

Algebraic Curves/Fall 2015

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4. Projective Plane Curves are hypersurfaces in the plane $\mathbb{CP}^2$. When nonsingular, they are Riemann surfaces, but we will also consider plane curves with singularities. We will prove Bézout's Theorem and Noether's Theorem together with a standard "classical" application. This treatment follows Fulton's *Algebraic Curves* Chapter 5.

A hyperplane $L \subset \mathbb{CP}^2$ is a projective *line*, and every cone in $\mathbb{CP}^2$ is a union of lines through a vertex point $p \in \mathbb{CP}^2$. Given finitely many points $p_1, \dots, p_n \in \mathbb{CP}^2$, there is a line L_0 that contains none of the points and lines L_i that contain the point p_i but none of the others. These are simple results that you should check for yourself.

$C = V(F) \subset \mathbb{CP}^2$ is an irreducible curve and:

$$D = \sum_{i=1}^m d_i C_i \quad (\text{for curves } C_i = V(F_i) \text{ and } d_i \geq 0)$$

is an effective *divisor* on $\mathbb{CP}^2$, with associated homogeneous polynomial $F = \prod F_i^{d_i}$ that is well-defined up to a scalar multiple. We will write:

$$\text{div}(F) = D \text{ and } \deg(D) = \deg(F) = \sum d_i \deg(F_i)$$

(compare this with the definitions for smooth curves). Let:

$$\mathbb{P}(d) := \mathbb{P}(\mathbb{C}[x_0, x_1, x_2]_d) = \mathbb{CP}^{\frac{d(d+3)}{2}}$$

be the projective space associated to the vector space $\mathbb{C}[x_0, x_1, x_2]_d$. The last equality is explained by:

$$\dim(\mathbb{C}[x_0, x_1, x_2]_d) = \binom{d+2}{2} = \frac{d(d+3)}{2} + 1$$

and via the div operation (and Theorem 3.1), we may interpret:

$$\mathbb{P}(d) = \{\text{effective divisors of degree } d \text{ on } \mathbb{CP}^2\}$$

Example 4.1. (Degree 2 divisors) Every "conic divisor" is projectively equivalent to one of the following:

$$\text{div}(x_0^2 + x_1^2 + x_2^2) = C, \text{ an irreducible smooth conic.}$$

$$\text{div}(x_0^2 + x_1^2) = \text{div}(x_0 + ix_1) + \text{div}(x_0 - ix_1) = L_1 + L_2$$

$$\text{div}(x_0^2) = 2\text{div}(x_0) = 2L, \text{ a "double" line.}$$

Exercise 4.1. Every degree 3 divisor on $\mathbb{CP}^2$ is equivalent to one of:

(a) An elliptic curve $E_\lambda = \text{div}(x_0x_2^2 - x_1(x_1 - x_0)(x_1 - \lambda x_0))$, $\lambda \neq 0, 1$.

One of the following two singular irreducible curves:

(b) $E_0 = \text{div}(x_0x_2^2 - x_1^2(x_1 - x_0))$ (the nodal cubic) or

(c) $E_\infty = \text{div}(x_0x_2^2 - x_1^3)$ (the cuspidal cubic)

(d) Or else a reducible divisor $L + D$ where D is a conic divisor.

The **multiplicity** of an effective divisor D at p :

$$\text{mult}_p(D)$$

is the degree of the polynomial defining the tangent cone to D at p . In projective coordinates chosen so that $p_0 = (1 : 0 : 0)$, $\text{mult}_{p_0}(D) \geq m$ if and only if the associated polynomial of degree d :

$$F = \sum_{|I|=d} c_I x_0^{i_0} x_1^{i_1} x_2^{i_2}$$

has the property that $c_I = 0$ for all $I = (i_0, i_1, i_2)$ satisfying $i_0 > d - m$.

Definition 4.1. Given a degree $d > 0$ and integers $m_i \geq 0$ at $p_i \in \mathbb{CP}^2$,

$$\mathbb{P}(d; m_1p_1, \dots, m_rp_r) \subset \mathbb{P}(d)$$

is the *linear series* of divisors $D \in \mathbb{P}(d)$ with each $\text{mult}_{p_i}(D) \geq m_i$.

