The Genus of a Riemann Surface in the Plane

Let $S \subset \mathbb{CP}^2$ be a plane curve of degree δ. We prove here that:

$$g = \frac{(\delta-1)(\delta-2)}{2}$$

We'll start with a version of the:

Bézout Theorem. If $S \subset \mathbb{CP}^n$ has degree δ and G is a homogeneous polynomial of degree d not vanishing identically on S (i.e. $G \notin I_d$), then

$$\deg(\mathrm{div}(G)) = d\delta$$

Proof. The intersection $V(G) \cap S$ is a **finite** set. We choose a linear form l that is non-zero on all the points of $V(G) \cap S$, and then $\phi = G/l^d$ is meromorphic when restricted to S, and $\mathrm{div}(G) - d \cdot \mathrm{div}(l) = \mathrm{div}(\phi)$ has degree zero. But $\mathrm{div}(l)$ has degree δ, by the definition of δ. □

Now let $S \subset \mathbb{CP}^2$ be a planar embedding of a Riemann surface, and let $F \in \mathbb{C}[x_0, x_1, x_2]_\delta$ generate the ideal I of S and assume, changing coordinates if necessary, that $(0:1:0) \notin S$ and that the line $x_2 = 0$ is *transverse* to S, necessarily meeting S in δ points.

Proof Sketch. Projecting from $p = (0:1:0)$ gives a holomorphic map:

$$\pi : S \to \mathbb{CP}^1$$

(as a map from $\mathbb{CP}^2$ to $\mathbb{CP}^1$, the projection is defined at every point except for p itself). In the open set $U_2 = \{(x_0 : x_1 : x_2) \mid x_2 \neq 0\} = \mathbb{C}^2$ with local coordinates $x = x_0/x_2$ and $y = x_1/x_2$, π is the projection onto the x-axis. In these coordinates, the line at infinity is $x_2 = 0$ and maps to $\infty \in \mathbb{CP}^1$ and does not contribute to the ramification of the map π. Then:

(i) The degree of the map π is δ.

(ii) The ramification divisor of the map π is:

$$\sum_p (e_p - 1) \cdot p = \mathrm{div}\left(\frac{\partial F}{\partial x_1}\right)$$

Keeping in mind that the partial derivative has degree $\delta - 1$, we get:

$$2g - 2 = \delta \cdot (-2) + \sum_p (e_p - 1) = -2\delta + \delta \cdot (\delta - 1)$$

from the Riemann-Hurwitz and Bézout Theorems, giving the genus formula.

Let's go into each of these in greater detail.

(o) Projection from $p = (0 : 1 : 0)$ is projection to the x-axis.

The lines through p all have equations: $ax_0 + cx_2 = 0$ which, with the exception of the line $x_2 = 0$, all have the form:

$$\frac{x_0}{x_2} = x = -\frac{c}{a}$$

in the local coordinates (x, y). Thus, the fibers of the projection are vertical.

(i) The degree of the map π is δ.

The degree of π is the number of points of the intersection of S with any line through p that intersects S transversely. This number is δ.

(ii) At each point $q = (x_0, y_0) \in S$, we have

$$e_q - 1 = \mathrm{ord}_q\left(\frac{\partial f}{\partial y}\right) = \mathrm{ord}_q\left(\frac{\partial F}{\partial x_1}\right)$$

where $f(x, y) = F(x_0, x_1, x_2)/x_2^\delta$ is the polynomial cutting out S in the open set $U_2 = \mathbb{C}^2$. The second equality follows immediately from the chain rule. The first equality is the crux of the matter.

We can assume $q = (0, 0) \in S$ by translating the x and y variables without affecting either side of the equation, and then:

$$f(x, y) = ax + by + \text{ higher order, with } ab \neq 0$$

and $ax + by = 0$ is the tangent line to S at q.

Suppose first that $b \neq 0$. Then $x = 0$ is **not** tangent to S, so $e_q = 1$. On the other hand, $(\partial f/\partial y)(q) = b \neq 0$, and so $\mathrm{ord}_q(\partial f/\partial y) = 0 = e_q - 1$. Check. Now suppose $b = 0$ (so $a \neq 0$). We collect all terms divisible by x and then factor out the largest power of y in the remaining terms to get:

$$f(x, y) = xg(x, y) + y^e h(x, y) \text{ for some } 1 < e \leq \delta$$

with $g(q) \neq 0$ and $h(q) \neq 0$. Then on S (where $f \equiv 0$), we have:

$$-xg(x, y) = y^e h(x, y), \text{ so } \mathrm{ord}_q(x) + \mathrm{ord}_q(g) = e \cdot \mathrm{ord}_q(y) + \mathrm{ord}_q(h)$$

But as functions on S, $\mathrm{ord}_q(g) = 0 = \mathrm{ord}_q(h)$ and $\mathrm{ord}_q(y) = 1$, since $y = 0$ is not the tangent line. Thus $e = \mathrm{ord}_q(x) = e_q$.

On the other hand, taking the derivative, we have:

$$\frac{\partial f}{\partial y} = x\frac{\partial g}{\partial y} + y^e\frac{\partial h}{\partial y} + ey^{e-1}h$$

and the first and second terms both have order $\geq e$ at q. It follows that:

$$\mathrm{ord}_q\left(\frac{\partial f}{\partial y}\right) = \mathrm{ord}_q(ey^{e-1}h) = e - 1$$

Assignment. 3 (1) Read this and please tell me where it is confusing.

(2) Finish the proof by explaining why

$$\frac{\partial F}{\partial x_1}(q) \neq 0$$

for all points q in the intersection of S with the line at infinity $x_2 = 0$, recalling that this line was assumed to intersect S transversally, so $e_q = 1$ for all these points.

(3) Celebrate that we now have Riemann surfaces of genus $3, 6, 10$ etc. as non-singular plane curves of degrees $4, 5, 6$ etc. in $\mathbb{CP}^2$.

Final Remark. In the last assignment, we saw that this genus formula is a consequence of the Riemann-Roch theorem. In fact, we will use it (and an improvement, in the next assignment) to **prove** the Riemann-Roch theorem!

Side Note. There are other ways to get the genus formula for plane curves (e.g. using the *adjunction formula*) that you'll see in an algebraic geometry course, (e.g. mine in the Fall). Explaining a proof of this formula is one of my favorite oral exam questions.