Riemann Surfaces and Graphs
3. Linear Series

We begin with the definition of complex projective space.

Definition 3.1. $\mathbb{CP}^n$ is the manifold of complex lines through $0 \in \mathbb{C}^{n+1}$.

A point $x \in \mathbb{CP}^n$ may be labelled with the coordinates of any non-zero point on the line represented by x, modulo the equivalence:

$$(x_0 : .... : x_n) \sim (\lambda x_0 : .... : \lambda x_n) \ \text{ for any } \lambda \neq 0$$

In other words, we think of the coordinate as a *ratio*. We realize $\mathbb{CP}^n$ as an n-dimensional complex manifold via the topology in which:

$$U_i = \{(x_0 : ... : x_i : .... : x_n) \mid x_i \neq 0\} = \mathbb{C}^n$$

are open sets, equipped with local coordinates: $z_k = x_k/x_i$ for each $k \neq i$. If $w_k = x_k/x_j, k \neq j$ are local coordinates for $U_j = \mathbb{C}^n$, then:

$$z_k = \frac{w_k}{w_i} \text{ for } k \neq i, j \text{ and } z_i = \frac{1}{w_i}$$

are the transitions expressing the z_k as holomorphic functions of the w_k.

Exercise. Check that this agrees with the earlier definition of $\mathbb{CP}^1$.

Suppose now that S is a Riemann surface and $\Omega[S]$ is the g-dimensional vector space of holomorphic differentials on S. For convenience we will **choose a basis:** $\omega_0 = \psi_0(z)dz, ..., \omega_{g-1} = \psi_{g-1}(z)dz \in \Omega[S]$.

Lemma 3.2. The map:

$$\Psi : S \to \mathbb{CP}^{g-1}; \ \Psi(p) = (\psi_0(p) : \cdots : \psi_{g-1}(p))$$

is well-defined as written at all points $p \in S$ for which there is a holomorphic differential $\omega = \psi(z)dz$ with $\psi(p) \neq 0$.

Proof. The meromorphic functions $\psi_i(z)$ transform according to:

$$\psi_i(z)dz = \psi_i(h(w))h'(w)dw$$

when z is replaced by the local coordinate w with $z = h(w)$. It follows that **all** the values $\psi_i(p)$ are multiplied by the same scalar $\lambda = h'(p)$ when transferring from one coordinate to the other. Thus Φ is well-defined, provided that $\psi_i(p) \neq 0$, which, since the $\psi_i(z)dz$ are a basis, is the same as saying that $\psi(p) \neq 0$ for some holomorphic differential $\omega = \psi(z)dz$.

It is useful to think about the map Ψ without the choice of basis in mind. In particular, Ψ should be thought of as a map to the projective space of *hyperplanes* in $\Omega[S]$ via:

$$\Psi(p) = \{\omega \in \Omega[S] \mid \omega(p) = 0\}$$

which is the space of lines through the origin in the dual vector space $\Omega[S]^\vee$. The coordinates of the map are the coefficients of the hyperplane:

$$\Psi(p) = \{\omega = \sum x_i\omega_i \in \Omega[S] \mid \omega_0(p)x_0 + \cdots + \omega_{g-1}(p)x_{g-1} = 0\}$$

(well-defined up to one scalar multiple). From this point of view, we see again that Ψ is well-defined at p precisely when $\Psi(p)$ not all of $\Omega[S]$, i.e. when $\omega(p) \neq 0$ for some $\omega \in \Omega[S]$.

Lemma 3.3. Assume Ψ is well-defined. Then it is injective if and only if:

$$\Psi(p) \cap \Psi(q) = \{\omega \in \Omega[S] \mid \omega(p) = 0 = \omega(q)\}$$

is a codimension two subspace of $\Omega[S]$ for all $p \neq q \in S$.

Proof. $\Psi(p) \neq \Psi(q)$ if and only if $\Psi(p) \cap \Psi(q) \neq \Psi(p)$.

In other words,

($*$) The images of p and q are distinct under Ψ if and only if there is a differential ω with the property that $\omega(p) = 0$ but $\omega(q) \neq 0$.

