5. **Vector Bundles on a Smooth Curve.** We will construct projective moduli spaces for semistable vector bundles on a smooth projective curve \( C \) by applying GIT to a suitable Grothendieck Quot scheme. The construction we present here is due to Carlos Simpson.

Let \( E \) be a vector bundle on a smooth projective curve \( C \) of genus \( g \).

**Definition:**
(a) The **slope** \( \mu(E) = \deg(E)/\rk(E) \).
(b) \( E \) is **stable** if \( \mu(F) < \mu(E) \) for all proper subbundles \( F \subset E \).
(c) \( E \) is **semistable** if \( \mu(F) \leq \mu(E) \) for all \( F \subset E \).

**Lemma 5.1:** If \( 0 \to F \to E \to G \to 0 \) is an exact sequence of vector bundles, then \( \mu(F) \geq \mu(E) \) (resp. \( > \)) if and only if \( \mu(E) \geq \mu(G) \) (resp. \( > \)).

**Proof:** Arithmetic! If \( a,b,c,d > 0 \), then \( \frac{a}{c} > \frac{a+b}{c+d} \) if and only if \( \frac{b}{d} < \frac{a+b}{c+d} \).

**Examples:**
(i) Every vector bundle on \( \mathbb{P}^1 \) splits as a sum of line bundles, so only the line bundles \( \mathcal{O}_{\mathbb{P}^1}(d) \) are stable, and only \( \mathcal{O}_{\mathbb{P}^1}(d)^{\oplus n} \) are semistable.
(ii) \( E \) is (semi-)stable iff the dual bundle \( E^* \) is (semi-)stable.
(iii) \( E \) is (semi-)stable iff \( E \otimes L \) is (semi-)stable for all line bundles \( L \).
(iv) If \( E \) is semistable of rank \( r \) and degree \( d \) and:
   (a) \( d < 0 \), then \( \mathcal{H}^0(C, E) = 0 \).
   (b) \( d > r(2g - 2) \), then \( \mathcal{H}^1(C, E) = 0 \).
   (c) \( d > r(2g - 1) \), then \( E \) is generated by its global sections.

**(Schur’s) Lemma 5.2:**
(a) If \( E \) and \( F \) are stable with the same slope, then any map \( f : E \to F \) is either 0 or an isomorphism.
(b) The only automorphism of a stable bundle \( E \) is scalar multiplication.
(c) (Jordan decomposition) If \( E \) is semistable, there is a filtration:

\[
0 = E_0 \subset E_1 \subset ... \subset E_n = E
\]

such that \( F_i := E_i/E_{i-1} \) is a stable vector bundle and each \( \mu(F_i) = \mu(E) \).

The filtration is not canonical, in general, but the **associated graded** bundle \( \oplus_{i=1}^n F_i \) is independent of the choice of filtration.
Proof: If \( f : E \to F \) is not zero, then both \( \ker(f) \) and \( E/\ker(f) \) are bundles. If \( f \) isn’t injective, then the stability of \( E \) implies \( \mu(\ker(f)) < \mu(E) \) and by Lemma 5.1, \( \mu(E/\ker(f)) > \mu(E) = \mu(F) \), contradicting the stability of \( F \). So \( f \) is injective, and surjective by the stability of \( F \). This gives (a).

If \( \alpha : E \to E \) is an automorphism, let \( \lambda_x \) be an eigenvalue of the restriction of \( \alpha \) to the fiber of \( E \) over \( x \in C \). Then \( \alpha - \lambda_x(\text{id}) \) drops rank at \( x \), so it is not an isomorphism, and must be zero by (a) and we have (b). Finally, (c) follows from (a) by the usual Jordan-H"older decomposition.

**Harder-Narasimhan** Lemma 5.3: If \( E \) is an any vector bundle on \( C \), then there is a filtration:

\[
0 = E_0 \subset E_1 \subset \ldots \subset E_n = E
\]

such that \( F_i := E_{i+1}/E_i \) are semistable vector bundles, with \( \mu(F_i) > \mu(F_{i+1}) \).

This filtration is uniquely determined by the property that if \( F \subset E \) is any sub-bundle with \( \mu(F) \geq \mu(E_i) \), then \( F \subset E_i \).

**Proof:** Let \( S = \{ a \mid a < \mu(E) \text{ and } a = \mu(Q) \text{ for some quotient } E \to Q \} \).

We claim first that \( S \) is a finite set. Indeed, let \( D \) be a divisor of large enough degree so that \( E(D) \) is generated by its sections. Then any \( Q(D) \) is also generated by its sections, so \( \deg(Q(D)) \geq 0 \) and \( \mu(Q) \geq -\deg(D) \). So the elements of \( S \) are bounded below (and above!) and since the denominators are bounded above by \( r \), it follows that \( S \) is finite.

Finiteness of \( S \) implies that the set of slopes of sub-bundles \( F \subset E \) is bounded from above. Let \( E_1 \subset E \) be the sub-bundle of maximal rank among those of maximal slope. Then \( E_1 \) is semi-stable and \( E/E_1 \) is a vector bundle. If \( F \subset E \) is another sub-bundle with \( \mu(F) = \mu(E_1) \), then the span of \( F \) and \( E_1 \) is yet another sub-bundle of the same slope (since the kernel of the map from \( F \oplus E_1 \) to the span must have the same slope). Since \( E_1 \) was of maximal rank, it follows that \( F \subset E_1 \) which then has the desired property.

