Mathematics 7800– Quantum Kitchen Sink – Spring 2002

5. Vector Bundles on a Smooth Curve. We will construct projective moduli spaces for semistable vector bundles on a smooth projective curve C by applying GIT to a suitable Grothendieck Quot scheme. The construction we present here is due to Carlos Simpson.

Let E be a vector bundle on a smooth projective curve C of genus g.

Definition: (a) The *slope* $\mu(E) = \deg(E)/\mathrm{rk}(E)$.

(b) E is *stable* if $\mu(F) < \mu(E)$ for all proper subbundles $F \subset E$.

(c) E is *semistable* if $\mu(F) \leq \mu(E)$ for all $F \subset E$.

Lemma 5.1: If $0 \to F \to E \to G \to 0$ is an exact sequence of vector bundles, then $\mu(F) \geq \mu(E)$ (resp.$>$) if and only if $\mu(E) \geq \mu(G)$ (resp. $>$).

Proof: Arithmetic! If $a, b, c, d > 0$, then $\frac{a}{c} > \frac{a+b}{c+d}$ if and only if $\frac{b}{d} < \frac{a+b}{c+d}$.

Examples: (i) Every vector bundle on $\mathbf{P}^1$ splits as a sum of line bundles, so only the line bundles $\mathcal{O}_{\mathbf{P}^1}(d)$ are stable, and only $\mathcal{O}_{\mathbf{P}^1}(d)^{\oplus n}$ are semistable.

(ii) E is (semi-)stable iff the dual bundle E^* is (semi-)stable.

(iii) E is (semi-)stable iff $E \otimes L$ is (semi-)stable for all line bundles L.

(iv) If E is semistable of rank r and degree d and:

(a) $d < 0$, then $\mathrm{H}^0(C, E) = 0$.

(b) $d > r(2g-2)$, then $\mathrm{H}^1(C, E) = 0$.

(c) $d > r(2g-1)$, then E is generated by its global sections.

(Schur's) Lemma 5.2: (a) If E and F are stable with the same slope, then any map $f : E \to F$ is either 0 or an isomorphism.

(b) The only automorphism of a stable bundle E is scalar multiplication.

(c) (Jordan decomposition) If E is semistable, there is a filtration:

$$0 = E_0 \subset E_1 \subset ... \subset E_n = E$$

such that $F_i := E_i/E_{i-1}$ is a stable vector bundle and each $\mu(F_i) = \mu(E)$. The filtration is not canonical, in general, but the *associated graded* bundle $\oplus_{i=1}^n F_i$ is independent of the choice of filtration.

Proof: If $f : E \to F$ is not zero, then both $\ker(f)$ and $E/\ker(f)$ are bundles. If f isn't injective, then the stability of E implies $\mu(\ker(f)) < \mu(E)$ and by Lemma 5.1, $\mu(E/\ker(f)) > \mu(E) = \mu(F)$, contradictng the stability of F. So f is injective, and surjective by the stability of F. This gives (a).

If $\alpha : E \to E$ is an automorphism, let λ_x be an eigenvalue of the restriction of α to the fiber of E over $x \in C$. Then $\alpha - \lambda_x(\mathrm{id})$ drops rank at x, so it is not an isomorphism, and must be zero by (a) and we have (b). Finally, (c) follows from (a) by the usual Jordan-Hölder decomposition.

(Harder-Narasimhan) Lemma 5.3: If E is an any vector bundle on C, then there is a filtration:

$$0 = E_0 \subset E_1 \subset ... \subset E_n = E$$

such that $F_i := E_{i+1}/E_i$ are *semistable* vector bundles, with $\mu(F_i) > \mu(F_{i+1})$. This filtration is uniquely determined by the property that if $F \subset E$ is any sub-bundle with $\mu(F) \geq \mu(E_i)$, then $F \subset E_i$.

Proof: Let $S = \{a \mid a < \mu(E) \text{ and } a = \mu(Q) \text{ for some quotient } \ E \to Q\}$. We claim first that S is a finite set. Indeed, let D be a divisor of large enough degree so that $E(D)$ is generated by its sections. Then any $Q(D)$ is also generated by its sections, so $\deg(Q(D)) \geq 0$ and $\mu(Q) \geq -\deg(D)$. So the elements of S are bounded below (and above!) and since the denominators are bounded above by r, it follows that S is finite.

Finiteness of S implies that the set of slopes of **sub**-bundles $F \subset E$ is bounded from above. Let $E_1 \subset E$ be the sub-bundle of maximal rank among those of maximal slope. Then E_1 is semi-stable and E/E_1 is a vector bundle. If $F \subset E$ is another sub-bundle with $\mu(F) = \mu(E_1)$, then the span of F and E_1 is yet another sub-bundle of the same slope (since the kernel of the map from $F \oplus E_1$ to the span must have the same slope). Since E_1 was of maximal rank, it follows that $F \subset E_1$ which then has the desired property.

Now suppose inductively that the lemma holds for $F = E/E_1$. We may use the Harder-Narasimhan filtration of F:

$$0 = F_0 \subset F_1 \subset .... \subset F_{n-1} = F = E/E_1$$

to uniquely define E_{i+1} by the condition that $E_{i+1}/E_1 = F_i$. And it follows that this filtration has the desired property.