Remark. The linear series involving a single point $\mathbb{P}(d; mp) \subset \mathbb{P}(d)$ are projective linear subspaces of dimension:

$$\frac{d(d+3)}{2} - \frac{(m+1)m}{2}$$

(or else empty) since as above, the linear series $\mathbb{P}(d; mp_0)$ is defined by:

$$c_I = 0 \text{ for } I = (d, 0, 0), (d-1, 1, 0), (d-1, 0, 1), \dots, (d-m+1, *, *)$$

and changing coordinates in the plane induces a (projective) linear transformation on each of the projective spaces $\mathbb{P}(d)$, so the result holds for an arbitrary point $p \in \mathbb{CP}^2$.

Corollary 4.1. By definition:

$$\mathbb{P}(d; m_1p_1, \dots, m_rp_r) = \mathbb{P}(d; m_1p_1) \cap \dots \cap \mathbb{P}(d; m_rp_r)$$

and so the linear series is either a projective linear space satisfying:

$$\dim(\mathbb{P}(d; m_1p_1, \dots, m_n p_n)) \geq \frac{d(d+3)}{2} - \sum_{i=1}^r \frac{(m_i+1)m_i}{2}$$

which is the “expected dimension” or else it is empty, which **can only occur** when the expected dimension is negative.

Example 4.2. $\mathbb{P}(1) = \mathbb{CP}^2$ is the (dual) projective plane of lines.

- (a) $\dim(\mathbb{P}(1; p)) = 1$ and $\mathbb{P}(1; 2p)$ is empty.
- (b) $\mathbb{P}(1; p, q)$ is the unique line through p and q .
- (c) $\mathbb{P}(1; p, q, r)$ is empty **unless** the three points are collinear.

Thus the dimension of a linear series may depend upon the location of the points. The problem of determining exact dimensions of linear series for a particular configuration of points is very subtle in general.

Exercise 4.2. $\mathbb{P}(2) = \mathbb{CP}^5$ is the projective space of conic divisors.

- (a) Show that $\dim(\mathbb{P}(2; p, q, r)) = 2$ for **all** points p, q, r .
- (b) Show that $\dim(\mathbb{P}(2; p_1, \dots, p_5)) = 0$, i.e. there is a unique conic through the five points **unless** four of the points are collinear.
- (c) Show that $\mathbb{P}(2; 2p, 2q)$ is a single point for all pairs p and q .

Remark. (c) is interesting because the expected dimension is -1 , and yet for all pairs $p, q \in \mathbb{CP}^2$, the linear series is not empty.

Proposition 4.1. Fix $m_1, \dots, m_r \geq 0$. Then for all sufficiently large d ,

$$\dim(\mathbb{P}(d; m_1 p_1, \dots, m_r p_r)) = \frac{d(d+3)}{2} - \sum_{i=1}^r \frac{(m_i+1)m_i}{2}$$

is as expected for all collections of (distinct) points $p_1, \dots, p_r \in \mathbb{CP}^2$.

Proof. First consider the case $m_i = 1$ for all i . It suffices to show:

$$\mathbb{P}(d) \supset \mathbb{P}(d; p_1) \supset \dots \supset \mathbb{P}(d; p_1, \dots, p_r)$$

are strict inclusions when d is large, since this forces the linear series to drop in dimension with the introduction of each additional point.

Recall the lines L_0 (containing none of the points) and L_i (containing one p_i but none of the others) from the top of this section. Then:

$$(d-i)L_0 + L_1 + \dots + L_i \in \mathbb{P}(d; p_1, \dots, p_i) - \mathbb{P}(d; p_1, \dots, p_{i+1})$$

exhibits the desired proper inclusions for each $i = 1, \dots, r-1$, provided that $d-i \geq 0$. Thus, $d \geq r-1$ is sufficiently large for this argument.

We reason similarly in the general case. We need to show:

$$\mathbb{P}(d) \supset \mathbb{P}(d; m_1 p_1) \supset \dots \supset \mathbb{P}(d; m_1 p_1, \dots, m_r p_r)$$

each has the expected codimension in the one before.

