Riemann Surfaces and Graphs

7. Special Divisors

If $D$ is a divisor on a Riemann surface $S$ of genus $g$, then we set

$$l(D) = \dim(V(D))$$

so $D$ is a special divisor (see §6) if $l(D) > 0$ and $l(K_S - D) > 0$.

**Lemma 7.1.** Every effective divisor $D$ of degree $\leq g - 1$ is special.

**Proof.** By the Riemann-Roch Theorem,

$$l(K_S - D) - l(D) = g - 1 - \deg(D)$$

so if $l(D) > 0$ and $\deg(D) \leq g - 1$, then $l(K_S - D) > 0$.

So this only imposes a constraint on the (effective) divisors of degree $\geq g$.

Let $D$ be an effective divisor on an embedded Riemann surface $S \subset \mathbb{CP}^n$.

We say that $D \subset H = V(L)$ if $D$ is dominated by the divisor $\text{div}(L)$.

**Definition 7.2.** The linear span $\langle D \rangle \subseteq \mathbb{CP}^n$ of $D$ is the intersection:

$$\langle D \rangle := \bigcap_{D \subset H} H$$

and if no hyperplane contains $D$ in this sense, then $D$ spans $\mathbb{CP}^n$.

**Examples.**

(a) $\langle p + q \rangle$ is the secant line in $\mathbb{CP}^n$ through $p$ and $q$.

(b) $\langle p + q + r \rangle$ is a plane unless the three points are collinear in $\mathbb{CP}^n$, in which case $\langle p + q + r \rangle = \langle p + q \rangle$ is called a tri-secant line.

(c) $\langle 2p \rangle$ is the tangent line to $S$ at $p$ in $\mathbb{CP}^n$, and $\langle 3p \rangle$ is a projective plane unless $\langle 2p \rangle = \langle 3p \rangle$, and $p \in S$ is an inflection point of the embedding.

**Lemma 7.3.** If $S$ is not hyperelliptic, then an effective divisor $D$ is special if and only if $\langle D \rangle \neq \mathbb{CP}^{g-1}$ for the canonical embedding $S \subset \mathbb{CP}^{g-1}$.

**Proof.** The effective divisors $\text{div}(L)$ for the canonical embedding of $S$ are precisely the elements of $|K_S|$, so $V(K_S - D) \neq 0$ if and only if $D$ is dominated by some divisor $\text{div}(L)$, i.e. $D \subset H = V(L)$.

Let $S$ be hyperelliptic (through Lemma 7.4) and let $\phi : S \to \mathbb{CP}^1$ be the degree two map and:

$$F = (g - 1)(\phi^*(\infty))$$

where $\phi^*(\infty)$ is the (degree two) divisor of poles of $\phi$.  

Then $1, \phi, \phi^2, \ldots, \phi^{g-1} \in V(F)$ are linearly independent meromorphic functions, so $l(F) \geq g$. On the other hand, by the Riemann-Roch Theorem,

$$l(F) - l(K_S - F) = g - 1$$

and since $\text{deg}(K_S - F) = 0$, it follows that:

(i) $l(F) = g$, and 
(ii) $l(K_S - F) = 1$, so $K_S \sim F$ via $\phi \in V(K_S - F)$.

Thus $1, \phi, \cdots, \phi^{g-1}$ is a basis for $V(K_S) \cong V(F)$, and:

$$S \xrightarrow{\phi} \mathbb{CP}^1 \rightarrow \mathbb{CP}^{g-1}$$

is the canonical map for $S$. All the canonical divisors on $S$ are of the form:

$$\phi^*(q_1 + \cdots + q_{g-1}) = (p_1 + \tau(p_1)) + \cdots + (p_{g-1} + \tau(p_{g-1}))$$

where $\tau$ is the involution of $S$ that exchanges the points of the fibers $\phi^{-1}(q)$.

**Lemma 7.4.** An effective divisor $D$ on $S$ is special if and only if

$$D = \phi^*(q_1 + \cdots + q_m) + p_{m+1} + \cdots + p_n$$

where $p_i \neq \tau(p_j)$ for all $i \neq j$ and $n \leq g - 1$. Moreover, $l(D) = m + 1$ for this divisor and, in particular, the points $p_i$ are the **base points** of $|D|$.

**Proof.** Given $D$ in the form above, let $q_i = \phi(p_i)$. Then $D$ is dominated by $\phi^*(q_1 + \cdots + q_m)$ which is dominated by a canonical divisor since $n \leq g - 1$ (it is a canonical divisor if $n = g - 1$). So such a divisor $D$ is special. Conversely, if $l(K_S - D) > 0$, then $D$ is dominated by a canonical divisor, i.e. $\phi^*(q_1 + \cdots + q_{g-1})$ for some $q_1, \ldots, q_{g-1}$, and then $D$ has the desired form.

Since $l(q_1 + \cdots + q_m) = m + 1$ (on $\mathbb{CP}^1$), we right away get $l(D) \geq m + 1$. In addition, since $l(\phi^*(q_1 + \cdots + q_{g-1})) = g$ for all $q_1, \ldots, q_{g-1}$, it follows that $l(\phi^*(q_1 + \cdots + q_n)) = n + 1$ for all $n \leq g - 1$. But

$$D = \phi^*(q_1 + \cdots + q_n) - \tau(p_{m+1}) - \cdots - \tau(p_n)$$

The subtraction of each point disallows a pole at $\tau(p_i)$ which reduces the dimension of the linear series by one (check this!), and

$$l(D) = n + 1 - (n - m) = m + 1$$
**Clifford’s Theorem.** If $D$ is a special divisor on $S$, then

$$2(l(D) - 1) \leq \deg(D)$$

with equality if and only if either $D = 0$ or $D = K_S$ or:

$S$ is hyperelliptic and $D = \phi^*(q_1 + \cdots + q_m)$ for $0 < m < g - 1$.