Thus every vector bundle on C is an extension of stable vector bundles, which are the "indecomposable" objects. They also have the smallest possible group of automorphisms, namely $\mathbf{C}^*$, though there are vector bundles with this automorphism group (called simple vector bundles) which are not stable.

The following theorem is due to Narasimhan and Seshadri:

Theorem 5.4: For each pair (r, d) of coprime positive integers, the functor:

Obj : schemes S with an equivalence class of vector bundles E on $C \times S$ with the property that each E_s is stable, of rank r and degree d
(and $E \sim F \Leftrightarrow E \cong F \otimes \pi^*\Lambda$ for some line bundle Λ on S)

Mor : morphisms $\phi : S \to S'$ such that $(\phi, \mathrm{id})^* E' \sim E$

is represented by a projective scheme $M_C(r, d)$ which is irreducible and smooth, of dimension $r^2(g-1)+1$.

Proof: We need two key lemmas, the first solving a GIT problem, and the second having to do with the boundedness of families of sheaves on C.

(GIT) Lemma 5.5: If V and W are vector spaces and M is an integer, let

$$G(V \otimes W, M)$$

be the Grassmannian of M-dimensional **quotients** of $V \otimes W$. Then a point $\psi \in G(V \otimes W, M)$ is semistable (resp. stable) with respect to the natural line bundle and linearization of $\mathrm{SL}(V)$ if and only if

$$\frac{\dim(H)}{\dim(V)} \leq \frac{\dim(\psi(H \otimes W))}{M} \quad \text{(resp. } <)$$

for every proper subspace $H \subset V$.

Proof: Let $N = \dim(V)$ and $R = \dim(W)$ with a fixed basis $w_1, ..., w_R$. An point $\psi \in G(V \otimes W, M)$ lifts to $\widetilde{\psi} = \wedge^M \psi \in \wedge^M (V \otimes W)^*$ in the natural linearization. Given any basis $e_1, ..., e_N$ of V and dual basis $x_1, ..., x_N$, we'll call $e_{i_1} \otimes w_{j_1} \wedge ... \wedge e_{i_M} \otimes w_{j_M}$ the induced basis of Plücker vectors. Thus the coordinates of $\widetilde{\psi}$ are the values:

$$\wedge^M \psi(e_{i_1} \otimes w_{j_1} \wedge ... \wedge e_{i_M} \otimes w_{j_M}) = \psi(e_{i_1} \otimes w_{j_1}) \wedge ... \wedge \psi(e_{i_M} \otimes w_{j_M}) \in \mathbf{C}$$

which are zero if and only if the $\psi(e_{i_k} \otimes w_{j_l})$ are not linearly independent.

If $\lambda = \mathrm{diag}\{t^{r_1}, ..., t^{r_N}\}$ is a 1-PS of $\mathrm{SL}(V)$ and $x_1, ..., x_N$ is the associated (dual) basis, we'll say the weight of the Plücker vector above is $\sum_{j=1}^{M} r_{i_j}$. Then ψ is λ-unstable for this λ if and only if $\wedge^M \psi$ vanishes on every Plücker vector of nonpositive weight.

Suppose $H \subset V$ has dimension n, $\dim(\psi(H \otimes W)) = m$ and $\frac{n}{N} > \frac{m}{M}$. Let $e_1, ..., e_n$ be a basis of H, extended to a basis $e_1, ..., e_N$ of V and let $\lambda = \mathrm{diag}\{t^{n-N}, ..., t^{n-N}, t^n, ..., t^n\}$ for the dual basis. For each Plücker vector, if $\wedge^M \psi(e_{i_1} \otimes w_{j_1} \wedge ... \wedge e_{i_M} \otimes w_{j_M}) \neq 0$, then $\psi(e_{i_1} \otimes w_1), ..., \psi(e_{i_M} \otimes w_M)$ must be linearly independent, so the e_{i_j} must involve at most m of the $e_1, ..., e_n$ vectors, thus its weight must be at least $m(n - N) + (M - m)n$. But $Mn - mN > 0$ by assumption, so $\wedge^M \psi$ is λ-unstable for this λ.

Conversely, let λ be any 1-PS, diagonalized as $\lambda = \mathrm{diag}\{t^{r_1}, ..., t^{r_N}\}$ for a basis $x_1, ..., x_N$. If ψ is λ- unstable, let H_n be the span of $e_1, ..., e_n$, and let $m_n = \dim(\psi(H_n \otimes W))$. Then λ-instability tells us:

$$(*) \quad r_1 m_1 + r_2(m_2 - m_1) + ... + r_N(M - m_{N-1}) > 0$$

because it is the minimal weight of a Plücker vector on which $\wedge^M \psi$ is nonzero. I claim that for some n, the "averaged" weights also satisfy:

$$\frac{1}{n}(r_1 + ... + r_n)m_n + \frac{1}{N-n}(r_{n+1} + ... + r_N)(M - m_n) > 0$$