This can be assured by finding projective linear subspaces

$$\mathbb{P}(V_i) \subset \mathbb{P}(d; m_1 p_1, \dots, m_i p_i) \subset \mathbb{P}(d)$$

with the property that the intersection $\mathbb{P}(V_i) \cap \mathbb{P}(d; m_{i+1}p_{i+1}) \subset \mathbb{P}(V_i)$ has the expected codimension. We do this by letting:

$$\mathbb{P}(V_i) = \{m_1L_1 + \dots + m_iL_i + D \mid D \in \mathbb{P}(d - m_1 - \dots - m_i)\}$$

where L_j are the **fixed** lines. Then: $\mathbb{P}(V_i) \cap \mathbb{P}(d; m_{i+1}p_{i+1}) =$

$$\{m_1L_1 + \dots + m_iL_i + D \mid D \in \mathbb{P}(d - m_1 - \dots - m_i; m_{i+1}p_{i+1})\}$$

reduces to the case of a single point linear series for divisors of degree $d - m_1 - \dots - m_i$, which has the expected codimension provided that $d - \sum m_j \geq m_{i+1} - 1$. For this argument, $d \geq m_1 + \dots + m_r - 1$ is sufficiently large, and is, in fact, optimal for the result. $\square$

Next, we turn to Bézout's Theorem, which is useful to think of as a comparison between globally and locally defined data coming from a pair of distinct irreducible plane curves $C, C' \subset \mathbb{CP}^2$ associated to prime homogeneous polynomials $A, A' \in \mathbb{C}[x_0, x_1, x_2]$ of degrees a, a' .

Global Data. The Hilbert function of the projective plane $\mathbb{CP}^2$ is:

$$h_{\mathbb{CP}^2}(d) = \dim \mathbb{C}[x_0, x_1, x_2]_d = \begin{cases} \binom{d+2}{2} & \text{for all } d \geq 0 \\ 0 & \text{for all } d < 0 \end{cases}$$

which is a **quadratic** polynomial in d for all $d \geq 0$.

The Hilbert function of the curve C is the eventually **linear** function:

$$h_C(d) = \dim(R_C)_d = \begin{cases} \binom{d+2}{2} - \binom{d+2-a}{2} = ad + 1 - \binom{a-1}{2} & \text{for all } d \geq a \\ \binom{d+2}{2} & \text{for all } 0 \leq d < a \\ 0 & \text{for all } d < 0 \end{cases}$$

where $R_C = \mathbb{C}[x_0, x_1, x_2]/\langle A \rangle$ is the *homogeneous coordinate ring* of C .

The Hilbert function of $C \cap C'$ is the eventually **constant** function:

$$h_{C \cap C'}(d) = \dim(R_{C \cap C'})_d = \begin{cases} h_C(d) - h_C(d - a') = aa' & \text{for all } d \geq a + a' \\ \text{other stuff} & \text{for } d < a + a' \end{cases}$$

where $R_{C \cap C'} = \mathbb{C}[x_0, x_1, x_2]/\langle A, A' \rangle$ is the homogeneous coordinate ring of the intersection $C \cap C'$ of the two curves.

Observation. The quotient ring R_C is a graded integral domain, and $R_{C \cap C'}$ is graded, but usually not a domain. The notation is a little misleading, since the **set** $C \cap C'$ is not enough information, in general, to determine $R_{C \cap C'}$ (the *scheme structure* on $C \cap C'$ is required).

In §5 we will prove that Hilbert functions are eventually polynomial functions in much greater generality, and the degree will be one of the ways of defining the dimension of a projective variety.

Remark. It follows already that the cardinality of the set $C \cap C'$ satisfies:

$$|C \cap C'| \leq aa'$$

hence in particular that $C \cap C'$ is finite. This is explained by Proposition 4.1, since each point $p \in C \cap C'$ supports a homogeneous polynomial $F_p \in \mathbb{P}(d; C \cap C' - p) - \mathbb{P}(d; C \cap C')$ that by definition vanishes at all points of $C \cap C'$ except p whenever $d \geq |C \cap C'| - 1$. These are evidently linearly independent vectors in $(R_{C \cap C'})_d$, which has dimension aa' .

Local Data. With suitable coordinates, x_0 does not divide A or A' and the intersection $C \cap C'$ is contained in the set $U_0 = \mathbb{CP}^2 - V(x_0)$ which may be identified with $\mathbb{C}^2$ with coordinates $y_i = x_i/x_0$. Let:

$$\alpha(y_1, y_2) = A/x_0^a \text{ and } \alpha'(y_1, y_2) = A'/x_0^{a'}$$

and consider the “affine” coordinate rings:

$$\mathbb{C}[y_1, y_2]/\langle \alpha \rangle \text{ and } \mathbb{C}[y_1, y_2]/\langle \alpha, \alpha' \rangle$$

associated to the affine curve and intersection of affine curves:

$$C^{\text{aff}} = C \cap U_0 = \{(q_1, q_2) \in \mathbb{C}^2 \mid \alpha(q_1, q_2) = 0\} \text{ and } C^{\text{aff}} \cap (C')^{\text{aff}} = C \cap C'$$

respectively. This “affine” data has the advantage that elements of the affine coordinate rings are **functions** on C^{aff} and $C \cap C'$ respectively.