Now suppose inductively that the lemma holds for \( F = E/E_1 \). We may use the Harder-Narasimhan filtration of \( F \):

\[
0 = F_0 \subset F_1 \subset \ldots \subset F_{n-1} = F = E/E_1
\]

to uniquely define \( E_{i+1} \) by the condition that \( E_{i+1}/E_1 = F_i \). And it follows that this filtration has the desired property.
Thus every vector bundle on $C$ is an extension of stable vector bundles, which are the “indecomposable” objects. They also have the smallest possible group of automorphisms, namely $\mathbb{C}^*$, though there are vector bundles with this automorphism group (called simple vector bundles) which are not stable.

The following theorem is due to Narasimhan and Seshadri:

**Theorem 5.4:** For each pair $(r, d)$ of coprime positive integers, the functor:

- **Obj**: schemes $S$ with an equivalence class of vector bundles $E$ on $C \times S$ with the property that each $E_s$ is stable, of rank $r$ and degree $d$
  (and $E \sim F \iff E \cong F \otimes \pi^* \Lambda$ for some line bundle $\Lambda$ on $S$)

- **Mor**: morphisms $\phi : S \to S'$ such that $(\phi, \text{id})^* E' \sim E$

is represented by a projective scheme $M_C(r, d)$ which is irreducible and smooth, of dimension $r^2(g-1) + 1$.

**Proof:** We need two key lemmas, the first solving a GIT problem, and the second having to do with the boundedness of families of sheaves on $C$.

**(GIT) Lemma 5.5:** If $V$ and $W$ are vector spaces and $M$ is an integer, let

$$G(V \otimes W, M)$$

be the Grassmannian of $M$-dimensional quotients of $V \otimes W$. Then a point $\psi \in G(V \otimes W, M)$ is semistable (resp. stable) with respect to the natural line bundle and linearization of $\text{SL}(V)$ if and only if

$$\frac{\dim(H)}{\dim(V)} \leq \frac{\dim(\psi(H \otimes W))}{M} \quad (\text{resp. } <)$$

for every proper subspace $H \subset V$.

**Proof:** Let $N = \dim(V)$ and $R = \dim(W)$ with a fixed basis $w_1, \ldots, w_R$. An point $\psi \in G(V \otimes W, M)$ lifts to $\bar{\psi} = \wedge^M \psi \in \wedge^M(V \otimes W)^*$ in the natural linearization. Given any basis $e_1, \ldots, e_N$ of $V$ and dual basis $x_1, \ldots, x_N$, we’ll call $e_{i_1} \otimes w_{j_1} \wedge \ldots \wedge e_{i_M} \otimes w_{j_M}$ the induced basis of Plücker vectors. Thus the coordinates of $\bar{\psi}$ are the values:

$$\wedge^M \psi(e_{i_1} \otimes w_{j_1} \wedge \ldots \wedge e_{i_M} \otimes w_{j_M}) = \psi(e_{i_1} \otimes w_{j_1}) \wedge \ldots \wedge \psi(e_{i_M} \otimes w_{j_M}) \in \mathbb{C}$$

which are zero if and only if the $\psi(e_{i_k} \otimes w_{j_l})$ are not linearly independent.
If $\lambda = \text{diag}\{t^1, \ldots, t^N\}$ is a 1-PS of $\text{SL}(V)$ and $x_1, \ldots, x_N$ is the associated (dual) basis, we’ll say the weight of the Plücker vector above is $\sum_{j=1}^{M} r_{ij}$. Then $\psi$ is $\lambda$-unstable for this $\lambda$ if and only if $\wedge^M \psi$ vanishes on every Plücker vector of nonpositive weight.

Suppose $H \subset V$ has dimension $n$, $\dim(\psi(H \otimes W)) = m$ and $\frac{n}{N} > \frac{m}{M}$. Let $e_1, \ldots, e_n$ be a basis of $H$, extended to a basis $e_1, \ldots, e_N$ of $V$ and let $\lambda = \text{diag}\{t^{n-N}, \ldots, t^{n-N}, t^n, \ldots, t^n\}$ for the dual basis. For each Plücker vector, if $\wedge^M \psi(e_{i_1} \otimes w_{j_1} \wedge \ldots \wedge e_{i_M} \otimes w_{j_M}) \neq 0$, then $\psi(e_{i_1} \otimes w_1), \ldots, \psi(e_{i_M} \otimes w_M)$ must be linearly independent, so the $e_{ij}$ must involve at most $m$ of the $e_1, \ldots, e_n$ vectors, thus its weight must be at least $m(n - N) + (M - m)n$. But $Mn - mN > 0$ by assumption, so $\wedge^M \psi$ is $\lambda$-unstable for this $\lambda$.

Conversely, let $\lambda$ be any 1-PS, diagonalized as $\lambda = \text{diag}\{t^1, \ldots, t^n\}$ for a basis $x_1, \ldots, x_N$. If $\psi$ is $\lambda$-unstable, let $H_n$ be the span of $e_1, \ldots, e_n$, and let $m_n = \dim(\psi(H_n \otimes W))$. Then $\lambda$-instability tells us:

\[
(*) \quad r_1 m_1 + r_2 (m_2 - m_1) + \ldots + r_N (M - m_{N-1}) > 0
\]

because it is the minimal weight of a Plücker vector on which $\wedge^M \psi$ is nonzero.

