Riemann Surfaces and Graphs
6. The Riemann Roch Theorem

The Riemann-Roch Theorem is a relation between two vector spaces associated to a divisor D of degree d on a Riemann surface S. Namely:

$V(D) = \{\phi \in \mathbb{C}(S) - \{0\} \text{ with } \mathrm{div}(\phi) + D \geq 0\} \cup \{0\}$ and

$W(-D) = \{\omega \in \Omega[S] - \{0\} \text{ with } \mathrm{div}(\omega) - D \geq 0\} \cup \{0\}$

If $D = \sum d_i p_i$ is an *effective* divisor, then $V(D)$ is the vector space of meromorphic functions on S with a pole of order $\leq d_i$ at each point p_i (and no other poles), and $W(-D)$ is the vector space of holomorphic differentials on S with zeroes of order $\geq d_i$ at each point p_i (and no poles).

Choose a holomorphic differential $\omega \in \Omega(S)$, and set:

$$K_S = \mathrm{div}(\omega)$$

This is called a "canonical" effective divisor of degree $2g-2$, though it isn't actually canonical. On the other hand, once ω is chosen, then the vector spaces $V(K_S - D)$ and $W(-D)$ are isomorphic via the map $\phi \mapsto \phi \cdot \omega$ so we will follow the literature and replace $W(-D)$ with $V(K_S - D)$, keeping in mind that it is really $W(-D)$ that we want to work with.

Note. As we saw earlier, graphs **do** have truly canonical divisors.

Riemann-Roch. The dimensions of $V(D)$ and $V(K_S - D)$ satisfy:

$$\dim(V(D)) - \dim(V(K_S - D)) = \deg(D) + 1 - g$$

where g is the genus of the Riemann surface S.

Note. We have assumed a case of the Riemann-Roch Theorem, namely:

$$\dim(V(0)) - \dim(V(K_S)) = 1 - g$$

since $V(0)$ are the constant functions and $V(K_S)$ is isomorphic to the vector space of holomorphic differentials, which we assumed to have dimension g. Moreover, switching the roles of 0 and K_S, we have another case:

$$\dim(V(K_S)) - \dim(V(0)) = g - 1 = \deg(K_S) + 1 - g$$

since we have seen that $\deg(K_S) = 2g - 2$.

We begin by using residues to prove a Riemann-Roch *inequality*:

Proposition 6.1. If D is linearly equivalent to an **effective** divisor, then:

$$\dim(V(D)) - \dim(V(K_S - D)) \leq \deg(D) + 1 - g$$

Proof: We may assume D itself is effective, since $V(D) \cong V(E)$ and $V(K_S - D) \cong V(K_S - E)$ whenever $D \sim E$. Note that D being linearly equivalent to an effective divisor is the **same** as the condition $V(D) \neq 0$, and when D is effective, then the constant functions are in $V(D)$, exhibiting the fact that $V(D)$ is not the zero space. Note also that when $V(D) \neq 0$, then $|D| \neq \emptyset$ and $\dim(|D|) = \dim(V(D) - 1$.

Next we introduce the vector space of Laurent tails, which we define (non-canonically) by choosing a local coordinate z_i near p_i with $z_i = 0$ at the point p_i and then setting $\mathrm{Laur}(D) = \{a_{i,-d_i} z_i^{-d_i} + \cdots + a_{i,-1} z_i^{-1} \mid a_{i,j} \in \mathbb{C}\}$, a vector space of dimension $d = \deg(D)$. We are interested in two maps:

(i) The "tail" map:

$$\lambda : V(D) \to \mathrm{Laur}(D)$$

expanding ϕ as a Laurent series in the variables z_i and truncating, and:

(ii) The (locally defined) "residue" pairing:

$$\mathrm{Laur}(D) \times \Omega[S] \to \mathbb{C}$$

expanding $\omega = \psi(z_i) dz_i \in \Omega[S]$ around each point p_i, multiplying by the Laurent tail, "reading" off the coefficients of $z_i^{-1} dz_i$, and taking their sum. This defines a linear map:

$$\rho : \mathrm{Laur}(D) \to \Omega[S]^*$$

(a) The kernel of λ is the vector space of constant functions.

(b) The image of ρ is the kernel of the map $\Omega[S]^* \to W(-D)^*$, and

(c) The composition $\rho \circ \lambda$ is the zero map. In other words:

$$0 \to \mathbb{C} \to V(D) \to \mathrm{Laur}(D) \to \Omega[S]^* \to V(K_S - D)^* \to 0$$

is a complex of vector spaces that is exact everywhere except possibly at the middle term, and then it follows that:

$$1 - \dim(V(D)) + d - g + \dim(V(K_S - D) \geq 0$$

which is the desired inequality. $\square$

Remarkably, a significant case of the Riemann-Roch Theorem follows!