We pass to local data via the fields of rational functions, which are the same whether viewed projectively or affinely:

$$K(\mathbb{CP}^2) = \left\{ \frac{F}{G} \mid F, G \in \mathbb{C}[x_0, x_1, x_2] \right\} \cong \left\{ \frac{f}{g} \mid f, g \in \mathbb{C}[y_1, y_2] \right\} = K(\mathbb{C}^2)$$

(with the usual equivalence relation on fractions) via the isomorphism:

$$\frac{F}{G} \mapsto \frac{F/x_0^d}{G/x_0^d} \text{ and } \frac{f}{g} \mapsto \frac{fx_0^n}{gx_0^n} \text{ for sufficiently large } n$$

Notice in particular that α (but not A) is an element of $K(\mathbb{CP}^2)$.

Definition 4.2. To each point $p \in \mathbb{CP}^2$, define the ring:

$$\mathcal{O}_{\mathbb{CP}^2, p} = \left\{ \frac{F}{G} \mid G(p) \neq 0 \right\} \subset K(\mathbb{CP}^2)$$

of local functions defined at p , which has a unique maximal ideal:

$$m_p = \{\phi \in \mathcal{O}_{\mathbb{CP}^2, p} \mid \phi(p) = 0\} \subset \mathcal{O}_{\mathbb{CP}^2, p} \subset K(\mathbb{CP}^2)$$

of local functions vanishing at p .

Remark. These may alternatively be defined with affine coordinates:

$$m_p \subset \mathcal{O}_{\mathbb{C}^2, p} = \mathcal{O}_{\mathbb{CP}^2, p} \subset K(\mathbb{C}^2)$$

whenever $p \in U_0 = \mathbb{C}^2$, and in these coordinates it is clear that m_p is generated by two elements. For example, when $p = (1 : 0 : 0) \in \mathbb{CP}^2$, then $p = (0, 0) \in \mathbb{C}^2$, and m_p is generated by the coordinates y_1, y_2 .

Similarly, we may define the fields of rational functions on C :

$$K(C) = \left\{ \frac{F}{G} \mid F, G \in (R_C)_d \right\} \cong \left\{ \frac{f}{g} \mid f, g \in \mathbb{C}[y_1, y_2]/\langle \alpha \rangle \right\} = K(C^{\text{aff}})$$

with local ring and maximal ideal for all $p \in C$ given by:

$$m_p = \{\phi \mid \phi(p) = 0\} \subset \mathcal{O}_{C, p} = \left\{ \frac{F}{G} \mid G(p) \neq 0 \right\} \subset K(C)$$

and if $p \in C^{\text{aff}}$, then $\mathcal{O}_{C, p} = \mathcal{O}_{C^{\text{aff}}, p}$.

Remark. The coordinate ring R_C (or $\mathbb{C}[y_1, y_2]/\langle \alpha \rangle$) of a plane curve C does not usually have unique factorization, in which case there is no natural “lowest terms” fraction representing a typical $\phi \in K(C)$.

Exercise 4.3. (a) Regarding $\alpha \in m_p \subset \mathcal{O}_{\mathbb{CP}^2, p}$ for $p \in C$, show that:

$$\mathcal{O}_{\mathbb{CP}^2, p}/\langle \alpha \rangle \cong \mathcal{O}_{C, p}$$

(b*) Suppose $I = \langle f_1, \dots, f_n \rangle \subset \mathbb{C}[y_1, y_2]$ is an ideal such that the set:

$$V(I) = \{p \in \mathbb{C}^2 \mid f_i(p) = 0 \forall f_i\} \subset \mathbb{C}^2$$

is finite. Show that the natural map:

$$\mathbb{C}[y_0, y_1]/I \mapsto \times_{p \in V(I)} \mathcal{O}_{\mathbb{CP}^2, p}/\langle f_1, \dots, f_n \rangle; \quad f \mapsto (f, \dots, f)$$

is an isomorphism onto the product of local rings.

