Abstract Algebra. Math 6320. Bertram/Utah 2022-23.
Generators and Commutators

Let $S \subset G$ be a subset of a group. The subgroup $H(S) \subset G$ generated by S is the smallest subgroup containing S. Since the intersection of (any number of) subgroups is a subgroup, $H(S)$ is the intersection of all subgroups that contain S. Constructively,

$$H(S) = \{g \in G \mid g = w(S)\}$$

is the set of finite-length *words* in the elements of S, where a word is a finite product of elements of S and their inverses (with repetitions).

Example. As we noted earlier, the symmetric group S_n is generated by

$$S = \{(1\ 2),\ (2\ 3), \cdots,\ (n-1\ n)\}$$

Remark. The permutation that requires the longest word is the transposition:

$$(1\ n) = (1\ 2)(2\ 3)...(n-1\ n)...(2\ 3)(1\ 2)$$

Definition. The *commutator subgroup* $[G,G] \subset G$ is the group generated by:

$$\{[a,b] := aba^{-1}b^{-1} \mid a, b \in G\}$$

the set of all "commutators" of elements a and b.

Example. The commutator subgroup of an abelian group is trivial. On the other hand, within the symmetric group, the commutator of overlapping transpositions:

$$[(i\ j),(j\ k)]\ = (i\ j)(j\ k)(i\ j)(j\ k) = (i\ k\ j)$$

is a three-cycle, so all three-cycles are commutators, and $[S_n, S_n] = A_n$.

Proposition 1. (a) The commutator subgroup $[G,G]$ is normal in G.

(b) The quotient "abelianization" group $G^{\mathrm{ab}} := G/[G,G]$ is abelian and universal in the sense that every homomorphism $f : G \to A$ to an abelian group factors through $\overline{f} : G^{\mathrm{ab}} \to A$. In particular, G is abelian if and only if $[G,G] = \{e\}$.

(c) G is solvable if and only if the sequence of commutator subgroups:

$$\cdots \subset G_2 \subset G_1 \subset G_0 = G \text{ with } G_{i+1} = [G_i, G_i]$$

is eventually trivial.

Proof. (a) Note that $[a,b][b,a] = (aba^{-1}b^{-1})(bab^{-1}a^{-1}) = e$, so the inverse of a commutator is a commutator and every word in commutators is a product of commutators. Moreover, commutators conjugate to commutators:

$$g[a,b]g^{-1} = [gag^{-1}, gbg^{-1}] = [c_g(a), c_g(b)]$$

from which it follows that $g[a_1,b_1]\cdots[a_n,b_n]g^{-1} = [c_g(a_1), c_g(b_1)]\cdots[c_g(a_n), c_g(b_n)]$ so conjugation preserves words in commutators, i.e. $[G,G]$ is normal.

(b) Let $N = [G,G]$. Then G/N is abelian, since: $abN = ab(b^{-1}a^{-1}ba)N = baN$ and if $f : G \to A$ is a group homomorphism to an abelian group, then:

$$f(aba^{-1}b^{-1}) = f(a)f(b)f(a)^{-1}f(b)^{-1} = e$$

so $[G,G] \subset \ker(f)$, and f factors through $G^{\mathrm{ab}} = G/[G,G]$.

(c) A solvable group has a string of normal subgroups (one in the next) with abelian quotient groups, and conversely, so this follows from (b).

Remark. There is no direct analogue for abelan *subgroups* of a group G. The only canonical abelian subgroup is the center $Z(G)$, but this not generally the largest subgroup, nor is there a single largest abelian subgroup, since unlike intersections, the unions of subgroups are not subgroups.

Examples. (i) The commutator subgroup of a simple group G is $\{e\}$ when $G = C_p$ and is G itself when G is (simple and) not abelian.

(ii) The commutator subgroup of a dihedral group satisfies:

$$D_{2n}/[D_{2n}, D_{2n}] = C_2 \text{ when } n \text{ is odd, and } C_2 \times C_2 \text{ when } n \text{ is even}$$

If $g \in C_n$ is a generator, and $h \in C_2$ is a generator, then $hgh^{-1}g^{-1} = g^{-2} \in D_{2n}$ (from the realization of D_{2n} as a semi-direct product), and g^{-2} generates C_n when n is odd, but only half of C_n when n is even.

(iii) The fundamental group π_1 and first homology group H_1 of a path connected topological space X satisfy:

$$\pi_1^{\mathrm{ab}}(X, x) = \pi_1(X, x)/[\pi_1(X, x), \pi_1(X, x)] = \mathrm{H}_1(X, \mathbb{Z})$$

Next, we turn to finitely generated groups.