Corollary 6.2. If D and $K_S - D$ are **both** linearly equivalent to effective divisors, then the Riemann-Roch equality holds for D (and $K_S - D$).

Proof. Apply the Proposition twice!

(1) $\dim(V(D)) - \dim(V(K_S - D)) \leq \deg(D) + 1 - g$ and

(2) $\dim(V(K_S - D)) - \dim(V(D)) \leq \deg(K_S - D) + 1 - g = -(\deg(D) + 1 - g)$

Taken together, these give the Riemann-Roch equality. □

Definition 6.3. A divisor D is **special** if D and $K_S - D$ are both linearly equivalent to effective divisors.

Thus we have the Riemann-Roch Theorem for special divisors.

Note. Because an effective divisor has non-negative degree, a special divisor must satisfy $0 \leq \deg(D) \leq 2g - 2$, and of course being special is symmetric; D is special if and only if $K_S - D$ is also special. We will see that most divisors in this degree range are **not** special.

Together with the results from §3, we get some nice consequences:

Proposition 6.4. If $g(S) \geq 1$, the linear series $|K_S|$ is base-point free.

Proof. When $g = 1$, then $K_S \sim 0$ and $|0|$ is base-point free.

When $g \geq 2$, then each point $p \in S$ is special as a divisor, since $K_S - p$ is also effective (we can always find a non-zero differential ω with $\omega(p) = 0$). Thus by Corollary 6.2., we have:

$$\dim(V(K_S - p)) - \dim(V(p)) = \deg(K_S) - 1 + 1 - g = g - 2$$

But $\dim(V(p)) = 1$, since $V(p)$ consists entirely of the constant functions, otherwise S would have a meromorphic function $\phi \in \mathbb{C}(S)$ with a single pole at p, which would determine a degree one **isomorphism** $\phi : S \to \mathbb{CP}^1$. Thus

$$\dim(V(K_S - p)) = g - 1 = \dim(V(K_S)) - 1$$

which is to say that p is not a base point of the linear series $|K_S|$.

Next, a definition:

Definition 6.5. A Riemann surface S of genus $g \geq 2$ is **hyperelliptic** if there is a meromorphic function $\phi \in \mathbb{C}(S)$ such that the holomorphic map $\phi : S \to \mathbb{CP}^1$ has degree two.

i.e. S is hyperelliptic if there is some $\phi \in \mathbb{C}(S)$ with two poles.

Example. Every Riemann surface S of genus two is hyperelliptic.

Indeed, if ω and τ are linearly independent holomorphic differentials on S (which exist in every genus $g \geq 2$), then $\omega = \phi \cdot \tau$ for a non-constant meromorphic function ϕ, which defines a holomorphic map $\phi : S \to \mathbb{CP}^1$ of degree $\leq 2g-2$ (it is smaller than $2g-2$ if ω and τ share common zeroes). When $g = 2$, this is therefore a map of degree exactly 2 (not 1, because in that case, it would define an isomorphism between S and $\mathbb{CP}^1$, as above).

Let S be a Riemann surface of genus $g \geq 3$.

Proposition 6.6. S is hyperelliptic if and only if the base-point-free linear series $|K_S|$ **fails to** embed S in $\mathbb{CP}^{g-1}$.

Proof. Suppose $\phi : S \to \mathbb{CP}^1$ is a map of degree two, and let:

$$\mathrm{div}(\phi) = p + q - r - s$$

(i.e. p and q are the zeroes of ϕ and r and s are the poles). Then:

$$\dim(V(r+s)) \geq 2 \text{ because } 1, \phi \in V(r+s)$$

But $r+s$ and $K_S - r - s$ are special divisors (because $\dim(V(K_S)) \geq 3$), therefore the Riemann-Roch Theorem applies, and we get:

$$\dim(V(K_S - r - s)) - \dim(V(r+s)) = (2g-4) + 1 - g = g - 3$$

so $\dim(V(K_S - r - s)) \geq g - 1 = \dim(V(K_S - r)) = \dim(V(K_S - w))$. Thus, by Proposition 3.13, the map defined by $|K_S|$ fails to be injective because r and s have the same image (or fails to be an immersion at r if $r = s$).