It then follows that $\frac{n}{N} > \frac{m_n}{M}$ holds for $H = H_n$. To see the claim, notice first that if $m_{i+1} - m_i \leq m_i - m_{i-1}$, then we may combine r_i and r_{i+1}, replacing them with their average $\frac{r_i + r_{i+1}}{2}$ without decreasing $(*)$. The averaged weights are the same as the original, so we may assume the sequence of differences is increasing: $\Delta_1 := m_1 < \Delta_2 := m_2 - m_1 < ... < \Delta_N := m_N - m_{N-1}$. Now consider the linear function:

$$L(t_1, t_2, ..., t_N) = r_1 t_1 + ... + r_N t_N$$

which by assumption satisfies $L(\Delta_1, ..., \Delta_N) > 0$, and consider its values at the points:

$$p_n := \left(\frac{\sum_{i=1}^{n} \Delta_i}{n}, ..., \frac{\sum_{i=1}^{n} \Delta_i}{n}, \frac{\sum_{i=n+1}^{N} \Delta_i}{N-n}, ..., \frac{\sum_{i=n+1}^{N} \Delta_i}{N-n}\right) \in \mathbf{R}^N$$

These points are linearly independent and $L(p_N) = 0$ (the r_i sum to zero). Thus they span the hyperplane $\{\sum t_i = \sum \Delta_i\} \subset \mathbf{R}^N$ and in particular, $(\Delta_1, ..., \Delta_N) = \sum_{i=1}^{N-1} y_i p_i - y_N p_N$ for **positive** values y_i, so **some** $L(p_i) > 0$.

Thus, we've shown that $\frac{n}{N} > \frac{m}{M}$ if and only if ψ is λ-unstable for some λ. By the numerical criterion, this proves the "semi-stable" part of the lemma, and the stable part is proved by replacing each "$>$" by a "$\geq$."

Lemma 5.6: Let $p \in C$, and $\mathcal{O}_C(1) := \mathcal{O}_C(p)$. If $n > 2g - 1 - \frac{d}{r}$ then:

(a) If E is semistable of rank r and degree d, then $\mathrm{H}^1(C, E(n)) = 0$, $E(n)$ is generated by global sections and for all subbundles $F \subset E$:

$$\frac{h^0(C, F(n))}{\mathrm{rank}(F)} \leq \frac{h^0(C, E(n))}{\mathrm{rank}(E)}$$

with equality if and only if F is semistable, $h^1(C, F(n)) = 0$ and for all m:

$$\frac{\chi(C, F(m))}{\mathrm{rank}(F)} = \frac{\chi(C, E(m))}{\mathrm{rank}(E)}$$

(b) If $\mathcal{E}$ is any coherent sheaf of the same Hilbert polynomial $\chi(C, \mathcal{E}(n)) = P(m) = rm + d - r(g-1)$ as a vector bundle of rank r and degree d, and if every vector bundle quotient $\mathcal{E} \to G$ satisfies:

$$\frac{h^0(C, G(n))}{\mathrm{rank}(G)} \geq \frac{P(n)}{r},$$

then $\mathcal{E}$ is itself a semistable vector bundle of rank r and degree d.

Proof: The key point is the following. If E is a semistable bundle of rank r and if $h^1(C, E) \neq 0$, then $h^0(C, E) \leq rg$ *independent of the degree of* E. This is well-known for line bundles, since every L with $h^1(C, L) \neq 0$ is a subsheaf of the canonical line bundle and $h^0(C, \omega_C) = g$. But here's a proof that generalizes. If $h^1(C, L) \neq 0$, then $\deg(L) \leq 2g - 2$. If $h^0(C, L) \geq g$, then there is a section $s \in H^0(C, L)$ vanishing at any $p_1, ..., p_{g-1} \in C$. If the p_i are "general," then $h^0(C, \mathcal{O}_C(\sum p_i)) = 1$ and from:

$$0 \to \mathcal{O}_C(\sum p_i) \to L \to \tau \to 0$$

and $\deg(\tau) \leq g - 1$ it follows that $h^0(C, L) \leq 1 + h^0(C, \tau) \leq g$.

If E is semistable of rank r and $h^1(C, E) \neq 0$, then by Example (iv) we have $\deg(E) \leq r(2g-2)$ and $\deg(F) \leq r'(2g-2)$ for any subbundle $F \subset E$ of rank r'. If $h^0(C, E) \geq rg$, then there is a section $s \in H^0(C, E)$ vanishing at any $g - 1$ points, and then we get $\mathcal{O}_C(\sum p_i) \subset E$ spanning a line bundle $L \subset E$ of degree $\leq 2g - 2$ satisfying $h^0(C, L) \leq g$ as above.

If $h^0(C,E) \geq rg$, then $h^0(C,E/L) \geq (r-1)g$ and we can find a section $s' \in H^0(C,E/L)$ which again can be chosen to vanish at $g-1$ general points. The two sections s, s' will span a sub-bundle $F \subset E$ of degree $\leq 2(2g-2)$ which then has at most $2+2(g-1) = 2g$ sections from the exact sequence:

$$0 \to \mathcal{O}_C(\sum p_i) \oplus \mathcal{O}_C(\sum p_i') \to F \to \tau \to 0$$

and then one considers sections of E/F, etc.

We already saw that the first part of (a) is satisfied in Example (iv). Notice that:

$$\frac{\chi(C,E(n))}{r} = \frac{P(n)}{r} = n + \frac{d}{r} - (g-1)$$

Thus any semistable bundle F of rank $r' \leq r$ and slope $\mu \leq \frac{d}{r}$ must satisfy $h^0(C,F(n)) = \chi(C,F(n)) \leq \frac{r'}{r}P(n)$, or else $h^0(C,F(n)) \leq r'g < \frac{r'}{r}P(n)$ by the key point above (and the lower bound $n > 2g-1-\frac{d}{r}$).

