Derived Functors

Let $\mathcal{A}$ be an abelian category.

**Definition.** (a) $\mathcal{A}$ has enough projectives if each object $A$ admits:

$$P \to A \to 0$$

an epimorphism from a projective object $P$ of $\mathcal{A}$.

(b) $\mathcal{A}$ has enough injectives if each object $A$ admits:

$$0 \to A \to I$$

a monomorphism to an injective object $I$ of $\mathcal{A}$.

Fortunately for us, the categories $\text{Mod}_R$ of $R$-modules have enough of both. Note that by iterating, we obtain exact complexes of projectives and of injectives:

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \to 0$$

and

$$0 \to A \xrightarrow{d_0} I_0 \xrightarrow{d_1} I_1 \xrightarrow{d_2} I_2 \to \cdots$$

We will use the first to construct left derived functors (of a right-exact functor) and the second to construct right derived functors (of a left-exact functor).

**Remark.** One might instead use superscripts for the terms of the injective resolution (which is a “cochain” since the indices increase as one moves to the right).

Recall that given an $R$-module $M$, the Hom (covariant) functor:

$$F_M := \text{Hom}_R(M, \cdot) : \text{Mod}_R \to \text{Mod}_R$$

is left-exact. The opposite Hom functor $F^M = \text{Hom}_R(\cdot, M)$ is also left-exact, but behaves more like a right-exact functor since it is contravariant. The tensor product defines a right-exact covariant functor as follows:

**Proposition 1.** Tensoring with a fixed $R$-module $M$ defines the functor:

$$T_M(N) = N \otimes_R M,$$

with

$$T_M(f : N \to N') = (f \otimes 1_M : N \otimes_R M \to N' \otimes_R M)$$

that is (covariant and) right-exact.

**Proof.** It is clear that this is a functor. Right-exactness is the issue. Let

$$(*) \quad N \xrightarrow{f} N' \xrightarrow{g} N'' \to 0$$

be a right-exact sequence of $R$-modules. Then:

(i) $g \otimes 1_M$ is surjective (this is obvious).

(ii) $(g \otimes 1_M) \circ (f \otimes 1_M) = (g \circ f) \otimes 1_M = 0$ (this is also obvious)

(iii) The morphism $g \otimes 1_M$ is the cokernel of $f \otimes 1_M$. Recall the universal properties UC and UT of the cokernel (in an arbitrary abelian category) and tensor product (in the category of $R$-modules) respectively, and consider:

$$N \times M \xrightarrow{(f,1_M)} N' \times M \xrightarrow{(g,1_M)} N'' \times M \to 0$$

the sequence of $R$-bilinear maps, with the analogue of the cokernel property:
UC: Any bilinear map \( b' : N' \times M \to L \) such that \( b' \circ (f, 1_M) = 0 \) is the composition \( b'' \circ (g, 1_M) \) for the unique bilinear map \( b'' : N'' \times M \to L \) defined by \( b''(g(n'), m) = b'(n', m) \). Coupling this with the universal property UT of the tensor product, we obtain the following:

An \( R \)-module homomorphism \( h' : N' \otimes_R M \to L \) with \( h' \circ (f \otimes 1_M) = 0 \) gives:
\[
b' : N' \times M \to N' \otimes_R M \to L \quad \text{with} \quad b' \circ (f, 1_M) = 0
\]
which therefore factors through a unique bilinear map \( b'' : N'' \times M \to L \) and, by UT, factors uniquely through an \( R \)-module homomorphism \( h'' : N'' \otimes_R M \to L \).

Thus \( g \otimes 1_M \) is the cokernel of \( f \otimes 1_M \) in the abelian category \( \mathcal{M}od_R \), which is to say that the sequence \((*) \otimes_R M\) is exact at the middle term. \( \square \)

Getting back to the projectives:

**Proposition 2.** In an arbitrary abelian category \( \mathcal{A} \), suppose:
\[
\begin{array}{cccccc}
\cdots & \to & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & A & \to & 0 \\
\cdots & \to & E_2 & \xrightarrow{\partial_2} & E_1 & \xrightarrow{\partial_1} & E_0 & \xrightarrow{\partial_0} & A' & \to & 0 \\
\end{array}
\]
are two exact sequences, the first made up of projectives. Then:

(a) There is an extension of \( f \) to a morphism of chain complexes:
\[
f_\bullet : P_\bullet \to E_\bullet
\]

(b) Any two such extensions \( f_\bullet \) and \( g_\bullet \) are homotopic maps of chain complexes.