The converse also holds. If the map associated to the linear series $|K_S|$ either fails to be injective or fails to be an immersion, then there is a divisor $p + q$ with the property that $V(K_S - p - q) = V(K_S - p) = V(K_S - q)$ and so by the Riemann Roch Theorem, $\dim(V(p+q)) \geq 2$, and there is a (non-constant) meromorphic function ϕ with poles only at p and q. $\square$

RR Assignment 1. Read this and make sense of it. Then:

Prove (a), (b) and (c) in Proposition 6.1 and then (more challenging) prove that if S is hyperelliptic, then the map to $\mathbb{CP}^{g-1}$ given by the canonical linear series **factors** through the degree two map $\phi : S \to \mathbb{CP}^1$, followed by an embedding of $\mathbb{CP}^1$. Conclude that the degree two map is **unique**, if it exists, i.e. a hyperelliptic Riemann surface is hyperelliptic in only one way.

What about the Riemann-Roch Theorem in general? For example, given an effective divisor D of positive degree on a Riemann surface of genus $g \geq 1$, $K_S + D$ is linearly equivalent to an effective divisor and $V(-D) = 0$, so:

$$\dim(V(K_S + D)) \leq g + \deg(D) - 1$$

by the Riemann-Roch inequality, but Corollary 6.2 does not apply since $-D = K_S - (K_S + D)$ is not linearly equivalent to an effective divisor.

This particular example is quite important, since equality would give the exactness of an analogue of the exact sequence in Proposition 6.1(c) for meromorphic **differentials** (as opposed to a meromorphic functions):

$$0 \to W(0) = \Omega[S] \to W(D) \to \mathrm{Laur}(D) \to \mathbb{C} \to 0$$

which, in terms of local coordinates z_i around the points p_i appearing in D, maps a meromorphic differential to its Laurent tail and maps a Laurent tail to its "residue," namely the sum of the coefficients $a_{i,-1}$ of each z_i^{-1}. Riemann-Roch in this context states that **every** Laurent tail satisfying the "zero residue" condition is the tail of a meromorphic differential. This will be used in the proof of Abel's Theorem.

Definition 6.7. A divisor E **dominates** D if $E - D$ is effective.

Proposition 6.8. If the Riemann-Roch Theorem holds for a divisor E that dominates D, then D satisfies the (opposite!) Riemann-Roch inequality:

$$\dim(V(D)) - \dim(V(K_S - D) \geq \deg(D) + 1 - g$$

Proof. Let $E = \sum e_i p_i$ and $D = \sum d_i p_i$ with $d_i \leq e_i$ by assumption, and let $\mathrm{Laur}(E, D)$ be the vector space of tails relative to D, i.e.

$$\mathrm{Laur}(E, D) = \{a_{i,-e_i} z^{-e_i} + \cdots + a_{i,-d_i-1} z^{-d_i-1}\}$$

given a choice of local coordinate z_i near each p_i. This is a vector space of dimension $\deg(E) - \deg(D)$ pairing with $W(-D)$ via the residue map:

$$\mathrm{Laur}(E, D) \times W(-D) \to \mathbb{C}$$

which, as in the proof of Proposition 6.1 gives a complex of vector spaces:

$$0 \to V(D) \to V(E) \to \mathrm{Laur}(E, D) \to W(-D) \to W(-E) \to 0$$

that is exact everywhere except possibly the middle term. Then:

$$\deg(E) - \deg(D) \geq \dim(V(E)) - \dim(V(D) + \dim(W(-D)) - \dim(W(-E))$$

But by assumption, the Riemann-Roch Theorem holds for E, so:

$$\deg(E) - \deg(D) \geq (\deg(E) + 1 - g) - (\dim(V(D)) - \dim(W(-D))$$

giving the desired inequality. $\square$

Corollary 6.9. Suppose the Riemann-Roch Theorem is known for a set of divisors on S that includes a divisor that dominates any given D. Then the Riemann-Roch Theorem follows for all divisors D.

Proof. Given D, then by assumption both D and $K_S - D$ are dominated by divisors for which the Riemann-Roch Theorem is known. Then as in Corollary 6.2, the two Riemann-Roch inequalities for D and for $K_S - D$ imply the Riemann-Roch equality.