I claim that for some $n$, the “averaged” weights also satisfy:

\[
\frac{1}{n} (r_1 + \ldots + r_n) m_n + \frac{1}{N - n} (r_{n+1} + \ldots + r_N) (M - m_n) > 0
\]

It then follows that $\frac{n}{N} > \frac{m_n}{M}$ holds for $H = H_n$. To see the claim, notice first that if $m_{i+1} - m_i \leq m_i - m_{i-1}$, then we may combine $r_i$ and $r_{i+1}$, replacing them with their average $\frac{r_i + r_{i+1}}{2}$ without decreasing $(*)$. The averaged weights are the same as the original, so we may assume the sequence of differences is increasing: $\Delta_1 := m_1 < \Delta_2 := m_2 - m_1 < \ldots < \Delta_N := m_N - m_{N-1}$. Now consider the linear function:

\[
L(t_1, t_2, \ldots, t_N) = r_1 t_1 + \ldots + r_N t_N
\]

which by assumption satisfies $L(\Delta_1, \ldots, \Delta_N) > 0$, and consider its values at the points:

\[
p_n := \left(\frac{\sum_{i=1}^{n} \Delta_i}{n}, \ldots, \frac{\sum_{i=n+1}^{N} \Delta_i}{n - n}, \frac{\sum_{i=n+1}^{N} \Delta_i}{N - n}, \ldots, \frac{\sum_{i=n+1}^{N} \Delta_i}{N - n}\right) \in \mathbb{R}^N
\]

These points are linearly independent and $L(p_N) = 0$ (the $r_i$ sum to zero). Thus they span the hyperplane $\{ \sum t_i = \sum \Delta_i \} \subset \mathbb{R}^N$ and in particular, $(\Delta_1, \ldots, \Delta_N) = \sum_{i=1}^{N-1} y_i p_i - y_N p_N$ for positive values $y_i$, so some $L(p_i) > 0$. 4
Thus, we’ve shown that $\frac{n}{N} > \frac{m}{M}$ if and only if $\psi$ is $\lambda$-unstable for some $\lambda$. By the numerical criterion, this proves the “semi-stable” part of the lemma, and the stable part is proved by replacing each “>” by a “≥.”

**Lemma 5.6:** Let $p \in C$, and $\mathcal{O}_C(1) := \mathcal{O}_C(p)$. If $n > 2g - 1 - \frac{d}{r}$ then:

(a) If $E$ is semistable of rank $r$ and degree $d$, then $H^1(C, E(n)) = 0$, $E(n)$ is generated by global sections and for all subbundles $F \subset E$:

$$\frac{h^0(C, F(n))}{\text{rank}(F)} \leq \frac{h^0(C, E(n))}{\text{rank}(E)}$$

with equality if and only if $F$ is semistable, $h^1(C, F(n)) = 0$ and for all $m$:

$$\frac{\chi(C, F(m))}{\text{rank}(F)} = \frac{\chi(C, E(m))}{\text{rank}(E)}$$

(b) If $\mathcal{E}$ is any coherent sheaf of the same Hilbert polynomial $\chi(C, \mathcal{E}(n)) = P(m) = rm + d - r(g - 1)$ as a vector bundle of rank $r$ and degree $d$, and if every vector bundle quotient $\mathcal{E} \to G$ satisfies:

$$\frac{h^0(C, G(n))}{\text{rank}(G)} \geq \frac{P(n)}{r},$$

then $\mathcal{E}$ is itself a semistable vector bundle of rank $r$ and degree $d$.

**Proof:** The key point is the following. If $E$ is a semistable bundle of rank $r$ and if $h^1(C, E) \neq 0$, then $h^0(C, E) \leq rg$ independent of the degree of $E$. This is well-known for line bundles, since every $L$ with $h^1(C, L) \neq 0$ is a subsheaf of the canonical line bundle and $h^0(C, \omega_C) = g$. But here’s a proof that generalizes. If $h^1(C, L) \neq 0$, then $\deg(L) \leq 2g - 2$. If $h^0(C, L) \geq g$, then there is a section $s \in H^0(C, L)$ vanishing at any $p_1, ..., p_{g-1} \in C$. If the $p_i$ are “general,” then $h^0(C, \mathcal{O}_C(\sum p_i)) = 1$ and from:

$$0 \to \mathcal{O}_C(\sum p_i) \to L \to \tau \to 0$$

and $\deg(\tau) \leq g - 1$ it follows that $h^0(C, L) \leq 1 + h^0(C, \tau) \leq g$.

If $E$ is semistable of rank $r$ and $h^1(C, E) \neq 0$, then by Example (iv) we have $\deg(E) \leq r(2g - 2)$ and $\deg(F) \leq r'(2g - 2)$ for any subbundle $F \subset E$ of rank $r'$. If $h^0(C, E) \geq rg$, then there is a section $s \in H^0(C, E)$ vanishing at any $g - 1$ points, and then we get $\mathcal{O}_C(\sum p_i) \subset E$ spanning a line bundle $L \subset E$ of degree $\leq 2g - 2$ satisfying $h^0(C, L) \leq g$ as above.
If \( h^0(C, E) \geq rg \), then \( h^0(C, E/L) \geq (r - 1)g \) and we can find a section \( s' \in H^0(C, E/L) \) which again can be chosen to vanish at \( g - 1 \) general points. The two sections \( s, s' \) will span a sub-bundle \( F \subset E \) of degree \( \leq 2(2g - 2) \) which then has at most \( 2 + 2(g - 1) = 2g \) sections from the exact sequence:

\[
0 \to \mathcal{O}_C(\sum p_i) \oplus \mathcal{O}_C(\sum p'_i) \to F \to \tau \to 0
\]

and then one considers sections of \( E/F \), etc.