If $F \subset E$ and E is semistable, then every F_i in the Harder-Narasimhan filtration of F has slope at most $\frac{d}{r}$, so each subquotient F_i satisfies

$$\frac{h^0(C,F_i(n))}{\mathrm{rk}(F_i)} \leq \frac{P(n)}{r}$$

and by Lemma 5.1, we have the same inequality for F. If equality holds, then it must hold for every F_i, and we conclude that every F_i has slope exactly $\frac{d}{r}$, so F is semistable, and $\frac{\chi(C,F(m))}{\mathrm{rk}(F)} = \frac{P(m)}{\mathrm{rk}(E)}$ for all m. This proves (a).

If $\mathcal{E}$ is the sheaf in (b), let $\mathcal{T} \subset \mathcal{E}$ be the torsion subsheaf, and let G be the (semistable) quotient of smallest rank in the Harder-Narasimhan filtration of $\mathcal{E}/\mathcal{T}$. Since $\mu(G) \leq \mu(\mathcal{E}/\mathcal{T}) \leq \frac{d}{r}$, it follows as above that:

$$\frac{\mathrm{h}^0(C,G(n))}{\mathrm{rk}(G)} \leq \frac{P(n)}{r}$$

with equality if and only if $\mathcal{T} = 0$ and $\mu(G) = \mu(\mathcal{E})$. But this means $\mathcal{E} = G$!

We are ready for the proof of Theorem 5.4 now.

Let $P(m) = rm + d - r(g-1)$ be the Hilbert polynomial of a bundle of rank r and degree d as in Lemma 5.6, and for fixed $n > 2g-1-\frac{d}{r}$, consider the Quot scheme

$$\mathrm{Quot}(V \otimes \mathcal{O}_C(-n), P(m))$$

where V is a vector space of rank $P(n)$ (with $\mathrm{SL}(V)$ action).

If E is any semistable bundle of rank r and degree d, then as we have already remarked, $E(n)$ is generated by global sections and $H^1(C, E(n)) = 0$, so $h^0(C, E(n)) = P(n)$ and the global section map $V \cong H^0(C, E(n)) \to E(n)$ twists to give a point $V \otimes \mathcal{O}_C(-n) \to E$ of the Quot scheme. Recall that the Quot scheme embeds in Grassmannians:

$$\iota_m : \mathrm{Quot}(V \otimes \mathcal{O}_C(-n), P(m)) \hookrightarrow G(V \otimes W, M)$$

for each $M = P(m)$ and sufficiently large m, and $W = H^0(C, \mathcal{O}_C(m-n))$.

We will consider the GIT quotient of the Quot scheme for the action of $\mathrm{SL}(V)$ induced from the Grassmannian (and linearized as in Lemma 5.5). For large enough m, the two notions of **vector bundle** (semi-)stability and **GIT** (semi-)stability will coincide. When n, d are coprime, semi-stability equals stability, and the GIT quotient will represent the functor. Deformation theory will then show that the quotient is smooth, of the indicated dimension.

If $x \in \mathrm{Quot}(V \otimes \mathcal{O}_C(-n), P(m))$, let $q_x : V \otimes \mathcal{O}_C(-n) \to \mathcal{E}_x$ be the corresponding quotient. Such a quotient induces a map $V \to H^0(C, \mathcal{E}_x(n))$ and for each (large enough) m, let $\psi_x : V \otimes W \to \mathrm{H}^0(C, \mathcal{E}_x(m))$ be the image point in the Grassmannian. Let $X_U(m), X_{SS}(m)$ and $X_S(m)$ be the loci of unstable, semistable and stable points for this embedding.

Step 1: For large enough m (independent of x), if

(i) $\mathcal{E}_x$ is a semistable vector bundle and

(ii) $V \to \mathrm{H}^0(C, \mathcal{E}_x(n))$ is an isomorphism, then $x \in X_{SS}(m)$.

Proof: If $x \in X_U(m)$, then by Lemma 5.5, there is an $H \subset V$ so that:

$$(*) \quad \frac{\dim(H)}{P(n)} > \frac{\dim(\psi_x(H \otimes W))}{P(m)}$$

and we need to show that the existence of such an H violates (i) or (ii).

For each $H \subset V$, let $\mathcal{F}_{x,H} \subset \mathcal{E}_x$ be the subsheaf generated by $H \otimes \mathcal{O}_C(-n)$. Assuming (ii), we see that $H \cong H^0(C, \mathcal{F}_{x,H}(n))$. Consider:

$$0 \to \mathcal{K}_{x,H} \to H \otimes \mathcal{O}_C(-n) \to \mathcal{F}_{x,H} \to 0$$

and choose m_0 so that $m \geq m_0$ implies that $\mathrm{H}^1(C, \mathcal{K}_{x,H}(m)) = 0$ and $\mathrm{H}^1(C, \mathcal{F}_{x,H}(m)) = 0$, for all $H \subset V$ and all x in the Quot scheme. Then $\psi_x(H \otimes W) = \mathrm{H}^0(C, \mathcal{F}_{x,H}(m))$ is of dimension $\chi(C, \mathcal{F}_{x,H}(m))$.