**Proof.** (a) The map \( f \circ d_0 : P_0 \to A' \) lifts to \( f_0 : P_0 \to E_0 \) using the facts that \( \partial_0 \) is surjective and \( P_0 \) is projective. Then \( f_0 \) maps \( \ker(d_0) = \im(d_1) \) to \( \ker(\partial_0) = \im(\partial_1) \) since \( f \circ d_0 = \partial_0 \circ f_0 \) and so we may once more lift \( f_0 \circ d_1 \) to \( f_1 : P_1 \to E_1 \) satisfying \( f_0 \circ d_1 = \partial_1 \circ f_1 \) and continue.

(b) Given two such extensions \( f_\bullet \) and \( g_\bullet \) each making the diagram commute:
\[
\begin{array}{cccccc}
\cdots & \to & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & A & \to & 0 \\
\cdots & \to & E_2 & \xrightarrow{\partial_2} & E_1 & \xrightarrow{\partial_1} & E_0 & \xrightarrow{\partial_0} & A' & \to & 0 \\
\end{array}
\]
we also define the homotopy between them inductively. First, we let:
\[
0 = h_{-1} : A \to E_0 \quad \text{so that} \quad f - f = \partial_0 \circ h_{-1}
\]
Then we notice that \( f_0 - g_0 \) maps \( P_0 \) to the kernel of \( \partial_0 \), so we may choose:
\[
h_0 : P_0 \to E_1 \quad \text{so that} \quad f_0 - g_0 = \partial_1 \circ h_0 = \partial_1 \circ h_0 + h_{-1} \circ d_0
\]
Then \( \partial_1(f_1 - g_1 - h_0 \circ d_1) = (f_0 - g_0) \circ d_1 - (\partial_1 \circ h_0) \circ d_1 = 0 \), so we choose:
\[
h_1 : P_1 \to E_2 \quad \text{so that} \quad f_1 - g_1 - h_0 \circ d_1 = \partial_2 \circ h_1
\]
and one more step gets us to the general case. We have \( \partial_2(f_2 - g_2 - h_1 \circ d_2) = (f_1 - g_1) \circ d_2 - (\partial_2 \circ h_1) \circ d_2 = (f_1 - g_1) \circ d_2 - (f_1 - g_1 - h_0 \circ d_1) \circ d_2 = 0 \) and this allows us to choose:
\[
h_2 : P_2 \to E_3 \quad \text{so that} \quad f_2 - g_2 - h_1 \circ d_2 = \partial_3 \circ h_2
\]
and off we go. In the end, we have the desired homotoy \( h_i : P_i \to E_{i+1} \) satisfying:
\[
f_i - g_i = \partial_{i+1} \circ h_i + h_{i-1} \circ d_i \quad \square
\]
Corollary. Let \( F : \mathcal{A} \to \mathcal{B} \) be a right-exact functor of abelian categories and assume \( \mathcal{A} \) has enough projectives. Then the sequence of left derived functors:

\[
L_i F(A) := H_i(F(P\bullet)) \quad \text{and} \quad L_i F(f : A \to B) = H_i(F(f) : F(P\bullet) \to F(Q\bullet))
\]

are well-defined (only up to isomorphism, unfortunately) by choosing projective resolutions \( P\bullet \) (for \( A \)) and \( Q\bullet \) (for \( B \)) and using Proposition 2.