We already saw that the first part of (a) is satisfied in Example (iv). Notice that:

\[
\chi(C, E(n)) = \frac{P(n)}{r} = n + \frac{d}{r} - (g - 1)
\]

Thus any semistable bundle \( F \) of rank \( r' \leq r \) and slope \( \mu \leq \frac{d}{r} \) must satisfy \( h^0(C, F(n)) = \chi(C, F(n)) \leq \frac{d}{r} P(n) \), or else \( h^0(C, F(n)) \leq r'g < \frac{d}{r} P(n) \) by the key point above (and the lower bound \( n > 2g - 1 - \frac{d}{r} \)).

If \( F \subset E \) and \( E \) is semistable, then every \( F_i \) in the Harder-Narasimhan filtration of \( F \) has slope at most \( \frac{d}{r} \), so each subquotient \( F_i \) satisfies

\[
\frac{h^0(C, F_i(n))}{\text{rk}(F_i)} \leq \frac{P(n)}{r}
\]

and by Lemma 5.1, we have the same inequality for \( F \). If equality holds, then it must hold for every \( F_i \), and we conclude that every \( F_i \) has slope exactly \( \frac{d}{r} \), so \( F \) is semistable, and \( \chi(C, F_i(n)) = \frac{P(n)}{r \text{rk}(F)} \) for all \( m \). This proves (a).

If \( E \) is the sheaf in (b), let \( T \subset E \) be the torsion subsheaf, and let \( G \) be the (semistable) quotient of smallest rank in the Harder-Narasimhan filtration of \( E/T \). Since \( \mu(G) \leq \mu(E/T) \leq \frac{d}{r} \), it follows as above that:

\[
\frac{h^0(C, G(n))}{\text{rk}(G)} \leq \frac{P(n)}{r}
\]

with equality if and only if \( T = 0 \) and \( \mu(G) = \mu(E) \). But this means \( E = G! \)

We are ready for the proof of Theorem 5.4 now.

Let \( P(m) = rm + d - r(g - 1) \) be the Hilbert polynomial of a bundle of rank \( r \) and degree \( d \) as in Lemma 5.6, and for fixed \( n > 2g - 1 - \frac{d}{r} \), consider the Quot scheme

\[
\text{Quot}(V \otimes \mathcal{O}_C(-n), P(m))
\]

where \( V \) is a vector space of rank \( P(n) \) (with \( \text{SL}(V) \) action).
If $E$ is any semistable bundle of rank $r$ and degree $d$, then as we have already remarked, $E(n)$ is generated by global sections and $H^1(C, E(n)) = 0$, so $h^0(C, E(n)) = P(n)$ and the global section map $V \cong H^0(C, E(n)) \to E(n)$ twists to give a point $V \otimes \mathcal{O}_C(-n) \to E$ of the Quot scheme. Recall that the Quot scheme embeds in Grassmannians:

$$
\iota_m : \text{Quot}(V \otimes \mathcal{O}_C(-n), P(m)) \hookrightarrow G(V \otimes W, M)
$$

for each $M = P(m)$ and sufficiently large $m$, and $W = H^0(C, \mathcal{O}_C(m-n))$.

We will consider the GIT quotient of the Quot scheme for the action of $SL(V)$ induced from the Grassmannian (and linearized as in Lemma 5.5). For large enough $m$, the two notions of vector bundle (semi-)stability and GIT (semi-)stability will coincide. When $n, d$ are coprime, semi-stability equals stability, and the GIT quotient will represent the functor. Deformation theory will then show that the quotient is smooth, of the indicated dimension.

If $x \in \text{Quot}(V \otimes \mathcal{O}_C(-n), P(m))$, let $q_x : V \otimes \mathcal{O}_C(-n) \to \mathcal{E}_x$ be the corresponding quotient. Such a quotient induces a map $V \to H^0(C, \mathcal{E}_x(n))$ and for each (large enough) $m$, let $\psi_x : V \otimes W \to H^0(C, \mathcal{E}_x(m))$ be the image point in the Grassmannian. Let $X_U(m), X_{SS}(m)$ and $X_S(m)$ be the loci of unstable, semistable and stable points for this embedding.

**Step 1:** For large enough $m$ (independent of $x$), if

(i) $\mathcal{E}_x$ is a semistable vector bundle and

(ii) $V \to H^0(C, \mathcal{E}_x(n))$ is an isomorphism, then $x \in X_{SS}(m)$.

**Proof:** If $x \in X_U(m)$, then by Lemma 5.5, there is an $H \subset V$ so that:

$$(*) \quad \frac{\dim(H)}{P(n)} > \frac{\dim(\psi_x(H \otimes W))}{P(m)}
$$

and we need to show that the existence of such an $H$ violates (i) or (ii).

