Riemann Surfaces and Graphs
6. The Riemann Roch Theorem

The Riemann-Roch Theorem is a relation between two vector spaces associated to a divisor D of degree d on a Riemann surface S. Namely:

$V(D) = \{\phi \in \mathbb{C}(S) - \{0\} \text{ with } \mathrm{div}(\phi) + D \geq 0\} \cup \{0\}$ and

$W(-D) = \{\omega \in \Omega[S] - \{0\} \text{ with } \mathrm{div}(\omega) - D \geq 0\} \cup \{0\}$

If $D = \sum d_i p_i$ is an *effective* divisor, then $V(D)$ is the vector space of meromorphic functions on S with poles of order $\leq d_i$ at each of the points p_i (and no other poles), and $W(-D)$ is the vector space of holomorphic differentials on S with zeroes of order $\geq d_i$ at each point p_i (and no poles).

Choose a holomorphic differential $\omega \in \Omega(S)$, and set:

$$K_S = \mathrm{div}(\omega)$$

This is called a "canonical" effective divisor of degree $2g - 2$, though it isn't actually canonical. On the other hand, once ω is chosen, then the vector spaces $V(K_S - D)$ and $W(-D)$ are isomorphic via the map $\phi \mapsto \phi \cdot \omega$ so we will follow the literature and replace $W(-D)$ with $V(K_S - D)$, keeping in mind that it is really $W(-D)$ that we want to work with.

Note. As we saw earlier, graphs **do** have truly canonical divisors.

Riemann-Roch. The dimensions of $V(D)$ and $V(K_S - D)$ satisfy:

$$\dim(V(D)) - \dim(V(K_S - D)) = \deg(D) + 1 - g$$

where g is the genus of the Riemann surface S.

Note. We have assumed a case of the Riemann-Roch Theorem, namely:

$$\dim(V(0)) - \dim(V(K_S)) = 1 - g$$

since $V(0)$ are the constant functions and $V(K_S)$ is isomorphic to the vector space of holomorphic differentials, which we assumed to have dimension g. Moreover, switching the roles of 0 and K_S, we have:

$$\dim(V(K_S)) - \dim(V(0)) = g - 1 = \deg(K_S) + 1 - g$$

since we have seen that $\deg(K_S) = 2g - 2$.

We begin by using residues to prove a Riemann-Roch *inequality*:

Proposition 6.1. If D is linearly equivalent to an **effective** divisor, then:

$$\dim(V(D)) - \dim(V(K_S - D)) \leq \deg(D) + 1 - g$$

Proof: We may assume D itself is effective, since $V(D) \cong V(E)$ and $V(K_S - D) \cong V(K_S - E)$ whenever $D \sim E$. Note that D being linearly equivalent to an effective divisor is the **same** as the condition $V(D) \neq 0$, and when D is effective, then the constant functions are in $V(D)$, exhibiting the fact that $V(D)$ is not the zero space. Note also that when $V(D) \neq 0$, then $|D| \neq \emptyset$ and $\dim(|D|) = \dim(V(D) - 1$.

Next we introduce the vector space of Laurent tails, which we define (non-canonically) by choosing a local coordinate z_i near p_i with $z_i = 0$ at the point p_i and then setting $\mathrm{Laur}(D) = \{a_{i,-d_i} z_i^{-d_i} + \cdots + a_{i,-1} z_i^{-1} \mid a_{i,j} \in \mathbb{C}\}$, a vector space of dimension $d = \deg(D)$. We are interested in two maps:

(i) The "tail" map:

$$\lambda : V(D) \to \mathrm{Laur}(D)$$

expanding ϕ as a Laurent series in the variables z_i and truncating, and:

(ii) The (locally defined) "residue" pairing:

$$\mathrm{Laur}(D) \times \Omega[S] \to \mathbb{C}$$

expanding $\omega = \psi(z_i) dz_i \in \Omega[S]$ around each point p_i, multiplying by the Laurent tail, "reading" off the coefficients of $z_i^{-1} dz_i$, and taking their sum. This defines a linear map:

$$\rho : \mathrm{Laur}(D) \to \Omega[S]^*$$

(a) The kernel of λ is the vector space of constant functions.

(b) The image of ρ is the kernel of the map $\Omega[S]^* \to W(-D)^*$, and

(c) The composition $\rho \circ \lambda$ is the zero map. In other words:

$$0 \to \mathbb{C} \to V(D) \to \mathrm{Laur}(D) \to \Omega[S]^* \to V(K_S - D)^* \to 0$$

is a complex of vector spaces that is exact everywhere except possibly at the middle term, and then it follows that:

$$1 - \dim(V(D)) + d - g + \dim(V(K_S - D) \geq 0$$

which is the desired inequality. □

Remarkably, a case of the Riemann-Roch Theorem follows!

