Riemann Surfaces and Graphs
7. Special Divisors

If D is a divisor on a Riemann surface S of genus g, then we set

$$l(D) = \dim(V(D))$$

so D is a **special** divisor (see §6) if $l(D) > 0$ and $l(K_S - D) > 0$.

Lemma 7.1. Every effective divisor D of degree $\leq g-1$ is special.

Proof. By the Riemann-Roch Theorem,

$$l(K_S - D) - l(D) = g - 1 - \deg(D)$$

so if $l(D) > 0$ and $\deg(D) \leq g-1$, then $l(K_S - D) > 0$. □

So this only imposes a constraint on the (effective) divisors of degree $\geq g$. Let D be an effective divisor on an embedded Riemann surface $S \subset \mathbb{CP}^n$. We say that $D \subset H = V(L)$ if D is dominated by the divisor $\mathrm{div}(L)$.

Definition 7.2. The *linear span* $\langle D\rangle \subseteq \mathbb{CP}^n$ of D is the intersection:

$$\langle D\rangle := \bigcap_{D\subset H} H$$

and if no hyperplane contains D in this sense, then D spans $\mathbb{CP}^n$.

Examples. (a) $\langle p+q\rangle$ is the *secant line* in $\mathbb{CP}^n$ through p and q.

(b) $\langle p+q+r\rangle$ is a plane unless the three points are collinear in $\mathbb{CP}^n$, in which case $\langle p+q+r\rangle = \langle p+q\rangle$ is called a tri-secant line.

(c) $\langle 2p\rangle$ is the *tangent line* to S at p in $\mathbb{CP}^n$, and $\langle 3p\rangle$ is a projective plane unless $\langle 2p\rangle = \langle 3p\rangle$, and $p \in S$ is an inflection point of the embedding.

Lemma 7.3. If S is not hyperelliptic, then an effective divisor D is special if and only if $\langle D\rangle \neq \mathbb{CP}^{g-1}$ for the canonical embedding $S \subset \mathbb{CP}^{g-1}$.

Proof. The effective divisors $\mathrm{div}(L)$ for the canonical embedding of S are precisely the elements of $|K_S|$, so $V(K_S - D) \neq 0$ if and only if D is dominated by **some** divisor $\mathrm{div}(L)$, i.e. $D \subset H = V(L)$. □

Let S be hyperelliptic (through Lemma 7.4) and let $\phi : S \to \mathbb{CP}^1$ be the degree two map and:

$$F = (g-1)(\phi^*(\infty))$$

where $\phi^*(\infty)$ is the (degree two) divisor of poles of ϕ.

Then $1, \phi, \phi^2, \ldots, \phi^{g-1} \in V(F)$ are linearly independent meromorphic functions, so $l(F) \geq g$. On the other hand, by the Riemann-Roch Theorem,

$$l(F) - l(K_S - F) = g - 1$$

and since $\deg(K_S - F) = 0$, it follows that:

(i) $l(F) = g$, and

(ii) $l(K_S - F) = 1$, so $K_S \sim F$ via $\phi \in V(K_S - F)$.

Thus $1, \phi, \cdots, \phi^{g-1}$ is a basis for $V(K_S) \cong V(F)$, and:

$$S \xrightarrow{\phi} \mathbb{CP}^1 \overset{(1:z:\cdots:z^{g-1})}{\longrightarrow} \mathbb{CP}^{g-1}$$

is the canonical map for S. All the canonical divisors on S are of the form:

$$\phi^*(q_1 + \cdots + q_{g-1}) = (p_1 + \tau(p_1)) + \cdots + (p_{g-1} + \tau(p_{g-1}))$$

where τ is the involution of S that exchanges the points of the fibers $\phi^{-1}(q)$.

Lemma 7.4. An effective divisor D on S is special if and only if

$$D = \phi^*(q_1 + \cdots + q_m) + p_{m+1} + \cdots + p_n$$

where $p_i \neq \tau(p_j)$ for all $i \neq j$ and $n \leq g - 1$. Moreover, $l(D) = m + 1$ for this divisor and, in particular, the points p_i are the **base points** of $|D|$.

Proof. Given D in the form above, let $q_i = \phi(p_i)$. Then D is dominated by $\phi^*(q_1 + \cdots + q_n)$ which is dominated by a canonical divisor since $n \leq g-1$ (it **is** a canonical divisor if $n = g - 1$). So such a divisor D is special. Conversely, if $l(K_S - D) > 0$, then D is dominated by a canonical divisor, i.e. $\phi^*(q_1 + \cdots + q_{g-1})$ for some $q_1, ..., q_{g-1}$, and then D has the desired form.

Since $l(q_1 + \cdots + q_m) = m+1$ (on $\mathbb{CP}^1$), we right away get $l(D) \geq m+1$. In addition, since $l(\phi^*(q_1 + \cdots + q_{g-1})) = g$ for all $q_1, ..., q_{g-1}$, it follows that $l(\phi^*(q_1 + \cdots + q_n)) = n + 1$ for all $n \leq g - 1$. But

$$D = \phi^*(q_1 + \cdots + q_n) - \tau(p_{m+1}) - .... - \tau(p_n)$$

The subtraction of each point disallows a pole at $\tau(p_i)$ which reduces the dimension of the linear series by one (check this!), and

$$l(D) = n + 1 - (n - m) = m + 1 \quad \square$$

Clifford's Theorem. If D is a special divisor on S, then

$$2(l(D)-1) \leq \deg(D)$$

with equality if and only if either $D = 0$ or $D = K_S$ or:

S is hyperelliptic and $D = \phi^*(q_1 + \cdots + q_m)$ for $0 < m < g-1$.