There is a similar criterion for the non-vanishing of the derivative of Ψ. Namely, suppose $\Psi(p) \in U_i$ and consider the equivalent map:

$$\Psi(z) = (\phi_0(z), ..., \phi_{g-1}(z)) \in U_i \text{ where } \phi_k = \psi_k/\psi_i$$

in a local coordinate z at p. Then:

$$\Psi'(z) = (\phi_0'(z), ..., \phi_{g-1}'(z)) = (0, ...0) \Leftrightarrow \psi_k'(z)\psi_i(z) = \psi_k(z)\psi_i'(z)$$

if and only if $(\psi_0(z) : \cdots : \psi_{g-1}(z)) = (\psi_0'(z) : \cdots : \psi_{g-1}'(z))$. I.e.

($**$) Ψ is an immersion at p if and only if there is a differential ω with $\omega(p) = 0$ and the order of vanishing of ω at p is one.

We will see with the help of the Riemann-Roch Theorem that if S is any Riemann surface, then either:

(a) S has a meromorphic function with two simple poles, or else:

(b) The map Ψ described above is an *embedding* of S in $\mathbb{CP}^{g-1}$.

This has the effect of “algebraizing” the Riemann surface because of:

Chow’s Lemma. If M is a compact complex manifold and $\Phi : M \to \mathbb{CP}^n$ is an embedding, then M is cut out by homogeneous polynomials.

Before we explain this, consider a generalization of the discussion above. Let $1, \phi_1, ..., \phi_n$ be linearly independent meromorphic functions on S. Then

$$\Phi(p) = (1 : \phi_1(p) : \cdots : \phi_n(p)) \in \mathbb{CP}^n$$

is a well-defined map from $S - \{\text{poles of the } \phi_i\}$ to $U_0 \subset \mathbb{CP}^n$. This map *extends* to a holomorphic map from S to $\mathbb{CP}^n$ as follows. If $q \in S$ is a pole of some ϕ_k, let ϕ_i be the meromorphic function for which the pole at q has maximal (negative) order. Then:

$$\Phi(z) = ((1/\phi_i)(z) : \cdots : (\phi_k/\phi_i)(z) : ...)$$

has the same image as Φ near q and extends the map holomorphically across q to an image point in $U_i - U_0$. This can evidently be done for all $q \in S$.

Again, if we let $V = \langle 1, \phi_1, ..., \phi_n \rangle$ then we can think of the map in a basis-free manner by setting $\Phi(p) = \{\phi \in V \mid \operatorname{ord}_p(\phi) > \min_i \operatorname{ord}_p(\phi_i)\}$ and then the analogues of $(*)$ and $(**)$ in this setting are:

$(*)$ $\Phi(p) \neq \Phi(q)$ if and only if there is a meromorphic function $\phi \in V$ such that $\operatorname{ord}_p(\phi) > \min_i \operatorname{ord}_p(\phi_i)$ but $\operatorname{ord}_q(\phi) = \min_i \operatorname{ord}_p(\phi_i)$.

$(**)$ Φ is an immersion at p if and only if there is a meromorphic function $\phi \in V$ such that $\operatorname{ord}_p(\phi) = \min_i \operatorname{ord}_p(\phi_i) + 1$.

Exercise. Extend the map from §2:

$$\Phi = (1 : \mathcal{P} : \mathcal{P}') : \mathbb{C}/\Lambda \to \mathbb{CP}^2$$

and prove that it is an embedding.

Equations. A homogeneous polynomial of degree d;

$$F(x_0, ..., x_n) = \sum_{|I|=d} c_I x^I; \quad \text{with } c_I \in \mathbb{C},\ x^I = x_0^{d_0} \cdots x_n^{d_n} \text{ for } I = (d_0, ..., d_n)$$

is not a function on $\mathbb{CP}^n$ since $F(\lambda x_0, ..., \lambda x_n) = \lambda^d F(x_0, ..., x_n)$, but

$$V(F) = \{(x_0 : .... : x_n) \in \mathbb{CP}^n \mid F(x_0, ..., x_n) = 0\} \subset \mathbb{CP}^n$$

cuts out a well-defined *hypersurface* in $\mathbb{CP}^n$.