Thus if $(*)$ holds, then:

$$\frac{\dim(\mathrm{H}^0(C, \mathcal{F}_{x,H}(n)))}{P(n)} > \frac{\chi(C, \mathcal{F}_{x,H}(m))}{P(m)}.$$

On the other hand, if we assume (i), then Lemma 5.6 (a) gives us:

$$\frac{\dim(\mathrm{H}^0(C, \mathcal{F}_{x,H}(n))}{P(n)} < \frac{\mathrm{rank}(\mathcal{F}_{x,H})}{r}$$

(equality would force equality in the previous formula). But $\chi(C, \mathcal{F}_{x,H}(m)) = r'm + d' - r'(g-1)$ for $r' = \mathrm{rk}(\mathcal{F}_{x,H})$ and $d' = \deg(\mathcal{F}_{x,H})$, so we are getting:

$$\frac{r'}{r} > \frac{\dim(\mathrm{H}^0(C, \mathcal{F}_{x,H}(n))}{P(n)} > \frac{r'(m + \frac{d'}{r'} - (g-1))}{r(m + \frac{d}{r} - (g-1))}$$

There are only finitely many d' and r', so since the right side approaches the left as $m \to \infty$, we obtain a contradiction when m is sufficiently large.

Step 2: After possibly increasing m again, if $x \in X_{SS}(m)$ then:

(a) The map $V \to \mathrm{H}^0(C, \mathcal{E}_x(n))$ is an isomorphism and

(b) The quotient $\mathcal{E}_x$ is a semistable vector bundle.

Proof of Step 2: By Lemma 5.5, if $x \in X_{SS}(m)$ (for any m), then $V \to \mathrm{H}^0(C, \mathcal{E}_x)$ must be injective, because any kernel would yield an H such that $\psi_x(H \otimes W) = 0$. Similarly, for all $H \subset V$, we must have:

$$(*) \quad \frac{\dim(H)}{\dim(\psi_x(H \otimes W))} \le \frac{P(n)}{P(m)}$$

Suppose $\mathcal{E}_x$ were not a bundle or not semistable. Then by Lemma 5.6(b), we could find a quotient bundle $\mathcal{E}_x \to G$ so that $\frac{h^0(C,G(n))}{\mathrm{rk}(G)} < \frac{P(n)}{r}$. Let H be the kernel of the map $V \to \mathrm{H}^0(C, G(n))$ for such a quotient, and let $\mathcal{F}_{x,H}$ be the image of H in $\mathcal{E}_x$. If $\mathcal{F}_{x,H}$ is torsion, then there is a universal bound on its length, say K, and we can choose m so that $\frac{P(n)}{P(m)} < \frac{1}{K}$ violating $(*)$.

Otherwise, by the arithmetic of Lemma 4.0, we have:

$$(**) \quad \frac{\dim(H)}{\mathrm{rank}(\mathcal{F}_{x,H})} > \frac{P(n)}{r}$$

where the rank of $\mathcal{F}_{x,H}$ is the generic rank, which is the coefficient of m in $\chi(C, \mathcal{F}_{x,H}(m))$. Since $\chi(C, \mathcal{F}_{x,H}(m)) = \dim(\psi_x(H \otimes W))$ (see Step 1) we get a contradiction to $(*)$, perhaps after boosting m again, from the fact that there is a uniform upper bound on the constant terms of the Hilbert polynomials of the $\mathcal{F}_{x,H}$. So $\mathcal{E}_x$ is semistable. Finally, since $\mathcal{E}_x$ is semistable, the map $V \to \mathrm{H}^0(C, \mathcal{E}_x(n))$, which we already saw was injective, must be an isomorphism by Lemma 5.6(a).

Step 3: For sufficiently large m

(a) $x \in X_S(m) \iff x \in X_{SS}(m)$ and $\mathcal{E}_x$ is stable.

(b) For any $x \in X_{SS}(m)$, the closed orbit $O(\widetilde{x}') \subset \overline{O(\widetilde{x})}$ corresponds to an $\mathcal{E}_{x'}$ that is isomorphic to the associated graded of $\mathcal{E}_x$.

Proof of Step 3: (a) is the same argument as Steps 1 and 2. For (b), if $x \in X_{SS}(m) - X_S(m)$, let $F \subset \mathcal{E}_x$ be a proper subbundle of the same slope, and let $H \subset V$ be the kernel of the map $V \to \mathrm{H}^0(C, G(n))$, where $G = \mathcal{E}_x/F$. Consider the induced extension:

$$(\dagger): \ 0 \to F \to \mathcal{E}_x \to G \to 0$$

of vector bundles of the same slope.

If we take $e_1, ..., e_n$ spanning H, extend to a basis of V, and consider the 1-PS subgroup $\lambda = \mathrm{diag}\{t^{n-N}, ..., t^{n-N}, t^n, ..., t^n\}$, then λ acts on the extension class of $\dagger$ in $\mathrm{H}^1(C, G^* \otimes F)$ by multiplication by t^N, taking it to the split extension in the limit as $t \to 0$. We can repeat the process until we get to the associated graded of $\mathcal{E}_x$. Since the associated graded is uniquely determined by Schur's Lemma, and there must be *some* closed orbit in the closure of the orbit of $\mathcal{E}_x$, this must be the one!