Proof. We show any two projective resolutions of \( A \) give isomorphic homologies:

\[
H_i(F(P\bullet)) \quad \text{and} \quad H_i(F(Q\bullet))
\]

To this end, we apply the Proposition twice to get:

\[
\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \to 0
\]

\[
\downarrow i_2 \quad \downarrow i_1 \quad \downarrow i_0 \quad \downarrow 1_A
\]

\[
\cdots \to P'_2 \xrightarrow{d_2'} P'_1 \xrightarrow{d_1'} P'_0 \xrightarrow{d_0'} A \to 0
\]

\[
\downarrow j_2 \quad \downarrow j_1 \quad \downarrow j_0 \quad \downarrow 1_A
\]

\[
\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \to 0
\]

and homotopies \( h_i : P_i \to P_{i+1} \) exhibiting \( j_i \circ i_i \sim 1_{P\bullet} \) (from the Proposition since both sides are lifts of \( 1_A \)), and \( h'_i : P'_i \to P'_{i+1} \) exhibiting \( i_i \circ j_i \sim 1_{P\bullet} \).

Now we apply \( F \) to everything, and get morphisms \((F \circ i)\)\( \bullet \) and \((F \circ j)\)\( \bullet \) and homotopies \((F \circ h)\)\( \bullet \) and \((F \circ h')\)\( \bullet \) exhibiting \((F \circ j) \circ (F \circ i) \sim 1_{F\circ P\bullet} \) and \((F \circ i) \circ (F \circ j) \sim 1_{F\circ P\bullet} \). Since homotopic maps of complexes induce the same maps on homology, it follows that each \( H_i(F \circ i) : H_i(F(P)) \to H_i(F(P')) \) is an isomorphism, with inverse \( H_i(F \circ j) \).

The Corollary then follows (except for the troubling isomorphism business) by applying Proposition 2 to \( P\bullet \) and \( Q\bullet \) and \( f : A \to B \). \( \square \)

Theorem 3. Given a right-exact functor \( F : \mathcal{A} \to \mathcal{B} \) and a short-exact sequence:

\[(*) \quad 0 \to A \to A' \to A'' \to 0\]

in an abelian category \( \mathcal{A} \) with enough projectives, there is a long exact sequence:

\[
\to L_1 F(A') \to L_1 F(A'') \to F(A) \to F(A') \to F(A'') \to 0
\]

of objects of \( \mathcal{B} \).

Proof. Choose projective resolutions: \( P\bullet \to A \) and \( P''\bullet \to A'' \). We will fashion a third projective resolution of \( A' \) that fits in a short exact sequence:

\[
0 \to P\bullet \to P''\bullet \to P''\bullet \to 0
\]

of chain complexes, which remains short-exact after applying the functor \( F \) to each of the complexes. The Zigzag Lemma then gives the desired long exact sequence among the homology objects of \( F(P\bullet) \), \( F(P''\bullet) \) and \( F(P''\bullet) \), which is the result.

In fact, the terms of \( P''\bullet \) are direct sums \( P_i'' = P_i \oplus P_i'' \) and the horizontal maps are the standard inclusions and projections:

\[
0 \to P_i \xrightarrow{i_i} P_i \oplus P_i'' \xrightarrow{j_i} P_i'' \to 0
\]

which explains why these horizontal sequences remain exact after applying \( F \).
In the diagram below, the map $d''_0 : P''_0 \to B''$ lifts (because $P''_0$ is a projective) to a map $P''_0 \to B'$, and then we obtain a commutative diagram:

\[
\begin{array}{c}
P_0 & \to & P_0 \oplus P'_0 & \to & P''_0 \\
\downarrow d_0 & \searrow & \downarrow d'_0 & \nearrow \downarrow d''_0 & \nearrow & \downarrow d''_0 \\
0 & \to & B & \to & B' & \to & B'' & \to & 0
\end{array}
\]

with $d'_0$ defined (and surjective by the five lemma...or rather the first four lemma!) using the universal property of the coproduct. Then we consider:

\[
0 \to \text{ker}(d_0) \to \text{ker}(d'_0) \to \text{ker}(d''_0) \to 0
\]

which is exact (by the snake lemma) giving a diagram just as the one above:

\[
\begin{array}{c}
P_1 & \to & P_1 \oplus P'_1 & \to & P''_1 \\
\downarrow d_1 & \searrow & \downarrow d'_1 & \nearrow \downarrow d''_1 & \nearrow & \downarrow d''_1 \\
0 & \to & \text{ker}(d_0) & \to & \text{ker}(d'_0) & \to & \text{ker}(d''_0) & \to & 0
\end{array}
\]