Example. (a) Linear equations $\sum_{i=0}^{n} a_i x_i = 0$ cut out vector subspaces $V \subset \mathbb{C}^{n+1}$ and linear projective subspaces of lines through the origin in V.

(b) If $f(z_1, ..., z_n) \in \mathbb{C}[z_1, ..., z_n]$ is a polynomial of degree d, then the *affine hypersurface* $\{z = (z_1, ..., z_n) \mid f(z) = 0\} \subset \mathbb{C}^n$ may be completed by setting $\mathbb{C}^n = U_0 \subset \mathbb{CP}^n$ and letting $F(x_0, ..., x_n) = x_0^d f(x_1/x_0, ..., x_n/x_0)$. Then $V(F) = V(f) \cup (V(F) \cap V(x_0))$ completes $V(f)$ with the additional points of the projective hypersurface $V(F(0, x_1, ..., x_n)) \subset V(x_0) = \mathbb{CP}^{n-1}$.

In two variables, the hypersurface $V(f) \subset \mathbb{C}^2$ is a plane curve, which is completed with a finite set of points in the projective line at infinity $\mathbb{CP}^2 - U_0$. For example, let $r_1, ..., r_d \in \mathbb{C}$ for $d \geq 2$, and consider:

$$f(z_1, z_2) = z_2^2 - (z_1 - r_1) \cdots (z_1 - r_d)$$

Then $F(z_0, z_1, z_2) = z_0^{d-2} z_2^2 - (z_1 - r_1 z_0) \cdots (z_1 - r_d z_0)$ and

$$V(F) \cap V(z_0) = \begin{cases} \{(0:0:1)\} \text{ when } d > 2 \\ \{(0:1:1), (0:1:-1)\} \text{ when } d = 2 \end{cases}$$

Lemma 3.4. Given F, if the locus of zeroes in $\mathbb{CP}^n$ of the *gradient*:

$$\nabla(F) = (\partial F/\partial x_0, \ldots, \partial F/\partial x_n)$$

is empty, then $V(F)$ is a complex submanifold of $\mathbb{CP}^n$ of dimension $n-1$.
Note. The locus of zeroes of ∇F is an intersection of hypersurfaces:

$$V(\partial F/\partial x_0) \cap \cdots \cap V(\partial F/\partial x_n)$$

which is expected to be empty because these are $n+1$ equation conditions.

Proof. On each open set U_i with coordinates $z_k = x_k/x_i$,

$$F(z_0, .., x_i/x_i = 1, .., z_n) = \frac{F(x_0, ..., x_n)}{x_i^d} \text{ and } \frac{\partial F}{\partial z_k} = \frac{\partial F}{\partial x_k}/x_i^{d-1} \text{ for } k \neq i$$

Thus if $p \in V(F) \cap U_i$ and some partial derivative $\partial F/\partial x_k(p) \neq 0$ for $k \neq i$, then $\partial F/\partial z_k(p) \neq 0$ and by the implicit function theorem, $V(F) \cap U_i$ is a complex submanifold of $U_i = \mathbb{C}^n$ in a neighborhood of p. On the other hand, if $p \in U_i$ and $\partial F/\partial x_k(p) = 0$ for all $k \neq i$, then from *Euler's identity*:

$$d \cdot F(p) = \sum_{k=0}^{n} x_k \cdot \frac{\partial F}{\partial x_k}(p)$$

it follows that $p \in V(F) \cap U_i$ if and only if $\partial F/\partial x_i(p) = 0$. $\square$

Example. The (ordinary) gradient of the affine hyperelliptic plane curve:

$$f(z_1, z_2) = z_2^2 - (z_1 - r_1) \cdots (z_1 - r_d) \text{ is}$$

$$\nabla f = \left(- \sum_i \prod_{j \neq i} (z_1 - r_j), 2z_2 \right)$$

which is therefore only zero at the point $(r_i, 0)$ for a root r_i with multiplicity two or more. Thus if all roots are distinct, then $V(f)$ is a Riemann surface. On the other hand, if $d > 2$ consider the point $(0 : 0 : 1) \in V(F) \cap U_2$. Then:

$$\nabla F = \left((d-2)x_0^{d-3}x_2^2 + \sum_i r_i \prod_{j \neq i} (x_1 - r_j x_0), - \sum_i \prod_{j \neq i} (x_1 - r_j x_0), 2x_0^{d-2}x_2 \right)$$

is non-zero at $(0 : 0 : 1)$ **if and only if** $d = 3$.