Proof. For a hyperelliptic Riemann surface S, we have:

$$\deg(D) = 2m + (n-m) = m + n \text{ and } l(D) = m+1$$

for the special divisors D in Lemma 7.4, and Clifford's Theorem for S follows.

For an arbitrary Riemann surface S, we claim first that:

$$l(D+E) \geq l(D) + l(E) - 1 \text{ for all effective divisors } D \text{ and } E$$

Given an effective divisor D, then for any choice of $p \in S$, we either have $l(D-p) = l(D)$ (for the finite set of base points) or else $l(D-p) = l(D)-1$. Thus if $l(D) = r+1$, then $l(D - p_1 - \cdots - p_r) > 0$ for all $p_1, ..., p_r$ and $l(D - p_1 - \cdots p_r) = 1$ for a "general" choice of points, in which case there is a **unique** divisor $D' \sim D$ dominating $p_1 + \cdots + p_r$. In addition, if $l(D) \leq r$, then $l(D - p_1 - \cdots - p_r) = 0$ for a general set of r points.

If $l(D) = r+1$ and $l(E) = s+1$ and $p_1, \cdots, p_{r+s}$ is a general set of points, then $l(D - p_1 - \cdots - p_r) = 1$ and $l(E - p_{r+1} - \cdots - p_{r+s}) = 1$, so there are unique divisors $D' \sim D$ and $E' \sim E$ dominating the sets of points, and then $D' + E'$ dominates $p_1 + \cdots + p_{r+s}$, so $l(D + E - p_1 - \cdots - p_{r+s}) > 0$. Thus:

$$l(D+E) \geq (r+s) + 1 = l(D) + l(E) - 1$$

and if equality holds, then every divisor in $|D+E|$ that contains a general set of $r+s$ points of S is of the form $D' + E'$ for $D' \in |D|$ and $E' \in |E|$.

When we let D be a special divisor and set $E = K_S - D$, this gives:

$$g = l(K_S) \geq l(D) + l(K_S - D) - 1 = l(D) + (l(D) - \deg(D) + g - 1) - 1$$

by the Riemann-Roch Theorem. This is the Clifford inequality! Moreover, if equality holds, then each canonical divisor $F \in |K_S|$ that dominates $g-1$ general points $p_1, ..., p_{g-1} \in S$ is a sum:

$$(*)\ F = D' + E' \text{ where } D' \sim D \text{ and } E' \sim K_S - D$$

Reality check: This holds when S is hyperelliptic and $D = \phi^*(q_1 + \cdots + q_m)$.

It only remains to show that $(*)$ is impossible when S is non-hyperelliptic. So assume D is a special divisor satisfying the Clifford equality.

First, the Riemann-Roch Theorem gives:

$$0 = \deg(D) - 2((l(D) - 1) = \deg(K_S - D) - 2(l(K_S - D) - 1)$$

so we may replace D if necessary by $K_S - D$ to assume that $\deg(D) \le g-1$. Next, we may assume $l(D) > 1$ since otherwise the equality gives $D = 0$.

Finally, there is a very useful geometric interpretation of $l(D)$ in terms of the linear span of the divisor D under the canonical embedding:

$$\dim(\langle D \rangle) = g - 1 - l(K_S - D) = \deg(D) - l(D)$$

In particular, since $l(D) > 1$, the points of D are linearly **dependent**.

The last piece of the puzzle is the following:

General Position Theorem. If $S \subset \mathbb{CP}^n$ is an embedded Riemann surface of degree δ not in a hyperplane, then the points of the transverse intersection:

$$S \cap H = \operatorname{div}(L) = p_1 + \cdots + p_\delta$$

of S with a general hyperplane are "algebraically indistinguishable." In particular, **every** collection of n of the points is linearly independent!

Assuming this, we finish the proof of Clifford's Theorem. If $p_1, \ldots, p_{g-1}$ are general points of S, then they span a general hyperplane H under the canonical embedding of S, and by the general position theorem, **every** set of $g - 1$ of the $2g - 2$ points of $F = \operatorname{div}(L) \sim K_S$ is linearly independent. But if the Clifford equality holds for the divisor D, then:

$$F = D' + E' \text{ for } D' \sim D \text{ and } E' \sim K_S - D$$

and the points of D' are linearly dependent since $l(D') = l(D) > 1$. □

Assignment. 1. Ask questions. (There are lots of new ideas here).

Definition 7.5. The **Clifford index** of a special divisor on S is:

$$\operatorname{Cliff}(D) = 2((l(D) - 1) - \deg(D) \ge 0$$

The Clifford index of a Riemann surface S is:

$$\operatorname{Cliff}(S) = \max\{\operatorname{Cliff}(D) \mid l(D) > 1,\ l(K_S - D) > 1\}$$

Thus hyperellitpic Riemann surfaces are those with $\operatorname{Cliff}(S) = 0$.

2. Notice that this max is taken over "super-special" divisors. Why?

3. Find two distinct classes of Riemann surfaces S with $\operatorname{Cliff}(S) = 1$.