with surjective vertical maps, etc. □

Example. The left derived functors of the tensor functor $T_M(N) = N \otimes_R M$ are:

$$
\text{Tor}_i^R(N, M) := L_i T_M(N)
$$

Thus, for example, to compute $\text{Tor}_i(M, k)$, we will use the (free) Koszul resolution:

\[
0 \to k[x, y] \xrightarrow{(-y,x)} k[x, y] \oplus k[x, y] \xrightarrow{x+y} k[x, y] \to k \to 0
\]

for $k$, and then we obtain $\text{Tor}_i(M, k)$ as the homologies of the sequence:

$$
M \xrightarrow{(-y,x)} M \oplus M \xrightarrow{x+y} M
$$

(since $M \otimes_R R = M$). Thus, for instance when $M = k$, all maps are zero(!) and:

$$
\text{Tor}_2(k, k) = k, \quad \text{Tor}_1(k, k) = k^2 \quad \text{and} \quad \text{Tor}_0(k, k) = k \otimes_R k = k
$$

When $M = k[y] = k[x, y]/\langle x \rangle$, only the $x$ map is zero, and we get:

$$
\text{Tor}_2(k[y], k) = 0, \quad \text{Tor}_1(k[y], k) = k \quad \text{and} \quad \text{Tor}_0(k[y], k) = k \otimes_R k[y] = k
$$

Or we could resolve $k[y]$ instead: $0 \to k[x, y] \xrightarrow{\delta} k[x, y] \to k[y] \to 0$ and then:

$$
M \xrightarrow{\delta} M
$$

computes $\text{Tor}_i(M, k[y])$, so e.g. $\text{Tor}_1(k, k[y]) = \text{Tor}_1(k[y], k)$. This is no accident.

Finally, given the short-exact sequence:

$$
0 \to k[y] \xrightarrow{\delta} k[y] \to k \to 0
$$

we can get a long exact sequence of Tor’s by applying the functor $\otimes k$. This gives:

$$
0 \to \text{Tor}_2(k, k) \to \text{Tor}_1(k[y], k) \xrightarrow{\delta} \text{Tor}_1(k[y], k) \to \text{Tor}_1(k, k) \to k \xrightarrow{0} k \to k \to 0
$$

Remark. If $R$ is a PID, then every finitely generated module $N$ resolves as:

$$
0 \to R^m \to R^n \to N \to 0
$$

for some free modules $R^m$ and $R^n$. It follows that $\text{Tor}_i(M \otimes N) = 0$ when $i > 1$. This is, in particular, the case for finitely generated abelian groups.

Meanwhile, over in Opposite Land...
By reversing all arrows and replacing projectives with injectives, we get:

**Theorem:** Given a left-exact functor $G : \mathcal{A} \to \mathcal{B}$ from an abelian category $\mathcal{A}$ with enough injectives, we obtain **right** derived functors:

$$R^iG(A) = H_i(G \circ I_\bullet) \quad \text{and} \quad R^iG(f : A \to A')$$

where $I_\bullet$ is an **injective** resolution of $A$, via **Proposition** applied with arrows reversed and injectives in place of projectives. Then every short-exact sequence:

$$0 \to A \to A' \to A'' \to 0$$

induces a long exact sequence of objects of $\mathcal{B}$:

$$0 \to G(A) \to G(A') \to G(A'') \to R^1G(A) \to R^1G(A') \to R^1G(A'') \to \cdots$$

Example. The right-derived functors of the left-exact $F_M = \text{Hom}_R(M, \cdot)$ are:

$$\text{Ext}^i_R(M, N) := R^iF_M(N)$$

Thus, for example, letting $M = N''$, we have a long exact sequence:

$$0 \to \text{Hom}(N'', N) \xrightarrow{\delta} \text{Hom}(N'', N') \xrightarrow{g_2} \text{Hom}(N'', N'') \xrightarrow{\delta_1} \text{Ext}^1(N'', N) \to \cdots$$

associated to any short exact sequence of the form

$$(*) \quad 0 \to N \to N' \to N'' \to 0$$

and the **extension class** $\epsilon(*) := \delta(1_{N''}) \in \text{Ext}^1(N'', N)$ of the sequence is zero if and only if $1_{N''}$ is in the image of $g_*$, if and only if the sequence $(*)$ splits.