Corollary 6.2. If D and $K_S - D$ are **both** linearly equivalent to effective divisors, then the Riemann-Roch equality holds for D (and $K_S - D$).

Proof. Apply the Proposition twice!

(1) $\dim(V(D)) - \dim(V(K_S - D)) \leq \deg(D) + 1 - g$ if D is effective.

(2) $\dim(V(K_S - D)) - \dim(V(D)) \leq \deg(K_S - D) + 1 - g = -(\deg(D) + 1 - g)$ if $K_S - D$ is linearly equivalent to an effective divisor.

Taken together, these give the Riemann-Roch equality. □

Definition 6.3. A divisor D is **special** if D and $K_S - D$ are both linearly equivalent to effective divisors.

Thus we have the Riemann-Roch Theorem for special divisors.

Note. Because an effective divisor has non-negative degree, a special divisor must satisfy $0 \leq \deg(D) \leq 2g - 2$, and of course being special is symmetric; D is special if and only if $K_S - D$ is also special. We will see that most divisors in this degree range are **not** special.

Together with the results from §3, we get some nice consequences:

Proposition 6.4. If $g(S) \geq 1$, the linear series $|K_S|$ is base-point free.

Proof. When $g = 1$, then $K_S \sim 0$ and $|0|$ is base-point free.

When $g \geq 2$, then each point $p \in S$ is special as a divisor, since $K_S - p$ is also effective (we can always find a non-zero differential ω with $\omega(p) = 0$). Thus by Corollary 6.2., we have:

$$\dim(V(K_S - p)) - \dim(V(p)) = \deg(K_S) - 1 + 1 - g = g - 2$$

But $\dim(V(p)) = 1$, since $V(p)$ consists entirely of the constant functions, otherwise S would have a meromorphic function $\phi \in \mathbb{C}(S)$ with a single pole at p, which would determine a degree one **isomorphism** $\phi : S \to \mathbb{CP}^1$. Thus

$$\dim(V(K_S - p)) = g - 1 = \dim(V(K_S)) - 1$$

which is to say that p is not a base point of the linear series $|K_S|$.

Next, a definition:

Definition 6.5. A Riemann surface S of genus $g \geq 2$ is **hyperelliptic** if there is a meromorphic function $\phi \in \mathbb{C}(S)$ such that the holomorphic map $\phi : S \to \mathbb{CP}^1$ has degree two.

Example. Every Riemann surface S of genus two is hyperelliptic.

Indeed, if ω and τ are linearly independent holomorphic differentials on S (which exist in every genus $g \geq 2$), then $\omega = \phi \cdot \tau$ for a non-constant meromorphic function ϕ, which defines a holomorphic map $\phi : S \to \mathbb{CP}^1$ of degree $\leq 2g-2$ (it is smaller than $2g-2$ if ω and τ share common zeroes). When $g = 2$, this is therefore a map of degree exactly 2.

Let S be a Riemann surface of genus $g \geq 3$.

Proposition 6.6. S is hyperelliptic if and only if the base-point-free linear series $|K_S|$ **fails to** embed S in $\mathbb{CP}^{g-1}$.

Proof. Suppose $\phi : S \to \mathbb{CP}^1$ is a map of degree two, and let:

$$\mathrm{div}(\phi) = p + q - r - s$$

(i.e. p and q are the zeroes of ϕ and r and s are the poles). Then:

$$\dim(V(r+s)) = 2$$

But $r+s$ and $K_S - r - s$ are special divisors (because $\dim(V(K_S)) \geq 3$), therefore the Riemann-Roch Theorem applies, and we get:

$$\dim(V(K_S - r - s)) - \dim(V(r+s)) = (2g-4) + 1 - g = g - 3$$

so $\dim(V(K_S - r - s)) = g - 1 = \dim(V(K_S - p)) = \dim(V(K_S - q))$. Thus, by Proposition 3.13, the map defined by $|K_S|$ fails to be injective because $\Phi(p) = \Phi(q)$ (or fails to be an immersion at p if $p = q$).

The converse also holds. If the map Φ associated to the linear series $|K_S|$ either fails to be injective or fails to be an immersion, then there is a divisor $p + q$ with the property that $V(K_S - p - q) = V(K_S - p) = V(K_S - q)$ and so by the Riemann Roch Theorem, $\dim(V(p+q)) \geq 2$, and there is a (non-constant) meromorphic function ϕ with poles only at p and q. □

Your (Revised) Assignment. Read this and make sense of it. Then:

5. (New) Prove that if S is hyperelliptic, then the map to $\mathbb{CP}^{g-1}$ given by the canonical linear series **factors** through the degree two map $\phi : S \to \mathbb{CP}^1$, followed by an embedding of $\mathbb{CP}^1$. Conclude that the degree two map is **unique**, if it exists, i.e. a hyperelliptic Riemann surface is hyperelliptic in only one way.