Remark. The divisors $K_S + D$ for arbitrary effective divisor D make up a set of divisors satisfying the criterion on Corollary 6.9, but as hinted at above, I don't know of a simple proof that these satisfy the Riemann-Roch equality. Instead, we will assume that the Riemann surface is embedded in projective space and use "hypersurface divisors" as the desired class. We have already seen that every Riemann surface of genus ≥ 3 that is **not** hyperelliptic has such an embedding. We consider instead the:

Hyperelliptic Riemann Surfaces. With the exception of finitely many points, a complex *plane curve* describes a hyperelliptic Riemann surface via:

$$C = \{(x, y) \in \mathbb{C}^2 \mid f(x, y) = y^2\alpha(x) + y\beta(x) + \gamma(x) = 0\} \subset \mathbb{C}^2$$

where $f(x, y)$ has degree two in y and degree $d = \max(\deg(\alpha, \beta, \gamma))$ in x. We will also assume that α, β and γ are all degree d polynomials with no multiple roots and no shared roots. Each fiber $\pi^{-1}(x_0)$ of the projection $\pi : C \to \mathbb{C}$ to the x-axis is the set of zeroes of the polynomial:

$$f(x_0, y) = y^2\alpha(x_0) + y\beta(x_0) + \gamma(x_0) \text{ which consists of:}$$

(i) Two distinct points if $\alpha(x_0) \neq 0$ and $\Delta(x_0) \neq 0$, where

$$\Delta(x) = \beta(x)^2 - 4\alpha(x)\gamma(x)$$

is the discrminant, which we'll assume to also be of maximal degree $2d$.

(ii) One (ramified) point if $\Delta(x_0) = 0$ but $\alpha(x_0) \neq 0$

(iii) One non-ramified point if $\alpha(x_0) = 0$ but $\Delta(x_0) = \beta(x_0)^2 \neq 0$.

The condition for C to be a (one-dimensional) complex manifold is:

$$f(x,y) \neq 0 \ \text{ or } \ \frac{\partial f}{\partial x}(x,y) \neq 0 \ \text{ or } \ \frac{\partial f}{\partial y}(x,y) \neq 0$$

for all points $(x,y) \in \mathbb{C}^2$. But $f(x_0,y_0) = 0 = (\partial f/\partial y)(x_0,y_0)$ if and only if:

$$\Delta(x_0) = 0 \ \text{ and } \ y_0 = -\frac{\beta(x_0)}{2\alpha(x_0)}$$

and the additional condition $(\partial f/\partial x)(x_0,y_0) = 0$ is equivalent to $\Delta'(x_0) = 0$.

Thus for $f(x,y)$ to define a Riemann surface, one only needs to be sure that the discriminant polynomial $\Delta(x)$ has no multiple roots. Next, we introduce a z variable to "homogenize the y variable" in $f(x,y)$ giving:

$$y^2\alpha(x) + yz\beta(x) + z^2\gamma(x)$$

the zeroes of which, in $\mathbb{C} \times \mathbb{CP}^1$ add a point to C over each root of $\alpha(x)$, which are the "missing" points of the projection map in case (iii) above. Finally, we introduce a w-variable to homogenize the x variable, replacing:

$$\alpha(x) \text{ by } A(x,w) = w^d \cdot \alpha(x/w), \text{ etc}$$

or equivalently, if $\alpha(x) = a(x-r_1)\cdots(x-r_d)$, then

$$A(x,w) = a(x-r_1w)\cdots(x-r_dw)$$

This further enlarges C, adding two more points and completing it to a closed Riemann surface S embedded in $\mathbb{CP}^1 \times \mathbb{CP}^1$. Notice that:

$$\pi^*(x) = \phi \text{ is the meromorphic function on } S \text{ with two poles}$$

defining S as a hyperelliptic Riemann surface, and:

$$\rho^*(y) = \psi \text{ is another meromorphic function on } S \text{ with } d \text{ poles}$$

where ρ is the "other" projection to the y-axis (extended to $\mathbb{CP}^1$).

Proposition 6.10. The genus of S is $d-1$.

Proof. By the Riemann-Hurwitz formula, since $\pi : S \to \mathbb{CP}^1$ ramifies only over the $2d$ zeroes of the discriminant $\Delta(x)$, we get:

$$2g-2 = 2(-2) + \sum(e_p - 1) = -4 + 2d \ \text{ and } \ g = d-1 \quad \square$$

In particular, the "extra" meromorphic function $\rho^*(y)$ has $g+1$ poles.

Proposition 6.11. Every hyperelliptic S is isomorphic to one of these.

Proof. If S, S' are hyperelliptic Riemann surfaces of genus g with maps:

$$\phi : S \to \mathbb{CP}^1 \text{ and } \psi : S' \to \mathbb{CP}^1$$

of degree two, ramified over the **same** set of points $x_1, ..., x_{2d+2} \in \mathbb{CP}^1$, then $S \cong S'$. Thus, to prove that our construction gives all hyperelliptic curves, we need to simply find, given distinct complex numbers $x_1, ..., x_{2d+2} \in \mathbb{C}$, three polynomials $\alpha(x), \beta(x)$ and $\gamma(x)$ each of degree $d+1$ such that:

$$\Delta(x) = \beta(x)^2 - 4\alpha(x)\gamma(x) = c(x-x_1)\cdots(x-x_{2d+2})$$

This is left to the reader as an exercise.

