Generators and Commutators

Let $S \subset G$ be a subset of a group. The subgroup $H(S) \subset G$ generated by $S$ is the smallest subgroup containing $S$. Since the intersection of (any number of) subgroups is a subgroup, $H(S)$ is the intersection of all subgroups that contain $S$. Constructively,

$$H(S) = \{ g \in G \mid g = w(S) \}$$

is the set of finite-length words in the elements of $S$, where a word is a finite product of elements of $S$ and their inverses (with repetitions).

Example. As we noted earlier, the symmetric group $S_n$ is generated by

$$S = \{(1 2), (2 3), \ldots, (n-1 \, n)\}$$

Remark. The permutation that requires the longest word is the transposition:

$$(1 \, n) = (1 2)(2 3)(n-1 \, n)(2 3)(1 2)$$

Definition. The commutator subgroup $[G,G] \subset G$ is the group generated by:

$$\{(a, b) : aba^{-1}b^{-1} \mid a, b \in G\}$$

the set of all “commutators” of elements $a$ and $b$.

Example. The commutator subgroup of an abelian group is trivial. On the other hand, within the symmetric group, the commutator of overlapping transpositions:

$$[(i \, j), (j \, k)] = (i \, j)(j \, k)(i \, j)(j \, k) = (i \, k \, j)$$

is a three-cycle, so all three-cycles are commutators, and $[S_n, S_n] = A_n$.

Proposition 1. (a) The commutator subgroup $[G,G]$ is normal in $G$.

(b) The quotient “abelianization” group $G^{ab} := G/[G,G]$ is abelian and universal in the sense that every homomorphism $f : G \to A$ to an abelian group factors through $\overline{f} : G^{ab} \to A$. In particular, $G$ is abelian if and only if $[G,G] = \{e\}$.

(c) $G$ is solvable if and only if the sequence of commutator subgroups:

$$\cdots \subset G_2 \subset G_1 \subset G_0 = G \text{ with } G_{i+1} = [G_i, G_i]$$

is eventually trivial.

Proof. (a) Note that $[a, b][b, a] = (aba^{-1}b^{-1})(bab^{-1}a^{-1}) = e$, so the inverse of a commutator is a commutator and every word in commutators is a product of commutators. Moreover, commutators conjugate to commutators:

$$g[a, b]g^{-1} = [gag^{-1}, gbg^{-1}] = [c_g(a), c_g(b)]$$

from which it follows that $g[a_1, b_1] \cdots [a_n, b_n]g^{-1} = [c_g(a_1), c_g(b_1)] \cdots [c_g(a_n), c_g(b_n)]$ so conjugation preserves words in commutators, i.e. $[G,G]$ is normal.

(b) Let $N = [G,G]$. Then $G/N$ is abelian, since: $abN = ab(b^{-1}a^{-1}ba)N = baN$ and if $f : G \to A$ is a group homomorphism to an abelian group, then:

$$f(aba^{-1}b^{-1}) = f(a)f(b)f(a)^{-1}f(b)^{-1} = e$$

so $[G,G] \subset \ker(f)$, and $f$ factors through $G^{ab} = G/[G,G]$.

(c) A solvable group has a string of normal subgroups (one in the next) with abelian quotient groups, and conversely, so this follows from (b).
Remark. There is no direct analogue for abelian subgroups of a group $G$. The only canonical abelian subgroup is the center $Z(G)$, but this not generally the largest subgroup, nor is there a single largest abelian subgroup, since unlike intersections, the unions of subgroups are not subgroups.

Examples. (i) The commutator subgroup of a simple group $G$ is $\{e\}$ when $G = C_p$ and is $G$ itself when $G$ is (simple and) not abelian.

(ii) The commutator subgroup of a dihedral group satisfies:
\[
D_{2n}/[D_{2n}, D_{2n}] = C_2 \quad \text{when } n \text{ is odd, and } \quad C_2 \times C_2 \quad \text{when } n \text{ is even}
\]
If $g \in C_n$ is a generator, and $h \in C_2$ is a generator, then $hgh^{-1}g^{-1} = g^{-2} \in D_{2n}$ (from the realization of $D_{2n}$ as a semi-direct product), and $g^{-2}$ generates $C_n$ when $n$ is odd, but only half of $C_n$ when $n$ is even.

(iii) The fundamental group $\pi_1$ and first homology group $H_1$ of a path connected topological space $X$ satisfy:
\[
\pi_1(X, x)/[\pi_1(X, x), \pi_1(X, x)] = H_1(X, \mathbb{Z})
\]

Next, we turn to finitely generated groups.

**Definition.** $G$ is finitely generated if $G = H(S)$ for a finite subset $S \subset G$.

Evidently, finite groups are finitely generated, as are the groups $\mathbb{Z}^n$.