Interestingly, there is a converse to this. Given $\epsilon \in \text{Ext}^1(N'', N)$, we can fashion a short exact sequence $(*)$ (in particular, constructing the module $N'$ in the middle) with $\epsilon(*) = \epsilon$. Starting with an injective resolution:

$$0 \to N \xrightarrow{d_3} I_0 \xrightarrow{d_1} I_1 \xrightarrow{d_2} I_2 \to \cdots$$

we have, by definition, that $\epsilon$ is an element of the middle homology of:

$$\text{Hom}(N'', I_0) \xrightarrow{d_2} \text{Hom}(N'', I_1) \xrightarrow{d_2} \text{Hom}(N'', I_2)$$

i.e. $\epsilon \in \text{Hom}(N'', \ker(d_2)) = \text{Hom}(N'', \text{im}(d_1))$ (modulo the image of $d_1$).

Now we add an injective resolution of $N''$ to the mix:

$$0 \to N'' \xrightarrow{d_3''} I_0'' \xrightarrow{d_1''} I_1'' \xrightarrow{d_2''} I_2'' \to \cdots$$

Then, using the injectivity of $I_1$, we obtain $f : I_0'' \to I_1$:

$$I_1 \xrightarrow{\epsilon} \xrightarrow{f} I_0''$$

which we use to define a homomorphism:

$$\Phi = \begin{bmatrix} d_1 & f \\ 0 & d_1'' \end{bmatrix} : I_0 \oplus I_0'' \to I_1 \oplus I_1''$$

and then obtain a commuting diagram:

$$\begin{array}{cccccc}
0 & \to & I_0 & \to & I_0 \oplus I_0'' & \to & I_0'' \\
\uparrow d_1 & & \uparrow \Phi & & \uparrow d_1'' & & \uparrow 0 \\
0 & \to & I_1 & \to & I_1 \oplus I_1'' & \to & I_1'' & \to & 0
\end{array}$$
with sequence (from the snake lemma):

$$0 \rightarrow N \rightarrow \ker(\Phi) \rightarrow N'' \xrightarrow{\delta} \coker(d_1) \rightarrow \coker(\Phi) \rightarrow \coker(d''_1) \rightarrow 0$$

But if $i_1, j_1 \in I_1$ and $(i_1, 0) - (j_1, 0) = 0$ as an element of $\coker(\Phi)$, then

$$(i_1, 0) - (j_1, 0) = (i_1 - j_1, 0) = \Phi(i_0, i''_0) = (d_1(i_0) + f(i''_0), d''_1(i''_0))$$

and then it follows that $d''_1(i''_0) = 0$, so $i''_0 = d''_0(n'')$ for some $n'' \in N''$ and also that $f(i''_0) = \epsilon(n'') \in \ker(d_2) = \im(d_1)$, so $i_1 - j_1 = d_1(i_0) + f(i''_0)$ is in the image of $d_1$ and $i_1 - j_1 = 0$ as an element of $\coker(d_1)$. All this is to say that the map following $\delta$ is injective, and so by exactness $\delta$ is the zero map! The truncated sequence:

$$(*) \quad 0 \rightarrow N \rightarrow N' = \ker(\Phi) \rightarrow N'' \xrightarrow{\delta} 0$$

is the desired short exact sequence with $\epsilon(*) = \epsilon$. \qed

**Problem.** It is difficult to work with injective resolutions.

For the Ext functors there is a convenient fix, which we give without proof.

**Theorem 4.** Instead of computing $\Ext^i(M, N)$ as

$$H^i(Hom(M, I_\bullet))$$

for an injective resolution $I_\bullet$ of $N$, we may instead compute it as:

$$H^i(Hom(P_\bullet, N))$$

for a projective resolution of $M$ (and the contravariant functor $F^N = Hom(\bullet, N)$).