For each $H \subset V$, let $\mathcal{F}_{x,H} \subset \mathcal{E}_x$ be the subsheaf generated by $H \otimes \mathcal{O}_C(-n)$. Assuming (ii), we see that $H \cong H^0(C, \mathcal{F}_{x,H}(n))$. Consider:

$$
0 \to \mathcal{K}_{x,H} \to H \otimes \mathcal{O}_C(-n) \to \mathcal{F}_{x,H} \to 0
$$

and choose $m_0$ so that $m \geq m_0$ implies that $H^1(C, \mathcal{K}_{x,H}(m)) = 0$ and $H^1(C, \mathcal{F}_{x,H}(m)) = 0$, for all $H \subset V$ and all $x$ in the Quot scheme. Then $\psi_x(H \otimes W) = H^0(C, \mathcal{F}_{x,H}(m))$ is of dimension $\chi(C, \mathcal{F}_{x,H}(m))$. 

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Thus if \((\ast)\) holds, then:

\[
\frac{\dim(H^0(C, F_{x,H}(n)))}{P(n)} > \frac{\chi(C, F_{x,H}(m))}{P(m)}.
\]

On the other hand, if we assume (i), then Lemma 5.6 (a) gives us:

\[
\frac{\dim(H^0(C, F_{x,H}(n)))}{P(n)} < \frac{\rank(F_{x,H})}{r} \tag{equality would force equality in the previous formula}. 
\]

But \(\chi(C, F_{x,H}(m)) = r'm + d' - r'(g - 1)\) for \(r' = \rank(F_{x,H})\) and \(d' = \deg(F_{x,H})\), so we are getting:

\[
\frac{r'}{r} > \frac{\dim(H^0(C, F_{x,H}(n)))}{P(n)} > \frac{r'(m + d') - (g - 1)}{r(m + d') - (g - 1)}
\]

There are only finitely many \(d'\) and \(r'\), so since the right side approaches the left as \(m \to \infty\), we obtain a contradiction when \(m\) is sufficiently large.

**Step 2:** After possibly increasing \(m\) again, if \(x \in X_{SS}(m)\) then:

(a) The map \(V \to H^0(C, E_x(n))\) is an isomorphism and

(b) The quotient \(E_x\) is a semistable vector bundle.

**Proof of Step 2:** By Lemma 5.5, if \(x \in X_{SS}(m)\) (for any \(m\)), then \(V \to H^0(C, E_x)\) must be injective, because any kernel would yield an \(H\) such that \(\psi_x(H \otimes W) = 0\). Similarly, for all \(H \subset V\), we must have:

\[
(\ast) \quad \frac{\dim(H)}{\dim(\psi_x(H \otimes W))} \leq \frac{P(n)}{P(m)}
\]

Suppose \(E_x\) were not a bundle or not semistable. Then by Lemma 5.6(b), we could find a quotient bundle \(E_x \to G\) so that \(\frac{h^0(C, G(n))}{\rank(G)} < \frac{P(n)}{r}\). Let \(H\) be the kernel of the map \(V \to H^0(C, G(n))\) for such a quotient, and let \(F_{x,H}\) be the image of \(H\) in \(E_x\). If \(F_{x,H}\) is torsion, then there is a universal bound on its length, say \(K\), and we can choose \(m\) so that \(\frac{P(n)}{P(m)} < \frac{1}{K}\), violating \((\ast)\).

Otherwise, by the arithmetic of Lemma 4.0, we have:

\[
(**) \quad \frac{\dim(H)}{\rank(F_{x,H})} > \frac{P(n)}{r}
\]
where the rank of $F_{x,H}$ is the generic rank, which is the coefficient of $m$ in $\chi(C, F_{x,H}(m))$. Since $\chi(C, F_{x,H}(m)) = \dim(\psi_x(H \otimes W))$ (see Step 1) we get a contradiction to (*), perhaps after boosting $m$ again, from the fact that there is a uniform upper bound on the constant terms of the Hilbert polynomials of the $F_{x,H}$. So $E_x$ is semistable. Finally, since $E_x$ is semistable, the map $V \to H^0(C, E_x(n))$, which we already saw was injective, must be an isomorphism by Lemma 5.6(a).

**Step 3:** For sufficiently large $m$

(a) $x \in X_S(m) \iff x \in X_{SS}(m)$ and $E_x$ is stable.

(b) For any $x \in X_{SS}(m)$, the closed orbit $O(\bar{x}') \subset \bar{O}(\bar{x})$ corresponds to an $E_{x'}$ that is isomorphic to the associated graded of $E_x$.

**Proof of Step 3:** (a) is the same argument as Steps 1 and 2. For (b), if $x \in X_{SS}(m) - X_S(m)$, let $F \subset E_x$ be a proper subbundle of the same slope, and let $H \subset V$ be the kernel of the map $V \to H^0(C, G(n))$, where $G = E_x/F$. Consider the induced extension:

$$\dagger : 0 \to F \to E_x \to G \to 0$$

of vector bundles of the same slope.

If we take $e_1, ..., e_n$ spanning $H$, extend to a basis of $V$, and consider the 1-PS subgroup $\lambda = \text{diag}\{t^{n-N}, ..., t^{n-N}, t^n, ..., t^n\}$, then $\lambda$ acts on the extension class of $\dagger$ in $H^1(C, G^* \otimes F)$ by multiplication by $t^N$, taking it to the split extension in the limit as $t \to 0$. We can repeat the process until we get to the associated graded of $E_x$. Since the associated graded is uniquely determined by Schur’s Lemma, and there must be some closed orbit in the closure of the orbit of $E_x$, this must be the one!