A Final Remark. By composing with the further embedding:

$$S \subset \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^3$$

of $\mathbb{CP}^1 \times \mathbb{CP}^1$ in $\mathbb{CP}^3$ as a quadric surface, we see that every hyperelliptic curve embeds in $\mathbb{CP}^3$. Indeed, **every** Riemann surface embeds in $\mathbb{CP}^3$.

RR Assignment 2. 1. Read this bit and give me feedback.

2. Find a graphing calculator (e.g. desmos.com) and play with equations:

$$y^2\alpha(x) + y\beta(x) + \gamma(x) = 0 \text{ of your choosing}$$

The curves you get can be quite intricate. Share your most inspired creations with me, and I'll forward them to the class.

For example, explain the features of the curve:

$$y^2(x^2-1) + y(x^2-6) + (x^2-9) = 0$$

(Keep in mind that we can't see all the features of the complex solutions in this set of real solutions.)

3. Tackle the Exercise in the final Proposition.

Embedded Riemann Surfaces. Let $S \subset \mathbb{CP}^n$ be a Riemann surface embedded in projective space. Note that by Propositions 6.6 and 6.11, every Riemann surface of genus ≥ 2 has such an embedding, as does the genus zero Riemann surface $\mathbb{CP}^1$ and the genus one surfaces $\mathbb{C}/\Lambda$ (via the Weierstrass $\mathcal{P}$ function and its derivative). In fact, this is the complete list of all closed Riemann surfaces (see Appendix A).

We prove the Riemann-Roch Theorem for embedded Riemann surfaces by looking at the linear series of hypersurface divisors of high degree. First, we take a detour into some graded commutative algebra. For $S \subset \mathbb{CP}^n$, let:

$$I_d = \{F \in \mathbb{C}[x_0, ..., x_n]_d \mid F(p) = 0 \text{ for all } p = (p_0 : ... : p_n) \in S\}$$

and $I(S) = \bigoplus_{d=0}^{\infty} I_d \subset \mathbb{C}[x_0, ..., x_n]$ be the homogeneous ideal of polynomials that "carve out" S in $\mathbb{CP}^n$ in the sense of Lemma 3.6. We will also assume that $S \subset \mathbb{CP}^n$ is not contained in any hyperplane, i.e. that $I_0 = I_1 = 0$.

The **homogeneous coordinate ring**:

$$\mathbb{C}[S] := \mathbb{C}[x_0, ..., x_n]/I(S) = \bigoplus_{d=0}^{\infty} \mathbb{C}[S]_d$$

is the quotient by this ideal, which is graded (since I is homogeneous), and:

$$h_S(d) = \dim\left(\mathbb{C}[S]_d\right)$$

is the **Hilbert function** of S. Note that $h_S(0) = 1$ and $h_S(1) = n + 1$.

Commutative Algebra Fact. The Hilbert function is a *linear* function for all values d larger than a fixed $d_0 \in \mathbb{Z}$.

Example. Suppose $S \subset \mathbb{CP}^2$ is embedded in the complex projective plane. In this case the homogeneous ideal is $I = F \cdot \mathbb{C}[x_0, x_1, x_2]$, generated by a single homogeneous polynomial $F \in \mathbb{C}[x_0, x_1, x_2]_\delta$ of degree δ and then:

$$h_{\mathbb{CP}^2}(d) = \dim(\mathbb{C}[x_0, x_1, x_2]_d) = \binom{d+2}{2} \text{ for } d \geq 0 \text{ and}$$

$$h_I(d) = \dim I_d = \dim\left(\mathbb{C}[x_0, x_1, x_2]_{d-\delta}\right) = \binom{d-\delta+2}{2} \text{ for } d \geq \delta$$

Subtracting, we get the linear function:

$$h_S(d) = \delta d + \left(1 - \frac{(\delta-1)(\delta-2)}{2}\right) \text{ for } d \geq \delta$$

Hypersurface Divisors. Let $L = c_0x_0 + \cdots + c_nx_n \in \mathbb{C}[x_0, ..., x_n]_1$ be a nonzero linear form. An effective divisor $\mathrm{div}(L)$ on the Riemann surface S is defined as follows. If $p = (p_0 : ... : p_n) \in S$ and $p_i \neq 0$, then:

$$\frac{L}{x_i} = c_0\frac{x_0}{x_i} + \cdots + c_n\frac{x_n}{x_i}$$

defines a nonzero meromorphic function on S, and $\mathrm{ord}_p(L) := \mathrm{ord}_p(L/x_i)$ which is in fact independent of the choice of i (provided that $p_i \neq 0$). Then

$$\mathrm{div}(L) := \sum_p \mathrm{ord}_p(L) \cdot p$$

(this is basically the same way that div was computed for differentials, and we'll eventually see that both generalize to the notion of div(s) for any *section* of a holomorphic line bundle on S).