**Remark.** The same drill as for the 3 Theorems allow one to conclude that $F^N$ has right derived functors, computed as $H_i(Hom(P_\bullet, N))$. The surprising part of the Theorem is that this yields the *same* modules $\Ext^i(M, N)$.

**Examples.** The following sequence of abelian groups is clearly not split:

$$(*) \quad 0 \rightarrow \mathbb{Z}^{(2;1)} \rightarrow \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \xrightarrow{1+4} \mathbb{Z}/6\mathbb{Z} \rightarrow 0$$

and so determines a nonzero class $\epsilon(*) \in \Ext^1(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z})$. We may compute this via:

$$0 \rightarrow \mathbb{Z} \xrightarrow{\epsilon} \mathbb{Z} \rightarrow \mathbb{Z}/6\mathbb{Z} \rightarrow 0$$

which we hit with the functor $F^\mathbb{Z}$ to get:

$$\mathbb{Z} = \Hom(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\epsilon} \Hom(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$$

from which we conclude that $\Ext^1(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/6\mathbb{Z}$.

We know that the zero extension class gives the split sequence, but:

**Question.** Which extension class(es) give:

$$(*) \quad 0 \rightarrow \mathbb{Z}^{(2;1)} \rightarrow \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \xrightarrow{1+4} \mathbb{Z}/6\mathbb{Z} \rightarrow 0?$$

and which extension class(es) give: $$(**) \quad 0 \rightarrow \mathbb{Z} \xrightarrow{\epsilon} \mathbb{Z} \rightarrow \mathbb{Z}/6\mathbb{Z} \rightarrow 0?$$

and are we missing any other short exact sequences?

**Popup Ad.** Let $X$ be a topological space, and consider the category $\mathcal{X}$ with:

The objects of $\mathcal{X}$ are the open subsets $U$ of $X$.

The morphisms of $\mathcal{X}$ are the inclusions $U \subseteq V$. 
A contravariant functor $A : \mathcal{X} \to \mathcal{Ab}$ (to the category of abelian groups) is a presheaf of abelian groups on $X$. In other words, a presheaf $\mathcal{F}$ consists of:

(i) An abelian group $A(U)$ attached to each open subset, and
(ii) Restriction maps $\rho_{V,U} : A(V) \to A(U)$ attached to each $U \subseteq V$ such that:
   • $\rho_{U,U} = 1_{A(U)}$ and:
   • $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U} : A(W) \to A(U)$ whenever $U \subseteq V \subseteq W$.

Example. The constant presheaf $A$ (for a fixed abelian group $A$) is defined by:

$$A(\emptyset) = 0, \quad A(U) = A \quad \text{and} \quad \rho_{V,U} = 1_{A(U)} \text{ for all } U \neq \emptyset$$

Rather amazingly, this is an interesting presheaf. It is associated to:

The locally constant sheaf $A^+$, defined by:

$$A^+(U) = \{ \text{continuous maps } f : U \to A \text{ for the discrete topology on } A \}.$$  

$A^+(U \subseteq V)$ is the restriction of continuous functions $f : V \to A$ to $f|_U : U \to A$.

Note that if $U$ is connected, then $A(U) = A^+(U)$, since the continuous maps from a connected set to a set with the discrete topology are the constant maps! But if $U$ has $n$ connected components, then $A^+(U) = A^n$ and the restriction maps to each connected component are the projections.

There is a lot to say about this, but suffice it for the purposes of this teaser to say that there is a category of sheaves of abelian groups on the fixed topological space $X$ with enough injectives, and that the covariant global section functor

$$\Gamma : \mathcal{A} \to \mathcal{Ab}; \quad \mathcal{A} \mapsto \mathcal{A}(X)$$

is left-exact, which then defines right derived functors of the global section functor, which are the cohomology groups:

$$H^i(X, A) := R^i \Gamma(X, A)$$

These may be computed by taking a “good open cover” of $X$, and are basically dual to the singular cohomology of $X$ (when $A = \mathbb{Z}$) that we discussed in an earlier popup topological ad.