**Definition.** The free group $F_S$ on a (finite) set $S$ is made up of equivalence classes of words $w(S)$ in the elements $s_1, \ldots, s_n \in S$ and their (formal) inverses $s_1^{-1}, \ldots, s_n^{-1}$.

Concatenation of words is multiplication (with the empty word as identity), and:
\[
w_1(S)s_1^{-1}w_2(S) \sim w_1(S)w_2(S)
\]
generates the equivalence relation, so that, for example:
\[
(s_1 \cdots s_n)^{-1} = s_n^{-1} \cdots s_1^{-1} \text{ is the two-sided inverse}
\]

Example. The abelianization of the free group on $n$ generators is $\mathbb{Z}^n$.

**Redefinition.** $G$ is generated by $n$ elements if there is a surjection:
\[
(*) \quad q : F_n \to G
\]
from the free group on $n$ generators, and a word $w \in F_n$ is a relation if $q(w) = e$.

**Definition.** The normal subgroup $N(S)$ generated by a subset $S \subset G$ is the smallest normal subgroup of $G$ containing $S$. Evidently, $H(S) \subset N(S)$, but if $H(S)$ fails to be normal, then evidently $N(S)$ is a larger subset.

**Definition.** A finite generation $(*)$ above of $G$ is a finite presentation if:
\[
\ker(q) = N(S) \quad \text{for a finite set of relations } S = \{w_1, \ldots, w_m\} \subset F_n
\]

In practice, each word $w \in F_n$ enlarges the equivalence relation via:
\[
uvw \sim_w uv \quad \text{for words } u, v \in F_n
\]
and $F_n/N(S)$ is the quotient group by the relation generated by the $\sim_w$.

Example. If $|G| = n$, then $G$ is finitely generated by all elements of $G$, and:
\[
w_{ij} = g_ig_j(g_j^{-1}g_i)^{-1} \quad \text{for } i \neq j
\]
is a finite set presenting $q : F_n \to G$. 
Note. There are finite generations of infinite groups that are not finite presentations.

**Notation.** Generators and relations finitely presenting a group \( G \) may be written:

\[
\langle x_1, \ldots, x_n \mid w_1, \ldots, w_r \rangle
\]

and many groups have particularly nice presentations.

Example. (a) \( \langle x \mid x^n \rangle \) is a presentation of the finite cyclic group \( C_n \).

(b) \( S_n \) is finitely generated by transpositions \( \tau_i = (i \ i + 1) \) with relations:

\[
\tau_i^2, (\tau_i \tau_{i+1})^3 \text{ and } (\tau_i \tau_j)^2 \text{ when } |i - j| \geq 2
\]

**Proposition 2.** If \( G = N \rtimes_\phi H \) is a semi-direct product, and:

(a) \( \langle x_1, \ldots, x_n \mid w_1, \ldots, w_r \rangle \) is a presentation of \( N \) and

(b) \( \langle y_1, \ldots, y_k \mid v_1, \ldots, v_s \rangle \) is a presentation of \( H \), then

\[
\langle x_1, \ldots, x_n, y_1, \ldots, y_k \mid w_1, \ldots, w_r, v_1, \ldots, v_s, \{y_i a_j y_i^{-1} \phi(j)(a) \} \rangle
\]

is a presentation of \( G \), where \( \phi(j)(a) \) are words in the \( a \)'s that map to \( \phi_n(a_j) \in N \).

**Proof.** Exercise.

**Corollary.** A presentation of the dihedral group \( D_{2n} \) is given by:

\[
\langle x, y \mid x^n, y^2, (yx)^2 \rangle
\]

since \( yxy^{-1} = x^{-1} \) in the semi-direct product (and \( y = y^{-1} \)).

More generally, a semi-direct product of cyclic groups \( C_n \rtimes C_m \) has presentation:

\[
\langle x, y \mid x^n, y^m, yxy^{-1}x^{-d} \rangle
\]

given that \( \phi_y(x) = x^d \) (and \( d \) solves \( x^m \equiv 1 \pmod{n} \)).

**Corollary.** A presentation of \( \mathbb{Z}^n \) is given by generators \( x_1, \ldots, x_n \) with relations:

\[
x_i x_j x_i^{-1} x_j^{-1}
\]

Sometimes a group may have a surprising set of generators and relations:

Example. The dihedral group \( D_{2n} \) is also generated by \( y \) and \( z = yx \) with:

\[
D_{2n} = \langle y, z \mid y^2, z^2, (yz)^n \rangle
\]

**Definition.** A (finite) **Coxeter group** has generators \( x_1, \ldots, x_n \) and relations \( x_i^2 \) (i.e. \( G \) is generated by reflections) with additional relations.

Coxeter groups are classified by their **Dynkin diagrams** and include \( D_{2n} \) and \( S_n \).

Possible additional topics: group cohomology, filtrations.