 0$ if $s_j \leq 0$, and $\widetilde{x}$ is not λ-semistable. This completes the proof of the theorem.

Remark: In the proof, the theory of elementary divisors was used to convert the valuative criterion into a 1-PS. A theorem of Iwahori says that the theory of elementary divisors also holds for any linearly reductive G, so the numerical criterion always applies.

Example 1 (Unordered Points on $\mathbf{P}^1$):

Choose the basis $x_0 = x^d, x_1 = x^{d-1}y, ..., x_d = y^d$ for $\mathbf{C}^{d+1} = \mathrm{Sym}^d(\mathbf{C}^2)^*$ (from a choice of basis x, y for $(\mathbf{C}^2)^*$) and consider the induced action of $\mathrm{SL}(2, \mathbf{C})$ on:

$$\mathbf{P}^d = \mathrm{Proj}(\mathbf{C}[x_0, ..., x_d])$$

A point of $\mathbf{P}^d$ "is" an effective divisor $p_1+...+p_d$ on $\mathbf{P}^1$, since the homogeneous polynomial $F(x, y)$ with roots $p_1, ..., p_d$ is linear in the $x_0, ..., x_d$. The set of unstable points for the action (with natural linearization to $L = \mathcal{O}_{\mathbf{P}^d}(1)$) is the set of common zeroes of the invariant polynomials:

$$\mathbf{C}[x_0, ..., x_d]_{>0}^{\mathrm{SL}(2.\mathbf{C})} = \mathrm{Sym}^{>0}(\mathrm{Sym}^d\mathbf{C}^2)^*$$

This ring is complicated! For example, when $d = 4$, it is generated by:

$$P = \frac{1}{6}(x_2^2 - 3x_1x_3 + 12x_0x_4) \text{ and}$$

$$Q = x_0x_2x_4 - \frac{3}{8}x_0x_3^2 - \frac{3}{8}x_1^2x_4 + \frac{1}{8}x_1x_2x_3 - \frac{1}{36}x_2^3$$

but we can simply describe the unstable (and semi-stable) loci with the numerical criterion. If λ is any 1-PS of $\mathrm{SL}(2, \mathbf{C})$, then $\lambda(t) = \mathrm{diag}\{t^{-r}, t^r\}$ for a (dual) basis $x, y \in \mathbf{C}^{2^*}$, and then λ acts by:

$$\lambda(t)(x_k) = t^{r(2k-d)}x_k$$

for $x_k = x^{d-k}y^k$ with respect to this particular basis.

Thus the numerical criterion says that $p_1+...+p_d$ is stable if and only if in **every** basis, $\lim_{t\to\infty} \lambda(t)(F(x, y)) = \infty$, which is to say that **some** monomial $x^{d-k}y^k$ satisfying $2k - d < 0$ has a non-zero coefficient in $F(x, y)$. That is, in every basis, there is no factorization $F(x, y) = y^kG(x, y)$ for $k \geq \frac{d}{2}$. But this says that $F(x, y)$ (in any basis) has no linear factor of multiplicity $\geq \frac{d}{2}$, or equivalently that the divisor $p_1 + ... + p_d$ has no point of multiplicity $\geq \frac{d}{2}$.

Thus $\mathbf{P}^d_S$ is the complement of the "deep diagonal" of divisors of the form $kp + p_{k+1} + ... + p_d$ for $k = \frac{d}{2}$ (or $\frac{d+1}{2}$ if d is odd). If d is even, then there will be semi-stable points that are not stable, when $k = \frac{d}{2}$ and no other $p_i = p$. Among all such d-tuples, there is a unique closed orbit, corresponding to the (unique!) orbit of the d-tuples of the form $\frac{d}{2}p + \frac{d}{2}q$.

You can see this with the numerical criterion. A polynomial

$$F(x,y) = cx^{\frac{d}{2}}y^{\frac{d}{2}} + (\text{higher order in } y)$$

will have $\lim_{t\to 0}\lambda(t)F(x,y) = cx^{\frac{d}{2}}y^{\frac{d}{2}}$ for the one-parameter subgroup above.