We have proved that for large $m$ (and arbitrary $(r, d)$!), the GIT quotient:

$$G(V \otimes W, M) \supset \text{Quot}(V \otimes O_C(-n), P(m)) \twoheadrightarrow M_C(r, d)$$

has the following properties:

(i) $M_C(r, d)$ is a projective scheme

(ii) The points of $M_C(r, d)$ correspond to associated gradeds of semistable vector bundles of rank $r$ and degree $d$.

The $M_C(r, d)$ are independent of the choice of (large enough) $m$ because they are all categorical quotients of the same open subscheme $X_{SS}(m)$!
Now take any vector bundle $E$ on $S \times C$ and consider the sheaf $\pi_{S*}(E(n))$, where $E(n) = E \otimes \pi_C^*O_C(n)$. If $E$ is a family of semi-stable bundles of rank $r$ and degree $d$ (and $n > 2g - 1 - \frac{d}{r}$), then $\pi_{S*}(E(n))$ is locally free of rank $P(n)$ and the natural map:

$$\pi_{S*}(E(n)) \otimes \pi_C^*O_C(-n) \to E$$

is a surjective map of vector bundles. Locally (on $S$) we may trivialize $\pi_{S*}(E(n))$ each trivialization determines $U_i \to \text{Quot}(V \otimes O_C(-n), P(m))$ which do not patch as maps to the Quot scheme, but do patch to:

$$\phi : S \to M_C(r,d)$$

So to prove that $M_C(r,d)$ represents the functor, we need only to find a universal vector bundle $U$ on $C \times M_C(r,d)$ with the property that any vector bundle $E$ as above satisfies $E \sim (\phi, 1)^*U$. We will use the following:

**(Descent) Lemma 5.7:** Given a linearized $G$-action on $(X, L)$ and a vector bundle $F$ on $X_{SS}(L)$ with a $G$-action, then $F$ descends to the GIT quotient:

$$\overline{f} : X_{SS}(L) \to X^G$$

if and only if for each closed orbit $O(\bar{x})$, the stabilizer $G_x \subset G$ also stabilizes the fiber $F_x$ of $F$ at $x \in X_{SS}(L)$.

**Proof (Kempf):** If $F$ descends, then by definition, $F = \overline{f}^*(\overline{\mathcal{F}})$ is the pull-back of the descended bundle $\overline{\mathcal{F}}$, so $G_x$ acts trivially on the fibers $F_x$.

To prove the converse, it suffices to find, for each $x' \in X_{SS}(L)$, an affine neighborhood $V'$ of $y := \overline{f}(x') \in X^G$ and a trivialization of $F|_{\overline{f}^{-1}(V')}$ by $G$-invariant sections. Given $x \in \overline{f}^{-1}(y)$ with closed $O(\bar{x})$, there are $r = \text{rk}(F)$ $G$-invariant sections of the restriction $F|_{O(\bar{x})}$ which trivialize $F$ along the orbit. Indeed, since we assumed that $G_x$ acts trivially on $F_x$, we can translate a basis $e_1, \ldots, e_r \in F_x$ by $G$ to obtain the desired sections $Ge_1, \ldots, Ge_r$.

Let $y \in V = D(h)^G$ for a homogeneous, invariant $h$ in the homogeneous coordinate ring of $X$. Then $\overline{f}^{-1}(V) = D(h) \subset X_{SS}(L)$ is also affine and, as in projective GIT, the map $D(h) \to D(h)^G$ is the affine GIT quotient. I claim that there is a Reynolds operator $E : H^0(D(h), F) \to H^0(D(h), F)^G$. To see this, it suffices to show that $G$ acts rationally on $H^0(D(h), F))$.
But if we choose an open affine $U \subset D(h)$ on which $F$ trivializes, then an $s \in H^0(U, F)$ gives rise to a regular function $\phi : G \times U \to \mathbf{C}^r$ defined by $(g, x) \mapsto gs(g^{-1}x)$. Then if $G = \text{Spec}(A)$ and $U = \text{Spec}(B)$, we have $\phi = \sum a_i \otimes \bar{b}_i$, where $a_i \in A$ and $\bar{b}_i : U \to \mathbf{C}^r$, and as before, $G_{s|U}$ is contained in the span, $W$, of the $\bar{b}_i$. Since the restriction of sections from $D(h)$ to $U$ is injective, we prove rationality by intersecting $W$ with $H^0(D(h), F)$.

Now take the sections $Ge_1, \ldots, Ge_r$ spanning $F|_{O(x)}$ and extend them to sections $s_1, \ldots, s_r$ of $F|_{D(h)}$, which is possible since $D(h)$ is affine. Apply the Reynolds operator to get invariant sections $E(s_1), \ldots, E(s_r)$, which still restrict to $Ge_1, \ldots, Ge_r$ on $O(x)$ (by property (i) of the Reynolds operator). Finally, consider the closed invariant subset $Z \subset D(h)$ where $E(s_1), \ldots, E(s_r)$ fail to span $F$. The image $\phi(Z) \subset D(h)^G$ is closed and does not contain $f(x')$, so we can shrink $V = D(h)$ to a smaller open neighborhood $x' \in V'$ for which $E(s_1), \ldots, E(s_r)$ do span, finishing the proof.