Definition. G is *finitely generated* if $G = H(S)$ for a finite subset $S \subset G$.

Evidently finite groups are finitely generated, as are the groups $\mathbb{Z}^n$.

Definition. The *free group* F_S on a (finite) set S is made up of equivalence classes of words $w(S)$ in the elements $s_1, ..., s_n \in S$ and their (formal) inverses $s_1^{-1}, ..., s_n^{-1}$.

Concatenation of words is multiplication (with the empty word as identity), and:

$$w_1(S)ss^{-1}w_2(S) \sim w_1(S)w_2(S) \text{ for words } w_1(S), w_2(S)$$

generates the equivalence relation, so that, for example:

$$(s_1 \cdots s_n)^{-1} = s_n^{-1} \cdots s_1^{-1} \text{ is the two-sided inverse}$$

Example. The abelianization of the free group on n generators is $\mathbb{Z}^n$.

Redefinition. G is generated by n elements if there is a surjection:

$$(*)\ q : F_n \to G$$

from the free group on n generators, and a word $w \in F_n$ is a *relation* if $q(w) = e$.

Definition. The *normal* subgroup $N(S)$ generated by a subset $S \subset G$ is the smallest normal subgroup of G containing S. Evidently, $H(S) \subset N(S)$, but if $H(S)$ fails to be normal, then evidently $N(S)$ is a larger subset.

Definition. A finite generation $(*)$ above of G is a *finite presentation* if:

$$\ker(q) = N(S) \text{ for a finite set of relations } S = \{w_1, ..., w_m\} \subset F_n$$

In practice, each word $w \in F_n$ enlarges the equivalence relation via:

$$uwv \sim_w uv \text{ for words } u, v \in F_n$$

and $F_n/N(S)$ is the quotient group by the relation generated by the $\sim_{w_i}$.

Example. If $|G| = n$, then G is finitely generated by all elements of G, and:

$$w_{ij} = g_i g_j (g_j \cdot g_j)^{-1} \text{ for } i \neq j$$

is a finite set presenting $q : F_n \to G$.

Note. There are finite generations of infinite groups that are not finite presentations.

Notation. Generators and relations finitely presenting a group G may be written:

$$\langle x_1, ..., x_n \mid w_1, ..., w_r \rangle$$

and many groups have particularly nice presentations.

Example. (a) $\langle x \mid x^n \rangle$ is a presentation of the finite cyclic group C_n.

(b) S_n is finitely generated by transpositions $\tau_i = (i \ i+1)$ with relations:

$$\tau_i^2, \ (\tau_i \tau_{i+1})^3 \text{ and } (\tau_i \tau_j)^2 \text{ when } |i-j| \geq 2$$

Proposition 2. If $G = N \rtimes_\phi H$ is a semi-direct product, and:

(a) $\langle x_1, ..., x_n \mid w_1, ..., w_r \rangle$ is a presentation of N and

(b) $\langle y_1, ..., y_k \mid v_1, ..., v_s \rangle$ is a presentation of H, then

$$\langle x_1, ..., x_n, y_1, ..., y_k \mid w_1, ..., w_r, v_1, ..., v_s, \{y_i a_j y_i^{-1} \phi_{ij}(a)^{-1}\} \rangle$$

is a presentation of G, where $\phi_{ij}(a)$ are words in the a's that map to $\phi_{h_i}(a_j) \in N$.

Proof. Exercise.

Corollary. A presentation of the dihedral group D_{2n} is given by:

$$\langle x, y \mid x^n, y^2, (yx)^2 \rangle$$

since $yxy^{-1} = x^{-1}$ in the semi-direct product (and $y = y^{-1}$).

More generally, a semi-direct product of cyclic groups $C_n \rtimes C_m$ has presentation:

$$\langle x, y \mid x^n, \ y^m, \ yxy^{-1}x^{-d} \rangle$$

given that $\phi_y(x) = x^d$ (and d solves $x^m \equiv 1 \pmod{n}$).

Corollary. A presentation of $\mathbb{Z}^n$ is given by generators $x_1, ..., x_n$ with relations:

$$x_i x_j x_i^{-1} x_j^{-1}$$

Sometimes a group may have a surprising set of generators and relations:

Example. The dihedral group D_{2n} is also generated by y and $z = yx$ with:

$$D_{2n} = \langle y, z \mid y^2, z^2, (yz)^n \rangle$$

Definition. A (finite) **Coxeter group** has generators $x_1, ..., x_n$ and relations x_i^2 (i.e. G is generated by *reflections*) with additional relations.

Coxeter groups are classified by their *Dynkin diagrams* and include D_{2n} and S_n.

Possible additional topics: group cohomology, filtrations.