Example 2: (Ordered Points on $\mathbf{P}^1$): Here, we want to consider instead:

$$\mathbf{P}^1 \times \mathbf{P}^1 \times ... \times \mathbf{P}^1 = (\mathbf{P}^1)^d$$

with the action of $\mathrm{SL}(2,\mathbf{C})$. In this case, unlike the previous one, there is some choice for the line bundle L. Namely, if

$$L = \mathcal{O}(a_1, a_2, ..., a_d)$$

with the obvious linearization, then we are looking at:

$$(\mathbf{P}^1)^d = \ \mathrm{Proj}(\mathbf{C}[\{x_1^{k_1}y_1^{a_1-k_1} \otimes \cdots \otimes x_d^{k_d}y_d^{a_d-k_d}\}_{k_i=0}^{a_i}])$$

which looks even more intimidating than the previous example. But the same analysis with the numerical criterion shows that $(p_1, p_2, ..., p_d) \in (\mathbf{P}^1)^d$ is stable (respectively semistable) for this linearized action if and only if:

$$\sum_j a_{i_j} < \ (\text{resp. } \leq) \ \ \frac{\sum_i a_i}{2}$$

whenever $p = p_{i_1} = ... = p_{i_j} = ...$ coincide. For example, if $d = 4$ then:

(a) The stable points for the "balanced" linearization $(1,1,1,1)$ are:

$$(\mathbf{P}^1)^4 - \cup\Delta_{ij}$$

the complement of the pairwise diagonals. The semi-stable points are:

$$(\mathbf{P}^1)^4 - \cup\Delta_{ijk}$$

the complement of the triple diagonals, and there are 3 closed orbits that are semi-stable, but not stable, given by:

$$\Delta_{12} \cap \Delta_{34} - \Delta_{1234}, \ \Delta_{13} \cap \Delta_{24} - \Delta_{1234}, \ \Delta_{14} \cap \Delta_{23} - \Delta_{1234}$$

This example, at least, is easy to analyze. The invariants are all generated by the "cross ratio" linear invariants:

$$F = (x_1 \otimes y_2 - x_2 \otimes y_1) \otimes (x_3 \otimes y_4 - y_3 \otimes x_4)$$

$$G = (x_1 \otimes y_3 - y_1 \otimes x_3) \otimes (x_2 \otimes y_4 - x_4 \otimes y_2)$$

(where the tensors in G reordered in the obvous way) and:

$$\overline{f} : (\mathbf{P}^1)^4 - - > \mathbf{P}^1 = \mathrm{Proj}(\mathbf{C}[F, G])$$

takes the three semi-stable (but not stable) orbits described above to the points $0 = (0:1), \infty = (1:0)$ and $1 = (1:1)$, respectively.

(b) The stable points for the linearization $(1,1,1,2)$ are:

$$(\mathbf{P}^1)^4 - (\Delta_{14} \cup \Delta_{24} \cup \Delta_{34} \cup \Delta_{123})$$

and all semistable points are stable.

(c) There are **no** stable points for the linearization $(1,1,1,3)$. The locus of semi-stable points is:

$$(\mathbf{P}^1)^4 - (\Delta_{14} \cup \Delta_{24} \cup \Delta_{34})$$

and there exactly one closed semi-stable orbit, namely $\Delta_{123} - \Delta_{1234}$.

Final Remark: In this last example, we see that by varying the linearizing line bundle L, we can change the GIT quotient. On the other hand, it is easy to see (for example using the numerical criterion) that replacing L by a tensor power $L^{\otimes m}$ (with induced linearization) does not change the GIT quotient. Since the set of stable points is open, it follows that any two GIT quotients (with stable points) are birational. The study of the precise relationship among such quotients was undertaken by Thaddeus and Dolgachev-Hu.