Riemann Surfaces in the Wild

We will prove the Riemann Roch Theorem for Riemann Surfaces that are embedded in projective space. That is, we will assume that the Riemann Surface is embedded by meromorphic functions as a subset $S \subset \mathbb{CP}^n$, for which the homogeneous ideal:

$$I = \bigoplus_{d=0}^{\infty} I_d$$

$$I_d = \{F \in \mathbb{C}[x_0, ..., x_n]_d \mid F(p) = 0 \text{ for all } p = (p_0 : ... : p_n) \in S\}$$

carves out S in the sense of Lemma 3.6. We will also assume that $S \subset \mathbb{CP}^n$ is not contained in any hyperplane, i.e. that $I_0 = I_1 = 0$.

The **homogeneous coordinate ring** of $S \subset \mathbb{CP}^n$:

$$\mathbb{C}[S] := \mathbb{C}[x_0, ..., x_n]/I = \bigoplus_{d=0}^{\infty} \mathbb{C}[S]_d$$

is the quotient ring, which is graded (since I is a homogeneous ideal), and:

$$h_S(d) = \dim\left(\mathbb{C}[S]_d\right)$$

is the **Hilbert function** of S. Note that $h_S(0) = 1$ and $h_S(1) = n + 1$.

Example. Suppose $S \subset \mathbb{CP}^2$ is embedded in the complex projective plane. In this case the homogeneous ideal is $I = F \cdot \mathbb{C}[x_0, x_1, x_2]$, generated by a single homogeneous polynomial $F \in \mathbb{C}[x_0, x_1, x_2]_\delta$ of degree δ and then:

$$h_{\mathbb{CP}^2}(d) = \dim(\mathbb{C}[x_0, x_1, x_2]_d) = \binom{d+2}{2} \text{ and}$$

$$h_I(d) = \dim I_d = \dim\left(\mathbb{C}[x_0, x_1, x_2]_{d-\delta}\right) = \binom{d-\delta+2}{2}$$

(the latter formula is only valid when $d \geq \delta$). Subtracting, we get:

$$h_S(d) = \delta d + \left(1 - \frac{(\delta-1)(\delta-2)}{2}\right), \text{ valid for all } d \geq \delta$$

a linear function of d (for all $d \geq \delta$).

Let $l = c_0x_0 + \cdots + c_nx_n \in \mathbb{C}[x_0, ..., x_n]_1$ be a linear form. Then we define an effective divisor $\mathrm{div}(l)$ on the Riemann surface locally as follows. If $p = (p_0 : ... : p_n) \in S$ and $p_i \neq 0$, then:

$$\frac{l}{x_i} = c_0\frac{x_0}{x_i} + \cdots + c_n\frac{x_n}{x_i}$$

defines a meromorphic function on S, and we define $\mathrm{ord}_p(l) := \mathrm{ord}_p(l/x_i)$ which is independent of the choice of $p_i \neq 0$ above. Then

$$\mathrm{div}(l) := \sum_p \mathrm{ord}_p(l) \cdot p \text{ is an effective divisor}$$

(this is basically the same way that div was computed for differential forms, and we'll eventually see that both generalize to the notion of div(s) for **sections** s of a holomorphic line bundle on S).

Some things to notice about $\mathrm{ord}_p(l)$ and $\mathrm{div}(l)$:

(a) $S \subset \mathbb{CP}^n$ **meets** the hyperplane $H = V(l) \subset \mathbb{CP}^n$ at p if $\mathrm{ord}_p(l) > 0$ and it is **tangent** to H at p if $\mathrm{ord}_p(l) > 1$.

(b) If S and H are **transverse**, i.e. there are no points of tangency, then $\deg(\mathrm{div}(l))$ is the number of intersection points (and it is always possible to find transverse hyperplanes to an embedded Riemann surface).

If l and l' are two linear forms, then:

$$\deg(\mathrm{div}(l)) - \deg(\mathrm{div}(l')) = \deg(\mathrm{div}(l/l')) = 0$$

since l/l' defines a meromorphic function on S. Thus, $\deg(\mathrm{div}(l)) = \delta$ is independent of the linear form. This is the **degree** of the embedding of S.

Our aim is to prove the Riemann Roch equality first for divisors

$$E = d \cdot \mathrm{div}(l)$$

for sufficiently large values of d. To that end, we'll show that:

(A) The Hilbert function of S is linear for large values of d, and in fact:

$$h_S(d) = \delta \cdot d + (1 - g) \text{ for large values of } d$$

where δ is the degree of the embedded S and g is the genus, which means that the constant term of $h_S(d)$ is independent of the embedding of S.

Note. It then will follow from the Example above that:

$$g(S) = \frac{(\delta - 1)(\delta - 2)}{2} \text{ for embedded plane curves } S \text{ of degree } \delta$$

(B) When d is sufficiently large, there is an isomorphism:

$$\mathbb{P}(\mathbb{C}[S]_d) \to |E|; \; G \pmod{I_d} \mapsto \mathrm{div}(G)$$

of projective spaces, where $\mathrm{div}(G)$ is defined for homogeneous polynomials of degree d not in I_d exactly as $\mathrm{div}(l)$ was defined.

Then by (B), $h_S(d) = \dim(V(E))$ and $V(K_S - E) = 0$ for E as above, and then (A) is precisely the Riemann Roch Theorem for E.

Assignment 2. (i) Check the computation of $h_S(d)$ in the Example.

(ii) Convince yourself that the degree of a Riemann surface embedded in the plane actually does agree with the degree of the homogeneous polynomial F that generates its homogeneous ideal. (This is a consequence of the computation in the Example and (A)).

(iii) Suppose $S \subset \mathbb{CP}^3$ and its homogeneous ideal I is generated by **two** homogeneous polynomials F and G, of degrees γ_1 and γ_2, respectively. Compute the degree δ and the genus g of S in terms of γ_1 and γ_2.