Some things to notice about $\mathrm{ord}_p(L)$ and $\mathrm{div}(L)$:

(a) $S \subset \mathbb{CP}^n$ intersects the hyperplane $H = V(L) \subset \mathbb{CP}^n$ at $p \in S$ if and only if $\mathrm{ord}_p(L) > 0$ and H is tangent to S at p if and only if $\mathrm{ord}_p(L) > 1$.

(b) If S is transverse to the hyperplane H, with no points of tangency (such hyperplanes can always be found), then $\deg(\mathrm{div}(L))$ is the number of points in the set $S \cap H$.

If L and L' are two linear forms, then:

$$\deg(\mathrm{div}(L)) - \deg(\mathrm{div}(L')) = \deg(\mathrm{div}(L/L')) = 0$$

since L/L' defines a meromorphic function on S. Thus, $\deg(\mathrm{div}(L)) = \delta$ is independent of the linear form. This is the **degree** of the embedding of S.

The same definition of $\mathrm{div}(F)$ can be made for $F \in \mathbb{C}[x_0, ..., x_n]_d - I_d$, or even better for the non-zero *image* of F in $\mathbb{C}[S]_d$. Namely,

(i) $\mathrm{ord}_p(F) = \mathrm{ord}_p(F/G)$ for any $G \in \mathbb{C}[x_0, ..., x_n]_d$ with $G(p) \neq 0$.

(ii) $\mathrm{div}(F) = \sum_p \mathrm{ord}_p(F) \cdot p$ is a divisor of degree $d\delta$ on S.

and finally, this defines an injective linear map of vector spaces:

$$(*)\ \mathbb{C}[S]_d \hookrightarrow V(\mathrm{div}(G)); \quad F \mapsto F/G = \phi \in \mathbb{C}(S)$$

(for any fixed nonzero $G \in \mathbb{C}[S]_d$).

Remark. In (ii), $\deg(\mathrm{div}(G)) = d\delta$ is an instance of **Bézout's Theorem**.

We call $\mathrm{div}(G)$ a hypersurface divisor on $S \subset \mathbb{CP}^n$. When d is large enough, $\deg(K_S - \mathrm{div}(G)) = 2g - 2 - d\delta < 0$, so $V(K_S - \mathrm{div}(G)) = 0$ and then $(*)$, Proposition 6.1 ($\mathrm{div}(G)$ is effective) and Bézout's Theorem give:

$$h_d(S) \leq \dim(V(\mathrm{div}(G))) \leq d\delta + 1 - g$$

The Riemann-Roch Theorem for these divisors will then follow from:

Theorem 6.12. For $S \subset \mathbb{CP}^n$ of degree δ and genus g,

$$h_d(S) = d\delta + 1 - g$$

for all sufficiently large values of d.

Proof. Let $H = V(L)$ be a hyperplane that is transverse to S, and let $H \cap S = \{p_1, ..., p_\delta\}$. Then for all $d \geq 1$, multiplication by L gives an injective map of vector spaces:

$$\mathbb{C}[S]_{d-1} \to \mathbb{C}[S]_d; \quad F \mapsto F \cdot L \ (\mathrm{mod}\ I_d)$$

to the subspace consisting of elements $G \in \mathbb{C}[S]_d$ with the property that $G(p_i) = 0$ for all i. We can rewrite this as a complex of vector spaces:

$$\mathbb{C}[S]_{d-1} \stackrel{\cdot L}{\to} \mathbb{C}[S]_d \stackrel{ev}{\to} \mathbb{C}^\delta$$

where ev is the evalution at the points p_i (with some choice of coordinate).