Exercise. Analyze the case $d = 2$.

Definition 3.5. A hypersurface $V(F) \subset \mathbb{CP}^2$ with non-vanishing gradient is a smooth plane curve. By Lemma 3.4, it is a closed Riemann surface.

Question. What is the genus of a smooth plane curve of degree d?

We will also have occasion to use plane curves that are not smooth. Suppose $0 \in V(f)$ and $\nabla(f)(0) = 0$. Then the expansion of f:

$$f(z_1, z_2) = f_0 + f_1 + f_2 + f_3 + \cdots + f_d$$

in homogeneous summands satisfies $f_0 = f_1 = 0$. Factoring:

$$f_e(z_1, z_2) = \prod_{i=1}^{e} (a_i z_1 - b_i z_2) \text{ for the first nonzero } f_e$$

determines e lines in $\mathbb{C}^2$ through 0 that make up the *tangent cone* to f at 0. When $e = 2$ is the first non-zero homogeneous summand, then:

(a) $V(f)$ has a *simple node* at 0 if the tangent cone is two distinct lines.

(b) $V(f)$ has an *ordinary cusp* at 0 if f_2 is a square.

Exercise. (i) What is the tangent cone of the point at infinity in the affine hyperelliptic curve example above when $d \geq 4$?

(ii) Find examples of homogeneous polynomials $F(x_0, x_1, x_2)$ in every degree with the property that $V(F)$ is a smooth projective plane curve.

Lemma 3.4 generalizes to *intersections* of hypersurfaces. For example:

Lemma 3.6. If $F_1, ..., F_{n-m}$ are homogeneous polynomials in the $x_0, ..., x_n$ (of various degrees) and if $p \in V(F_1) \cap \cdots \cap V(F_{n-m})$ and the *Jacobian:*

$$\det \left(\frac{\partial F_i}{\partial x_j} \right) (p) \text{ has rank } n - m$$

then in a neighborhood of $p \in \mathbb{CP}^n$, the intersection $V(F_1) \cap \cdots \cap V(F_{n-1})$ cuts out a complex submanifold of dimension m.

Remark. Chow's Lemma asserts that if $\Phi : M \to \mathbb{CP}^n$ is an embedding of a compact complex manifold of dimension m, then for each $p \in M$, there are homogeneous polynomials $F_1, ..., F_{n-m}$ as in the Lemma above that (necessarily) cut out the image of M in a neighborhood of p. However, as the following example shows, there need not be homogeneous polynomials that cut out the image of M for **all** points $p \in M$ at once.

Example. Consider the three quadratic polynomials:

$$Q_1 = x_0x_2 - x_1^2, Q_2 = x_0x_3 - x_1x_2 \text{ and } Q_3 = x_1x_3 - x_2^2$$

that collectively cut out the embedded *twisted cubic* curve:

$$\Phi : \mathbb{CP}^1 \to C \subset \mathbb{CP}^3; \Phi(z) = (1 : z : z^2 : z^3) = (w^3 : w^2 : w : 1)$$

Note that the quadratic polynomial Q_2 itself cuts out a submanifold of dimension two, since $\nabla Q_2 = (x_3, -x_2, -x_1, x_0)$ is never zero but Q_1 and Q_2 are singular at $(0 : 0 : 0 : 1)$ and $(1 : 0 : 0 : 0)$, respectively. In fact, the map:

$$\mathbb{CP}^1 \times \mathbb{CP}^1 \to V(Q_2) \subset \mathbb{CP}^2;\ ((a_0 : a_1), (b_0 : b_1)) \mapsto (a_0b_0 : a_1b_0 : a_0b_1 : a_1b_1)$$

is an isomorphism of complex surfaces, mapping each $\mathbb{CP}^1 \times \{(b_0 : b_1)\}$ and $\{(a_0 : a_1)\} \times \mathbb{CP}^1$ to a pair of intersecting lines in $\mathbb{CP}^3$. If Q, Q' are linearly independent in the vector space $\langle Q_1, Q_2, Q_3 \rangle$, then $V(Q) \cap V(Q') = C \cup l$, where l is a line intersecting C in two points. But l depends on the choice of Q, Q', and altogether $V(Q_1) \cap V(Q_2) \cap V(Q_3)$ cut out the twisted cubic.