Remark. Exercise 4.3 (b) is starred because it is both challenging and extremely important. The local rings should be thought of as “germs” of functions, and the Exercise shows that an arbitrary choice of germs can be pieced together to come from a single polynomial (mod I). This isn’t even obvious when $V(I)$ is a single point!

Proposition 4.2. (a) If $p \in C$ is a non-singular point, then $m_p \subset \mathcal{O}_{C, p}$ is a principal ideal, i.e. $\mathcal{O}_{C, p}$ is a discrete valuation ring.

(b) For any $p \in C$, the dimensions of the quotient vector spaces:

$$\dim_{\mathbb{C}} m_p^n / m_p^{n+1}$$

are constant for large n , equal to the multiplicity of $C \subset \mathbb{CP}^2$ at $p \in C$. In particular, if $p \in C$ is a singular point, then $\mathcal{O}_{C, p}$ is **not** a DVR.

Proof. For (a), we may choose coordinates that $p = (1 : 0 : 0)$ and the Zariski tangent line to C at p is $x_1 = 0$, so that:

$$\alpha(y_1, y_2) = y_1 f + y_2^2 g \in \mathbb{C}[y_1, y_2]; \text{ with } f(0, 0) \neq 0$$

Since y_1, y_2 generate $m_p \subset \mathcal{O}_{\mathbb{CP}^2, p}$ and $y_1 f = \alpha - y_2^2 g$ and f is a **unit** in $\mathcal{O}_{\mathbb{CP}^2, p}$, it follows from Exercise 4.3 (a) that $\bar{y}_1 \in \langle \bar{y}_2 \rangle \subset \mathcal{O}_{C, p}$ and $\bar{y}_1, \bar{y}_2$ generate $m_p \subset \mathcal{O}_{C, p}$, so $\bar{y}_2$ by itself is a generator of $m_p \subset \mathcal{O}_{C, p}$.

(b) From the isomorphism:

$$m_p^n / m_p^{n+1} = (\mathcal{O}_{C, p} / m_p^{n+1}) / (\mathcal{O}_{C, p} / m_p^n)$$

it suffices to prove that $\dim_{\mathbb{C}} \mathcal{O}_{C, p} / m_p^n$ is eventually linear, of the form $\text{mult}_p(C)d + \text{constant}$. Choose coordinates so that $p = (1 : 0 : 0)$ and let $I = \langle y_0, y_1 \rangle \subset \mathbb{C}[y_0, y_1]$. Then the natural map:

$$\mathbb{C}[y_1, y_2] / \langle \alpha, I^n \rangle \rightarrow \mathcal{O}_{C, p} / m_p^n = \mathcal{O}_{\mathbb{CP}^2, p} / \langle \alpha, I^n \rangle$$

is an isomorphism by Exercise 4.3 (a) and (b). Let $m = \text{mult}_p(C)$. Then multiplication by α maps I^{n-m} into I^n , and:

$$0 \rightarrow \mathbb{C}[y_0, y_1] / I^{n-m} \xrightarrow{\cdot \alpha} \mathbb{C}[y_0, y_1] / I^n \rightarrow \mathbb{C}[y_0, y_n] / \langle \alpha, I^n \rangle \rightarrow 0$$

is an exact sequence of $\mathbb{C}[y_0, y_1]$ -modules, hence:

$$\dim_{\mathbb{C}} \mathcal{O}_{C, p} / m_p^n = \dim_{\mathbb{C}} \mathbb{C}[y_0, y_1] / I^n - \dim_{\mathbb{C}} \mathbb{C}[y_0, y_1] / I^{n-m}$$

for all $n \geq m$, and it is a quick check to see that:

$$\dim_{\mathbb{C}} \mathbb{C}[y_0, y_1] / I^n = 1 + 2 + \dots + n = \binom{n+1}{2}$$

from which it follows that for all $n \geq m$,

$$\dim_{\mathbb{C}} \mathcal{O}_{C, p} / m_p^n = \binom{n+1}{2} - \binom{n+1-m}{2} = mn - \binom{m}{2}$$

as desired. $\square$

Remark. This is a second occurrence of a Hilbert function. Here it is attached to the local ring $\mathcal{O}_{C, p}$ as the (eventually) linear function:

$$\bigoplus_{n=0}^{\infty} \dim_{\mathbb{C}} \mathcal{O}_{C, p} / m_p^n$$

Let us return now to our two distinct, irreducible plane curves

$$C \cap C' = C^{\text{aff}} \cap (C')^{\text{aff}}$$

Definition 4.3. The *intersection number* at $p \in C \cap C'$ is:

$$I_p(C \cap C') := \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{CP}^2, p} / \langle \alpha, \alpha' \rangle$$