We have proved that for large m (and arbitrary (r, d)!), the GIT quotient:

$$G(V \otimes W, M) \supset \mathrm{Quot}(V \otimes \mathcal{O}_C(-n), P(m)) \overset{\overline{f}}{- -\!>} M_C(r, d)$$

has the following properties:

(i) $M_C(r, d)$ is a projective scheme

(ii) The points of $M_C(r, d)$ correspond to associated gradeds of semistable vector bundles of rank r and degree d.

The $M_C(r, d)$ are independent of the choice of (large enough) m because they are all **categorical** quotients of the same open subscheme $X_{SS}(m)$!

Now take any vector bundle E on $S \times C$ and consider the sheaf $\pi_{S*}(E(n))$, where $E(n) = E \otimes \pi_C^* \mathcal{O}_C(n)$. If E is a family of semi-stable bundles of rank r and degree d (and $n > 2g - 1 - \frac{d}{r}$), then $\pi_{S*}(E(n))$ is locally free of rank $P(n)$ and the natural map:

$$\pi_S^* \pi_{S*}(E(n)) \otimes \pi_C^* \mathcal{O}_C(-n) \to E$$

is a surjective map of vector bundles. Locally (on S) we may trivialize $\pi_{S*}(E(n))$ each trivialization determines $U_i \to \mathrm{Quot}(V \otimes \mathcal{O}_C(-n), P(m))$ which do not patch as maps to the Quot scheme, but **do** patch to:

$$\phi : S \to M_C(r, d)$$

So to prove that $M_C(r, d)$ represents the functor, we need only to find a universal vector bundle $\mathcal{U}$ on $C \times M_C(r, d)$ with the property that any vector bundle E as above satisfies $E \sim (\phi, 1)^* \mathcal{U}$. We will use the following:

(Descent) Lemma 5.7: Given a linearized G-action on (X, L) and a vector bundle F on $X_{SS}(L)$ with a G-action, then F descends to the GIT quotient:

$$\overline{f} : X_{SS}(L) \to X^G$$

if and only if for each closed orbit $O(\widetilde{x})$, the stabilizer $G_x \subset G$ also stabilizes the fiber F_x of F at $x \in X_{SS}(L)$.

Proof (Kempf): If F descends, then by definition, $F = \overline{f}^*(\overline{F})$ is the pull-back of the descended bundle $\overline{F}$, so G_x acts trivially on the fibers F_x.

To prove the converse, it suffices to find, for each $x' \in X_{SS}(L)$, an affine neighborhood V' of $y := \overline{f}(x') \in X^G$ and a trivialization of $F|_{\overline{f}^{-1}(V')}$ by G-invariant sections. Given $x \in \overline{f}^{-1}(y)$ with closed $O(\widetilde{x})$, there are $r = \mathrm{rk}(F)$ G-invariant sections of the restriction $F|_{O(x)}$ which trivialize F along the orbit. Indeed, since we assumed that G_x acts trivially on F_x, we can translate a basis $e_1, ..., e_r \in F_x$ by G to obtain the desired sections $Ge_1, ..., Ge_r$.

Let $y \in V = D(h)^G$ for a homogeneous, invariant h in the homogeneous coordinate ring of X. Then $\overline{f}^{-1}(V) = D(h) \subset X_{SS}(L)$ is also affine and, as in projective GIT, the map $D(h) \to D(h)^G$ is the affine GIT quotient. I claim that there is a Reynolds operator $E : \mathrm{H}^0(D(h), F) \to \mathrm{H}^0(D(h), F)^G$. To see this, it suffices to show that G acts rationally on $\mathrm{H}^0(D(h), F))$.

But if we choose an open affine $U \subset D(h)$ on which F trivializes, then an $s \in \mathrm{H}^0(U, F)$ gives rise to a regular function $\phi : G \times U \to \mathbf{C}^r$ defined by $(g, x) \mapsto gs(g^{-1}x)$. Then if $G = \mathrm{Spec}(A)$ and $U = \mathrm{Spec}(B)$, we have $\phi = \sum a_i \otimes \vec{b}_i$, where $a_i \in A$ and $\vec{b}_i : U \to \mathbf{C}^r$, and as before, $Gs|_U$ is contained in the span, W, of the $\vec{b}_i$. Since the restriction of sections from $D(h)$ to U is injective, we prove rationality by intersecting W with $\mathrm{H}^0(D(h), F)$.

Now take the sections $Ge_1, ..., Ge_r$ spanning $F|_{O(x)}$ and extend them to sections $s_1, ..., s_r$ of $F|_{D(h)}$, which is possible since $D(h)$ is affine. Apply the Reynolds operator to get **invariant** sections $E(s_1), ..., E(s_r)$, which still restrict to $Ge_1, ..., Ge_r$ on $O(x)$ (by property (i) of the Reynolds operator). Finally, consider the closed invariant subset $Z \subset D(h)$ where $E(s_1), ..., E(s_r)$ fail to span F. The image $\phi(Z) \subset D(h)^G$ is closed and does not contain $\overline{f}(x')$, so we can shrink $V = D(h)$ to a smaller open neighborhood $x' \in V'$ for which $E(s_1), ..., E(s_r)$ do span, finishing the proof.