Question. A **pair** of non-constant meromorphic functions ϕ, ψ on S give

$$\Phi := (\phi, \psi) : S \to \mathbb{CP}^1 \times \mathbb{CP}^1 = V(Q_2)$$

When do ϕ and ψ determine an injective map? An immersion? If ϕ, ψ have degrees d, e and Φ is an embedding, what is the genus of S?

Definition 3.7. A pair of divisors $D, D' \in \mathbb{Z}[S]_d$ are *linearly equivalent* (written $D \sim D'$) if $D + \text{div}(\phi) = D'$ for some (non-zero) $\phi \in \mathbb{C}(S)$.

Lemma 3.8. Linear equivalence is an equivalence relation.

Proof. (a) $D + \text{div}(1) = D$, so the relation is reflexive.

(b) If $D+\text{div}(\phi) = D'$, then $D'+\text{div}(1/\phi) = D$, so the relation is symmetric.

(c) If $D + \text{div}(\phi) = D'$ and $D' + \text{div}(\psi) = D''$, then $D + \text{div}(\phi\psi) = D''$ so the relation is transitive.

Definition 3.9. For any divisor D of degree d, let:

$$|D| = \{D' \mid D \sim D' \text{ and } D' \text{ is effective}\} \subset S_d$$

Lemma 3.10. $|D| = \emptyset$ or else $|D| = \mathbb{CP}^r$ for some $r \geq 0$.

Proof. We may assume that D is effective. Let:

$$D = \sum d_p p \in S_d$$

Then $D + \text{div}(\phi) = D'$ is an effective divisor if and only if $\text{ord}_p(\phi) \geq d_p$ for all $p \in S$, but if $\text{ord}_p(\phi) \geq m$ and $\text{ord}_p(\psi) \geq m$ then $\text{ord}_p(\phi + \psi) \geq m$, so the set of such meromorphic functions ϕ (together with zero) is a *vector space* V of dimension $\leq d + 1$. Finally, a divisor $D' \in |D|$ *determines* the meromorphic function ϕ with $\text{div}(\phi) = D' - D$ up to a scalar multiple, so $|D|$ is the projective space of lines through the origin in V.

Corollary 3.11. $r = r(D)$ agrees with Definition 1.9 (adapted from graphs).

Proof. Given $E = \sum e_p p \in S_e$, let $V(-E) \subset V$ be the vector subspace of meromorphic functions with the property that $\text{ord}_p(\phi) \geq d_p - e_p$. Then:

(i) The projective space of lines through $0 \in V(-E)$ is:

$$|D - E| = \{D' \in |D| \mid D' = E + \text{effective}\} \text{ and}$$

(ii) $\dim(V(-E)) \geq r + 1 - e$ for **all** E and $= r + 1 - e$ for **some** E. $\square$

Definition 3.12. A point $p \in S$ is a *base point* of $|D|$ if $r(D - p) = r(D)$. The linear series $|D|$ is *base point free* if it has no base points.

In other words, $|D|$ is base point free if $\dim(V(-p)) = r$ for all $p \in S$.

Proposition 3.13. If the linear series $|D|$ is base point free, then:

$$\Phi_D : S \to \mathbb{CP}^r;\ \Phi_D(p) = V(-p)$$

defines a map to the projective space of hyperplanes in V that:

($*$) Separates p and q if $r(D-p-q)) = r-2$ and

($**$) Is an immersion at p if $r(D-2p) = r-2$.

Example. Let $S = \mathbb{CP}^1$ and D be any divisor of degree $d \geq 0$. Then $|D|$ is base-point free, and if $d \geq 1$, then Φ_D is an embedding.