Now suppose $(r, d)$ are coprime and consider the universal quotient:

$$V \otimes \mathcal{O}_{C \times X_{SS}}(-n) \to \mathcal{E}$$

on $C \times X_{SS} \subset C \times \text{Quot}(V \otimes \mathcal{O}_C(-n), P(m))$. It is a consequence of flatness that $\mathcal{E}$ is a vector bundle of relative rank $r$ and degree $d$ over $X_{SS}$, and $\mathcal{E}$ is an $\text{SL}(V)$-bundle by virtue of the fact that the Quot scheme represents the functor. That is, the action on $\mathcal{E}$ is obtained by pulling back the universal quotient under the action of $\text{SL}(V)$ on $C \times X_{SS}$. Since $(r, d)$ are coprime, each of the bundles $\mathcal{E}_x$ is stable (there is no smaller $r'$ with $\frac{d}{r'} = \frac{d}{r}$) and since $\text{Aut}(\mathcal{E}_x) = \mathbf{C}^*$, it follows that each stabilizer $\text{SL}(V)_x = \text{SL}(V) \cap \mathbf{C}^* \cong \mu_{P(n)}$ is the group of $P(n)$-th roots of unity. Again, since $(r, d)$ are coprime, it follows that $P(n) = nr + d - r(g - 1)$ and $r$ are coprime, so we can solve(!)

$$1 + ar = bP(n)$$

and it follows that the action of the stabilizers $\text{SL}(V)_x$ on $\mathcal{E} \otimes (\wedge^r \mathcal{E})^{\otimes a}$ is trivial, and this bundle, at least, descends.

In the rank $r = 1$ case, take $b = 1$ and $a = P(n) - 1$, and this gives us a bundle $\mathcal{L}_n$ on $C \times \text{Pic}^d(C)$ for the Picard scheme $\text{Pic}^d(C) = M_C(1, d)$, which has the property that $(\mathcal{L}_n)_x$ is the $P(n)$th tensor power of the line bundle associated to $x \in M_C(1, d)$. But we may descend for two consecutive values of $n$, which gives consecutive values of $P(n) = n + d - (g - 1)$ and we get $\mathcal{L} = \mathcal{L}_{n+1} \otimes \mathcal{L}_n^*$ which then has the desired universal property.
In the arbitrary rank case, we have the “determinant” morphism:

\[ X_{SS} \to M_C(r, d) \xrightarrow{\text{det}} M_C(1, d) \]

coming from the family of line bundles \( \wedge^r \mathcal{E} \) on \( C \times X_{SS} \) which factors through \( M_C(r, d) \) because it is a categorical quotient! Thus we may take the vector bundle \( \mathcal{E}_n \) on \( C \times M_C(r, d) \) descended from \( \mathcal{E} \otimes (\wedge^r \mathcal{E})^{\otimes a} \) and “tensor back” by \((1, \text{det})^* \mathcal{L})^{\otimes -a}\) to obtain \( \mathcal{U} \) which has the desired universal property.

**Claim:** Each \( M_C(r, d) \) is irreducible.

**Proof:** First of all, notice that there are isomorphisms:

\[ \cdots \otimes_{\mathcal{O}_C(1)} M_C(r, d) \xrightarrow{\otimes_{\mathcal{O}_C(1)}} M_C(r, d + r) \xrightarrow{\otimes_{\mathcal{O}_C(1)}} \cdots \]

which, in case \((r, d)\) are coprime are obtained by taking the tensor product \( \mathcal{U} \otimes \pi_C^* \mathcal{O}_C(1) \) (and in the non-relatively prime case are obtained by considering \( \mathcal{E} \otimes \pi_C^* \mathcal{O}_C(1) \) on the Quot scheme and using the categorical quotient). Thus we may assume that \( d > r(2g - 1) \) so all bundles are generated by sections.

When \( r = 1 \), this gives us a surjective map (in fact a projective bundle)

\[ u_d : \text{Sym}^d(C) \to M_C(1, d); \quad D \mapsto \mathcal{O}_C(D) \]

defined rigorously by using the “universal” Cartier divisor \( \mathcal{D} \subset C \times \text{Sym}^d(C) \) and using it to construct the family \( \mathcal{O}_{C \times \text{Sym}^d(C)}(\mathcal{D}) \) of line bundles. Evidently, \( \text{Sym}^d(C) \) is irreducible, as it is the quotient of \( C^d \) by the permutation group.

In rank \( r \), a choice of \( r + 1 \) general sections of a bundle \( E \) of rank \( r \) and degree \( d \) gives a surjection \( \mathcal{O}_C^{r+1} \to E \) with kernel \( (\wedge^r E)^* \). Dually, this means we can exhibit \( E^* \) as the kernel of a map:

\[ 0 \to E^* \to \mathcal{O}_C^{r+1} \to L \to 0 \]

where \( L = (\wedge^r E) \) (and then dualize to get \( E \)). It turns out that a general choice of \( r + 1 \) sections of a line bundle \( L \) of degree \( d \) has a semi-stable \( E \) as its kernel, and this gives a surjective morphism:

\[ G(r + 1, \pi_* \mathcal{L}) \subset U \to M_C(r, d) \]