Now suppose (r, d) are **coprime** and consider the universal quotient:

$$V \otimes \mathcal{O}_{C \times X_{SS}}(-n) \to \mathcal{E}$$

on $C \times X_{SS} \subset C \times \mathrm{Quot}(V \otimes \mathcal{O}_C(-n), P(m))$. It is a consequence of flatness that $\mathcal{E}$ is a vector bundle of relative rank r and degree d over X_{SS}, and $\mathcal{E}$ is an $\mathrm{SL}(V)$-bundle by virtue of the fact that the Quot scheme represents the functor. That is, the action on $\mathcal{E}$ is obtained by pulling back the universal quotient under the action of $\mathrm{SL}(V)$ on $C \times X_{SS}$. Since (r, d) are coprime, each of the bundles $\mathcal{E}_x$ is **stable** (there is no smaller r' with $\frac{d'}{r'} = \frac{d}{r}$) and since $\mathrm{Aut}(\mathcal{E}_x) = \mathbf{C}^*$, it follows that each stabilizer $\mathrm{SL}(V)_x = \mathrm{SL(V)} \cap \mathbf{C}^* \cong \mu_{P(n)}$ is the group of $P(n)$-th roots of unity. Again, since (r, d) are coprime, it follows that $P(n) = nr + d - r(g-1)$ and r are coprime, so we can solve(!)

$$1 + ar = bP(n)$$

and it follows that the action of the stabilizers $\mathrm{SL}(V)_x$ on $\mathcal{E} \otimes (\wedge^r \mathcal{E})^{\otimes a}$ is trivial, and this bundle, at least, descends.

In the rank $r = 1$ case, take $b = 1$ and $a = P(n) - 1$, and this gives us a bundle $\mathcal{L}_n$ on $C \times \mathrm{Pic}^d(C)$ for the Picard scheme $\mathrm{Pic}^d(C) = M_C(1, d)$, which has the property that $(\mathcal{L}_n)_x$ is the $P(n)$th tensor power of the line bundle associated to $x \in M_C(1, d)$. But we may descend for two consecutive values of n, which gives consecutive values of $P(n) = n + d - (g-1)$ and we get $\mathcal{L} := \mathcal{L}_{n+1} \otimes \mathcal{L}_n^*$ which then has the desired universal property.

In the arbitrary rank case, we have the "determinant" morphism:

$$X_{SS} \to M_C(r,d) \stackrel{\det}{\to} M_C(1,d)$$

coming from the family of line bundles $\wedge^r \mathcal{E}$ on $C \times X_{SS}$ which factors through $M_C(r,d)$ because it is a categorical quotient! Thus we may take the vector bundle $\mathcal{E}_n$ on $C \times M_C(r,d)$ descended from $\mathcal{E} \otimes (\wedge^r \mathcal{E})^{\otimes a}$ and "tensor back" by $((1,\det)^* \mathcal{L})^{\otimes -a}$ to obtain $\mathcal{U}$ which has the desired universal property.

Claim: Each $M_C(r,d)$ is irreducible.

Proof: First of all, notice that there are **isomorphisms:**

$$\cdots \stackrel{\otimes \mathcal{O}_C(1)}{\to} M_C(r,d) \stackrel{\otimes \mathcal{O}_C(1)}{\to} M_C(r,d+r) \stackrel{\otimes \mathcal{O}_C(1)}{\to} \cdots$$

which, in case (r,d) are coprime are obtained by taking the tensor product $\mathcal{U} \otimes \pi_C^* \mathcal{O}_C(1)$ (and in the non-relatively prime case are obtained by considering $\mathcal{E} \otimes \pi_C^* \mathcal{O}_C(1)$ on the Quot scheme and using the categorical quotient). Thus we may assume that $d > r(2g-1)$ so all bundles are generated by sections.

When $r = 1$, this gives us a surjective map (in fact a projective bundle)

$$u_d : \mathrm{Sym}^d(C) \to M_C(1,d); \quad D \mapsto \mathcal{O}_C(D)$$

defined rigorously by using the "universal" Cartier divisor $\mathcal{D} \subset C \times \mathrm{Sym}^d(C)$ and using it to construct the family $\mathcal{O}_{C \times \mathrm{Sym}^d(C)}(\mathcal{D})$ of line bundles. Evidently, $\mathrm{Sym}^d(C)$ is irreducible, as it is the quotient of C^d by the permutation group.

In rank r, a choice of $r+1$ general sections of a bundle E of rank r and degree d gives a surjection $\mathcal{O}_C^{r+1} \to E$ with kernel $(\wedge^r E)^*$. Dually, this means we can exhibit E^* as the kernel of a map:

$$0 \to E^* \to \mathcal{O}_C^{r+1} \to L \to 0$$

where $L = (\wedge^r E)$ (and then dualize to get E). It turns out that a general choice of $r+1$ sections of a line bundle L of degree d has a semi-stable E as its kernel, and this gives a surjective morphism:

$$G(r+1, \pi_* \mathcal{L}) \subset U \to M_C(r,d)$$

from an open subset U of the Grassmann bundle over $M_C(1,d)$ to $M_C(r,d)$. Since $M_C(1,d)$ is irreducible, it then follows that $M_C(r,d)$ is irreducible too.