Then ev is *surjective* for $d \geq \delta - 1$. Indeed, there are linear forms L_i that vanish only at p_i and none of the other points, and then $\mathrm{ev}(L_1 \cdots \widehat{L}_i \cdots L_\delta)$ are a basis for $\mathbb{C}^\delta$. To get larger degree, simply multiply by any polynomial **not** vanishing at any of the points. Thus for large values of d,

$$h_{d-1}(S) + \delta \leq h_d(S)$$

which immediately implies that for large values of d,

$$d\delta + \text{constant} \leq h_d(S) \leq d\delta + 1 - g$$

and since $h_d(S)$ is eventually a linear function in d, we have $h_d(S) = d\delta + k$ for all sufficiently large values of d and some $k \leq 1 - g$.

Our aim is to prove that $k = 1 - g$ (to be continued....)

RR Assignment 3. (i) Check the computation of $h_S(d)$ in the Example.

(ii) Convince yourself that the degree of a Riemann surface embedded in the plane does agree with the degree of the homogeneous polynomial F that generates its homogeneous ideal.

(iii) Suppose $S \subset \mathbb{CP}^3$ and its homogeneous ideal I is generated by **two** homogeneous polynomials F and G, of degrees γ_1 and γ_2, respectively. Compute the degree δ and the genus g of S in terms of γ_1 and γ_2.

Riemann Surfaces in the Projective Plane: An Extended Example. Let $S \subset \mathbb{CP}^2$ be a plane curve of degree δ. We prove here that:

$$g = \frac{(\delta-1)(\delta-2)}{2}$$

With the earlier computation of the Hilbert function of S, this proves the Riemann-Roch Theorem for hypersurface divisors of large degree on S.

Let $F \in \mathbb{C}[x_0, x_1, x_2]_\delta$ generate the ideal I of S and assume, changing coordinates if necessary, that $(0:1:0) \notin S$ and that the line $x_2 = 0$ is *transverse* to S, necessarily meeting S in δ points.

Proof Sketch. Projecting from $p = (0:1:0)$ gives a holomorphic map:

$$\pi : S \to \mathbb{CP}^1$$

(as a map from $\mathbb{CP}^2$ to $\mathbb{CP}^1$, the projection is defined at every point except for p itself). In the open set $U_2 = \{(x_0 : x_1 : x_2) \mid x_2 \neq 0\} = \mathbb{C}^2$ with local coordinates $x = x_0/x_2$ and $y = x_1/x_2$, π is the projection onto the x-axis. In these coordinates, the line at infinity is $x_2 = 0$ and maps to $\infty \in \mathbb{CP}^1$ and does not contribute to the ramification of the map π. Then:

(i) The degree of the map π is δ.

(ii) The ramification divisor of the map π is:

$$\sum_p (e_p - 1) \cdot p = \mathrm{div}\left(\frac{\partial F}{\partial x_1}\right)$$

Keeping in mind that the partial derivative has degree $\delta - 1$, we get:

$$2g - 2 = \delta \cdot (-2) + \sum_p (e_p - 1) = -2\delta + (\delta - 1) \cdot \delta$$

from the Riemann-Hurwitz and Bézout Theorems, giving the genus formula.

Let's go into each of these in greater detail.

(o) Projection from $p = (0 : 1 : 0)$ becomes projection to the x-axis in U_2.

The lines through p all have equations: $ax_0 + cx_2 = 0$ which, with the exception of the line $x_2 = 0$, all have the form:

$$\frac{x_0}{x_2} = x = -\frac{c}{a}$$

in the local coordinates (x, y). Thus, the fibers of the projection are vertical.

(i) The degree of the map π is δ.

The degree of π is the number of points of the intersection of S with any line through p that intersects S transversely. This number is δ.

(ii) At each point $q = (x_0, y_0) \in S$, we have

$$e_q - 1 = \mathrm{ord}_q\left(\frac{\partial f}{\partial y}\right) = \mathrm{ord}_q\left(\frac{\partial F}{\partial x_1}\right)$$

where $f(x, y) = F(x_0, x_1, x_2)/x_2^\delta$ is the polynomial cutting out S in the open set $U_2 = \mathbb{C}^2$. The second equality follows immediately from the chain rule. The first equality is the crux of the matter.

We can assume $q = (0, 0) \in S$ by translating the x and y variables without affecting either side of the equation, and then:

$$f(x, y) = ax + by + \text{ higher order, with } ab \neq 0$$

and $ax + by = 0$ is the tangent line to S at q.