from an open subset \( U \) of the Grassmann bundle over \( M_C(1, d) \) to \( M_C(r, d) \). Since \( M_C(1, d) \) is irreducible, it then follows that \( M_C(r, d) \) is irreducible too.
For smoothness and the dimension count, we will use deformation theory. Given a stable bundle $E$ of rank $r$ and degree $d$, the Zariski tangent space is the space of (equivalence classes of) vector bundles $E_{\epsilon}$ on $C \times \text{Spec}(k[\epsilon])$ with the property that $E_{\epsilon}|_C \cong E$. We may trivialize $E$ on an open cover $\cup U_i = C$ with intersections $U_{ij} = U_i \cap U_j$, and then $E$ is determined by transition functions:

$$G_{ij} \in GL(\mathcal{O}_C(U_{ij}))$$

satisfying the cocycle condition:

$$G_{jk} G_{ij} = G_{ik}$$
on triple intersections $U_{ijk}$. An extension of $E$ is given by an extension of the transition functions:

$$G_{ij} + \epsilon H_{ij} \in GL(\mathcal{O}_C(U_{ij})[\epsilon])$$

(the invertibility puts no constraint on the matrix $H_{ij}$) satisfying:

$$(G_{jk} + \epsilon H_{jk})(G_{ij} + \epsilon H_{ij}) = G_{ik} + \epsilon H_{ik}$$
or the original cocycle condition together with:

$$H_{jk} G_{ij} + G_{jk} H_{ij} = H_{ik}$$
on triple intersections $U_{ijk}$. But if we regard the $H_{ij}$ as sections of the (trivialized!) bundle End($E$), then this is precisely the cocycle condition to define an element of:

$$H^1(C, \text{End}(E))$$
(as the $G_{ij}$ transition the $H_{ij}$ to allow us to compare them on $U_{ik}$). And coboundaries are cocycles that give trivial deformations of $E$, so this is indeed the tangent space. Similarly, one checks that the obstruction space is:

$$H^2(C, \text{End}(E)) = 0$$
Thus on a curve $C$, there is no obstruction space, so $M_C(r, d)$ is smooth, and:

$$\dim(M_C(r, d)) = \dim(H^1(C, \text{End}(E))) = \chi(C, \text{End}(E)) + 1 = r^2(g - 1) + 1$$
by Riemann-Roch (and Schur: $h^0(C, \text{End}(E)) = 1$ since $E$ is stable).
Finally, I want to use descent to describe:

**The Determinantal Line Bundle:** Going back to the semi-stable points of the Quot scheme $X_{SS} \subset \text{Quot}(V \otimes \mathcal{O}_C(-n), P(m))$ let’s assume that, in fact, far from being relatively prime, we actually have:

$$d = r(g - 1)$$

**Proposition 4.5:** There is a scheme structure on the subset

$$\Theta := \{E \mid H^0(C, E) \neq 0\} \subset \mathcal{M}^{r,r(g-1)}(C)$$

making it an ample Cartier divisor.

**Proof:** Let $n$ be chosen as in Lemma 4.4, let $X_{SS} \subset \text{Quot}_p(V \otimes \mathcal{O}_C(-D)/C)$ be the semistable locus, where $P(m) = mr$, and $D = \sum_{i=1}^n p_i$ is a divisor on $C$ consisting of distinct points. If $U$ is the universal quotient on $C \times X_{SS}$, then pushing down the exact sequence:

$$0 \to U \to U(D) \to \bigoplus_{i=1}^n U(D)_{p_i} \to 0$$

yields the sequence:

$$0 \to \pi_{X_{SS}}U \to \pi_{X_{SS}}(D) \xrightarrow{f} \bigoplus_{i=1}^n U(D)_{p_i} \to R^1\pi_{X_{SS}}U \to 0$$

where the middle two sheaves are both locally free of rank $N = rn$. Moreover, since there exist semistable bundles $E$ of degree $r(g - 1)$ with $H^1(C, E) = 0$, (e.g. $E = \bigoplus^r L$ where $H^1(C, L) = 0$), the first sheaf vanishes! Finally, the map $f$ is $G$-invariant, so $f$ descends, and $\wedge^N(f)$, a (nonzero) section of the line bundle $L := \text{Hom}(\wedge^n \pi_{X_{SS}}U(D), \bigoplus_{i=1}^n \wedge^r U(D)_{p_i})$ descends to a section $s$ which vanishes precisely on $\Theta$. If $m > M$ is fixed, then $\mathcal{O}_X(1) := \wedge^{mr} \pi_*U(m)$ is the linearization used in Theorem 4VB to define $X_{SS}$. In particular, some power of $\mathcal{O}(1)$ descends to an ample line bundle on $\mathcal{M}^{r,d}(C)$. We claim that there are integers $a$ and $b$ such that $L^a$ and $\mathcal{O}(b)$ differ by the pullback of a line bundle from $\text{Pic}^d(C)$. This implies that $\Theta$ is ample.

But $\wedge^n \pi_{X_{SS}}U(D)$ is trivial, naturally isomorphic to $\wedge^N V \otimes \mathcal{O}$, and the difference between $\wedge^c \pi_{X_{SS}}U(c)$ and $\wedge^{(c+1)} \pi_{X_{SS}}U(c + 1)$ is a translate of the bundle $\wedge^p \mathcal{O}_p$ by the pullback of a line bundle from $\text{Pic}^d(C)$ ($p \in C$ is an arbitrary point). The result is therefore immediate, since up to translation, $L$ and $\mathcal{O}(1)$ are powers of the same line bundle.