For smoothness and the dimension count, we will use deformation theory. Given a stable bundle E of rank r and degree d, the Zariski tangent space is the space of (equivalence classes of) vector bundles E_ϵ on $C \times \mathrm{Spec}(k[\epsilon])$ with the property that $E_\epsilon|_C \cong E$. We may trivialize E on an open cover $\cup U_i = C$ with intersections $U_{ij} = U_i \cap U_j$, and then E is determined by transition functions:

$$G_{ij} \in GL(\mathcal{O}_C(U_{ij}))$$

satisfying the cocycle condition:

$$G_{jk}G_{ij} = G_{ik}$$

on triple intersections U_{ijk}. An extension of E is given by an extension of the transition functions:

$$G_{ij} + \epsilon H_{ij} \in GL(\mathcal{O}_C(U_{ij})[\epsilon])$$

(the invertibility puts no constraint on the matrix H_{ij}) satisfying:

$$(G_{jk} + \epsilon H_{jk})(G_{ij} + \epsilon H_{ij}) = G_{ik} + \epsilon H_{ik}$$

or the original cocycle condition together with:

$$H_{jk}G_{ij} + G_{jk}H_{ij} = H_{ik}$$

on triple intersections U_{ijk}. But if we regard the H_{ij} as sections of the (trivialized!) bundle $\mathrm{End}(E)$, then this is **precisely** the cocycle condition to define an element of:

$$H^1(C, \mathrm{End}(E))$$

(as the G_{ij} transition the H_{ij} to allow us to compare them on U_{ik}). And coboundaries are cocycles that give trivial deformations of E, so this is indeed the tangent space. Similarly, one checks that the obstruction space is:

$$H^2(C, \mathrm{End}(E)) = 0$$

Thus on a curve C, there is no obstruction space, so $M_C(r, d)$ is smooth, and:

$$\dim(M_C(r, d)) = \dim(H^1(C, \mathrm{End}(E))) = \chi(C, \mathrm{End}(E)) + 1 = r^2(g - 1) + 1$$

by Riemann-Roch (and Schur: $h^0(C, \mathrm{End}(E)) = 1$ since E is stable).

Finally, I want to use descent to describe:

The Determinantal Line Bundle: Going back to the semi-stable points of the Quot scheme $X_{SS} \subset \mathrm{Quot}(V \otimes \mathcal{O}_C(-n), P(m))$ let's assume that, in fact, far from being relatively prime, we actually have:

$$d = r(g-1)$$

Proposition 4.5: There is a scheme structure on the subset

$$\Theta := \{E \mid \mathrm{H}^0(C, E) \neq 0\} \subset \mathcal{M}^{r,r(g-1)}(C)$$

making it an ample Cartier divisor.

Proof: Let n be chosen as in Lemma 4.4, let $X^{SS} \subset \mathrm{Quot}_P(V \otimes \mathcal{O}_C(-D)/C)$ be the semistable locus, where $P(m) = mr$, and $D = \sum_{i=1}^n p_i$ is a divisor on C consisting of distinct points. If $\mathcal{U}$ is the universal quotient on $C \times X^{SS}$, then pushing down the exact sequence:

$$0 \to \mathcal{U} \to \mathcal{U}(D) \to \oplus_{i=1}^n \mathcal{U}(D)_{p_i} \to 0$$

yields the sequence:

$$0 \to \pi_{X_*^{ss}} \mathcal{U} \to \pi_{X_*^{ss}} \mathcal{U}(D) \xrightarrow{f} \oplus_{i=1}^n \mathcal{U}(D)_{p_i} \to R^1 \pi_{X_*^{ss}} \mathcal{U} \to 0$$

where the middle two sheaves are both locally free of rank $N = rn$. Moreover, since there exist semistable bundles E of degree $r(g-1)$ with $\mathrm{H}^1(C, E) = 0$, (e.g. $E = \oplus^r L$ where $\mathrm{H}^1(C, L) = 0$), the first sheaf vanishes! Finally, the map f is G-invariant, so f descends, and $\wedge^N(f)$, a (nonzero) section of the line bundle $\mathcal{L} := Hom(\wedge^N \pi_{X_*^{ss}} \mathcal{U}(D), \otimes_{i=1}^n \wedge^r \mathcal{U}(D)_{p_i})$ descends to a section s which vanishes precisely on Θ. If $m > M$ is fixed, then $\mathcal{O}_X(1) := \wedge^{mr} \pi_* \mathcal{U}(m)$ is the linearization used in Theorem 4VB to define X^{SS}. In particular, some power of $\mathcal{O}(1)$ descends to an ample line bundle on $\mathcal{M}^{r,d}(C)$. We claim that there are integers a and b such that $\mathcal{L}^a$ and $\mathcal{O}(b)$ differ by the pullback of a line bundle from $\mathrm{Pic}^d(C)$. This implies that Θ is ample.

But $\wedge^N \pi_{X_*^{ss}} \mathcal{U}(D)$ is trivial, naturally isomorphic to $\wedge^N V \otimes \mathcal{O}$, and the difference between $\wedge^{cr} \pi_{X_*^{ss}} \mathcal{U}(c)$ and $\wedge^{(c+1)r} \pi_{X_*^{ss}} \mathcal{U}(c+1)$ is a translate of the bundle $\wedge^r \mathcal{U}_p$ by the pullback of a line bundle from $\mathrm{Pic}^d(C)$ ($p \in C$ is an arbitrary point). The result is therefore immediate, since up to translation, $\mathcal{L}$ and $\mathcal{O}(1)$ are powers of the same line bundle.