Suppose first that $b \neq 0$. Then $x = 0$ is **not** tangent to S, so $e_q = 1$. On the other hand, $(\partial f/\partial y)(q) = b \neq 0$, and so $\mathrm{ord}_q(\partial f/\partial y) = 0 = e_q - 1$. Check. Now suppose $b = 0$ (so $a \neq 0$). We collect all terms divisible by x and then factor out the largest power of y in the remaining terms to get:

$$f(x, y) = xg(x, y) + y^e h(x, y) \text{ for some } 1 < e \leq \delta$$

with $g(q) \neq 0$ and $h(q) \neq 0$. Then on $S \cap U_2$ (where $f \equiv 0$), we have:

$$-xg(x, y) = y^e h(x, y), \text{ so } \mathrm{ord}_q(x) + \mathrm{ord}_q(g) = e \cdot \mathrm{ord}_q(y) + \mathrm{ord}_q(h)$$

But as functions on S, $\mathrm{ord}_q(g) = 0 = \mathrm{ord}_q(h)$ and $\mathrm{ord}_q(y) = 1$, since $y = 0$ is not the tangent line. Thus $e = \mathrm{ord}_q(x) = e_q$.

On the other hand, taking the derivative, we have:

$$\frac{\partial f}{\partial y} = x\frac{\partial g}{\partial y} + y^e\frac{\partial h}{\partial y} + ey^{e-1}h$$

and the first and second terms both have order $\geq e$ at q. It follows that:

$$\operatorname{ord}_q\left(\frac{\partial f}{\partial y}\right) = \operatorname{ord}_q(ey^{e-1}h) = e - 1$$

RR Assignment 4. (1) Finish the proof by explaining why $\frac{\partial F}{\partial x_1}(q) \neq 0$ for all points q in the intersection of S with the line $x_2 = 0$, recalling that this line was assumed to intersect S transversally, so $e_q = 1$ for all these points.

(2) Celebrate that we now have Riemann surfaces of genus $3, 6, 10$ etc. as non-singular plane curves of degrees $4, 5, 6$ etc. in $\mathbb{CP}^2$.

Remark. Earlier, we saw that we can find all hyperelliptic Riemann surfaces of every genus $g \geq 2$ embedded in $\mathbb{CP}^1 \times \mathbb{CP}^1$, a "compactification" of $\mathbb{C}^2$. In the projective plane $\mathbb{CP}^2$, which is another compactification of $\mathbb{C}^2$, there are only embedded Riemann surfaces of genus $0, 1, 3, 6, 10$ etc.

Back to the Proof of Theorem 6.12. Assume $S \subset \mathbb{CP}^n$ for $n \geq 3$.

The Big Idea. Project from a point $p \notin S$.

(i) If $n \geq 4$, then $p \in \mathbb{CP}^n - S$ can be chosen so that the projection:

$$\pi : S \to \mathbb{CP}^{n-1}$$

remains an embedding (of the same degree δ)

(ii) If $n = 3$, then $p \in \mathbb{CP}^n - S$ can be chosen so that the projection:

$$\pi : S \to \mathbb{CP}^2$$

is an immersion and an embedding away from m pairs of points $p_i, q_i \in S$ that map to distinct points $\nu_i = \pi(p_i) = \pi(q_i)$ so that the tangent lines to S at p_i and q_i map to distinct lines in $\mathbb{CP}^2$ meeting at ν_i.

Remark. This is achieved with a *dimension count.* The union of the secant and tangent lines to $S \subset \mathbb{CP}^n$ are the loci of points $p \in \mathbb{CP}^n$ from which projection is **not** injective (resp not an immersion). The former has dimension ≤ 3 and the latter has dimension ≤ 2. This explains (i), and (ii) is only slightly more delicate.

After projecting (in stages) to $\mathbb{CP}^2$, S maps to a *singular* curve:

$$X \subset \mathbb{CP}^2 \text{ cut out by } F \in \mathbb{C}[x_0, x_1, x_2]_\delta$$

In local coordinates, each of the m points $\nu_i \in X$ is a *simple node*. Then:

(i) As before, we have the following computation for all $d \geq \delta$:

$$h_X(d) = d\delta + 1 - g(X) \quad \text{where}$$

$g(X) = \dfrac{(\delta-1)(\delta-2)}{2}$ is (by definition) the "arithmetic" genus of $X \subset \mathbb{CP}^2$

(ii) By a local computation as before, we project from a point in $\mathbb{CP}^2$ and use the Riemann-Hurwitz and Bézout Theorems to obtain:

$$g(S) = g(X) - m \text{ where } g(S) \text{ is the (ordinary) genus of } S$$

(iii) When d is sufficiently large,

$$h_S(d) \geq h_X(d) + m = d\delta + 1 - g(S)$$

which completes the proof of Theorem 6.12. $\square$

Proof of the Riemann-Roch Theorem. The hypersurface divisors of sufficiently large degree dominate any given divisor, so Corollary 6.9 applies.