**Proof.** For a hyperelliptic Riemann surface $S$, we have:

$$\deg(D) = 2m + (n - m) = m + n$$

and $l(D) = m + 1$ for the special divisors $D$ in Lemma 7.4, and Clifford’s Theorem for $S$ follows.

For an arbitrary Riemann surface $S$, we claim first that:

$$l(D + E) \geq l(D) + l(E) - 1$$

for all effective divisors $D$ and $E$.

Given an effective divisor $D$, then for any choice of $p \in S$, we either have $l(D - p) = l(D)$ (for the finite set of base points) or else $l(D - p) = l(D) - 1$. Thus if $l(D) = r + 1$, then $l(D - p_1 - \cdots - p_r) > 0$ for all $p_1, \ldots, p_r$ and $l(D - p_1 - \cdots - p_r) = 1$ for a “general” choice of points, in which case there is a unique divisor $D' \sim D$ dominating $p_1 + \cdots + p_r$. In addition, if $l(D) \leq r$, then $l(D - p_1 - \cdots - p_r) = 0$ for a general set of $r$ points.

If $l(D) = r + 1$ and $l(E) = s + 1$ and $p_1, \ldots, p_{r+s}$ is a general set of points, then $l(D - p_1 - \cdots - p_r) = 1$ and $l(E - p_{r+1} - \cdots - p_{r+s}) = 1$, so there are unique divisors $D' \sim D$ and $E' \sim E$ dominating the sets of points, and then $D' + E'$ dominates $p_1 + \cdots + p_{r+s}$, so $l(D + E - p_1 - \cdots - p_{r+s}) > 0$. Thus:

$$l(D + E) \geq (r + s) + 1 = l(D) + l(E) - 1$$

and if equality holds, then every divisor in $|D + E|$ that contains a general set of $r + s$ points of $S$ is of the form $D' + E'$ for $D' \in |D|$ and $E' \in |E|$.

When we let $D$ be a special divisor and set $E = K_S - D$, this gives:

$$g = l(K_S) \geq l(D) + l(K_S - D) - 1 = l(D) + (l(D) - \deg(D) + g - 1) - 1$$

by the Riemann-Roch Theorem. This is the Clifford inequality! Moreover, if equality holds, then each canonical divisor $F \in |K_S|$ that dominates $g - 1$ general points $p_1, \ldots, p_{g-1} \in S$ is a sum:

$$F = D' + E'$$

where $D' \sim D$ and $E' \sim K_S - D$.

Reality check: This holds when $S$ is hyperelliptic and $D = \phi^*(q_1 + \cdots + q_m)$. 

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It only remains to show that \((\ast)\) is impossible when \(S\) is non-hyperelliptic. So assume \(D\) is a special divisor satisfying the Clifford equality.

First, the Riemann-Roch Theorem gives:

\[
0 = \deg(D) - 2((l(D) - 1) = \deg(K_S - D) - 2(l(K_S - D) - 1)
\]

so we may replace \(D\) if necessary by \(K_S - D\) to assume that \(\deg(D) \leq g - 1\). Next, we may assume \(l(D) > 1\) since otherwise the equality gives \(D = 0\).

Finally, there is a very useful geometric interpretation of \(l(D)\) in terms of the linear span of the divisor \(D\) under the canonical embedding:

\[
\dim(\langle D \rangle) = g - 1 - l(K_S - D) = \deg(D) - l(D)
\]

In particular, since \(l(D) > 1\), the points of \(D\) are linearly dependent.

The last piece of the puzzle is the following:

**General Position Theorem.** If \(S \subset \mathbb{CP}^n\) is an embedded Riemann surface of degree \(\delta\) not in a hyperplane, then the points of the transverse intersection:

\[
S \cap H = \text{div}(L) = p_1 + \cdots + p_\delta
\]

of \(S\) with a general hyperplane are “algebraically indistinguishable.” In particular, every collection of \(n\) of the points is linearly independent!

Assuming this, we finish the proof of Clifford’s Theorem. If \(p_1, \ldots, p_{g-1}\) are general points of \(S\), then they span a general hyperplane \(H\) under the canonical embedding of \(S\), and by the general position theorem, every set of \(g - 1\) of the \(2g - 2\) points of \(F = \text{div}(L) \sim K_S\) is linearly independent. But if the Clifford equality holds for the divisor \(D\), then:

\[
F = D' + E' \text{ for } D' \sim D \text{ and } E' \sim K_S - D
\]

and the points of \(D'\) are linearly dependent since \(l(D') = l(D) > 1\). \(\square\)

**Assignment.** 1. Ask questions. (There are lots of new ideas here).

**Definition 7.5.** The **Clifford index** of a special divisor on \(S\) is:

\[
\text{Cliff}(D) = 2((l(D) - 1) - \deg(D) \geq 0
\]

The Clifford index of a Riemann surface \(S\) is:

\[
\text{Cliff}(S) = \max\{\text{Cliff}(D) \mid l(D) > 1, \ l(K_S - D) > 1\}
\]

Thus hyperelliptic Riemann surfaces are those with \(\text{Cliff}(S) = 0\).

2. Notice that this max is taken over “super-special” divisors. Why?

3. Find two distinct classes of Riemann surfaces \(S\) with \(\text{Cliff}(S) = 1\).