Bézout's Theorem 4.1. The globally defined “Hilbert polynomial” aa' of the graded ring $R = \mathbb{C}[x_0, x_1, x_2]/\langle A, A' \rangle$ is the sum of locally defined intersection numbers:

$$aa' = \sum_{p \in C \cap C'} I_p(C \cap C')$$

Proof. In our coordinates so that the line $x_0 = 0$ does not contain any of the (finitely many!) intersection points of C and C' ,

$$\sum_p I_p(C \cap C') = \dim_{\mathbb{C}} \mathbb{C}[y_1, y_2]/\langle \alpha, \alpha' \rangle$$

by Exercise 4.3 (b) with α, α' defined as earlier, so the proof amounts to showing that the maps $R_d \rightarrow \mathbb{C}[y_1, y_2]/\langle \alpha, \alpha' \rangle$; $\overline{F} \mapsto f = \overline{F}/x_0^d$ are isomorphisms for sufficiently large d .

First we show that the multiplication maps

$$\mu_{x_0} : R_d \rightarrow R_{d+1}; \mu_{x_0}(\overline{F}) = \overline{F}x_0$$

are injective for all d and therefore isomorphisms for all $d \geq a + b$.

To see this, suppose $x_0H = FA + F'A'$ for some F, F' . Then:

$$0 = F_0A_0 + F'_0A'_0 \in \mathbb{C}[x_1, x_2] = \mathbb{C}[x_0, x_1, x_2]/\langle x_0 \rangle$$

where $F_0 = F(0, x_1, x_2)$, etc. But $A_0, A'_0 \in \mathbb{C}[x_1, x_2]$ are relatively prime (because $C \cap C' \cap V(x_0) = V(A_0) \cap V(A'_0) = \emptyset$), so $F'_0 = EA_0$ and $F_0 = -EA'_0$ for some $E \in \mathbb{C}[x_1, x_2]$. Let:

$$G = F + EA' \text{ and } G' = F' - EA$$

Then $x_0H = GA + G'A'$ and both G and G' are divisible by x_0 , by construction, so $H = (G/x_0)A + (G'/x_0)A'$, proving injectivity.

Now fix $d \geq a + a'$ and $F_1, \dots, F_{aa'} \in \mathbb{C}[x_0, x_1, x_2]_d$ whose images in R_d are a basis. Let $f_i = F_i/x_0^d$, as usual. Then we need to show:

(*) The images of $f_1, \dots, f_{aa'} \in \mathbb{C}[y_1, y_2]$ in $\mathbb{C}[y_1, y_2]/\langle \alpha, \alpha' \rangle$ are a basis

Suppose $\sum \lambda_i f_i = g\alpha + h\alpha' \in \mathbb{C}[y_1, y_2]$ for some $\lambda_i \in \mathbb{C}$. Multiplying by a sufficiently large power of x_0 gives $\sum \lambda_i x_0^r F_i = x_0^s GA + x_0^t HA'$ for some r, s, t and homogeneous polynomials G, H , which shows that $\sum \lambda_i x_0^r F_i = 0$ in R_{d+r} and $\sum \lambda_i F_i = 0$ in R_d by the injectivity of μ_{x_0} . So the f_i are linearly independent.

Suppose $\gamma \in \mathbb{C}[y_1, y_2]$ and let $\Gamma = x_0^n \gamma \in \mathbb{C}[x_0, x_1, x_2]_n$ for $n = d + r$. Then $\Gamma = \sum \lambda_i x_0^r F_i + GA + HA'$ since the $x_0^r F_i$ are a basis for R_{d+r} , and then $\gamma = \sum \lambda_i f_i + g\alpha + h\alpha'$, so $\overline{\gamma} = \sum \lambda_i f_i$ in $\mathbb{C}[y_1, y_2]/\langle \alpha, \alpha' \rangle$. This proves surjectivity, and completes the proof of (*) and therefore of Bézout's Theorem. $\square$

Bézout's theorem begs the following:

Question. How do we compute the intersection numbers:

$$I_p(C \cap C') = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{CP}^2, p} / \langle \alpha, \alpha' \rangle?$$

This is answered in many cases by:

Proposition 4.3. (a) Suppose $p \in C$ is a non-singular point. Then:

$I_p(C \cap C')$ is the order of vanishing of the element α' in the DVR $\mathcal{O}_{C, p}$

(b) Suppose $p \in C \cap C'$ is an arbitrary intersection point. Then:

$$I_p(C \cap C') \geq \text{mult}_p(C) \cdot \text{mult}_p(C')$$

with equality if and only if the tangent cones to C and C' at p are “transverse,” i.e. they do not share a common line.

Proof. (a) is straightforward. $\mathcal{O}_{\mathbb{CP}^2, p} / \langle \alpha, \alpha' \rangle = (\mathcal{O}_{\mathbb{CP}^2, p} / \langle \alpha \rangle) / \langle \alpha' \rangle = \mathcal{O}_{C, p} / \langle \alpha' \rangle$ by Exercise 4.3 (a), and the dimension of $\mathcal{O}_{C, p} / \langle \alpha' \rangle$ is the order of α' (the value of d so that $\langle \alpha' \rangle = m_p^d$) in the DVR $\mathcal{O}_{C, p}$.

For (b), assume as usual that $p = (1 : 0 : 0)$ and let $m = \text{mult}_p(C)$ and $m' = \text{mult}_p(C')$ and consider the surjective map:

$$q : \mathcal{O}_{\mathbb{CP}^2, p} / \langle \alpha, \alpha' \rangle \rightarrow \mathcal{O}_{\mathbb{CP}^2, p} / \langle \alpha, \alpha', m_p^{m+m'} \rangle = \mathbb{C}[y_1, y_2] / \langle \alpha, \alpha', I^{m+m'} \rangle$$

where $I^{m+m'}$ is the ideal generated by monomials of degree $m + m'$ in y_1, y_2 . (The last equality, as usual, comes from Exercise 4.3 (b).)

There is an exact sequence:

$$\mathbb{C}[y_1, y_2] / I^m \oplus \mathbb{C}[y_1, y_2] / I^{m'} \xrightarrow{k} \mathbb{C}[y_1, y_2] / I^{m+m'} \rightarrow \mathbb{C}[y_1, y_2] / \langle \alpha, \alpha', I^{m+m'} \rangle \rightarrow 0$$

where $k(f, g) = f\alpha' - g\alpha$, from which we conclude the first part of (b):

$$\begin{aligned} I_p(C \cap C') &= \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{CP}^2, p} / \langle \alpha, \alpha' \rangle \geq \dim \mathbb{C}[y_1, y_2] / \langle \alpha, \alpha', I^{m+m'} \rangle \\ &\geq \binom{m+m'+1}{2} - \binom{m+1}{2} - \binom{m'+1}{2} = mm' \end{aligned}$$

with equality if and only if q, k are both injective. We finish the proof with two observations about q and k .

(i) k is injective if and only if C, C' do not share a tangent line at p .

Suppose, on the contrary, that their tangent cones at $(0, 0) \in \mathbb{C}^2$ share a common tangent line $l = 0$. Then the Taylor expansions are:

$$\alpha = l\alpha_{m-1} + \alpha_{m+1} + \dots + \alpha_a \text{ and } \alpha' = l\alpha'_{m'-1} + \alpha'_{m'+1} + \dots + \alpha'_a$$

and $k(\alpha_{m-1}, \alpha'_{m'-1}) = \alpha_{m-1}\alpha' - \alpha'_{m'-1}\alpha \in I^{m+m'}$ exhibits a non-zero element of the kernel of k .

Conversely, if $(f, g) \in \ker(k)$ and $f = f_d + \dots$ and $g = g_e + \dots$, then $f_d \alpha_{m'} - g_e \alpha_m = 0$ and $d < m, e < m'$ imply that the polynomials $\alpha'_{m'}, \alpha_m$ defining the tangent cones share a common factor.

(ii) If C, C' do not share a common tangent, then q is injective.

First, notice that because $I_p(C \cap C') = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{CP}^2, p} / \langle \alpha, \alpha' \rangle$ is finite, it follows that the map:

$$q_N : \mathcal{O}_{\mathbb{CP}^2, p} / \langle \alpha, \alpha' \rangle \rightarrow \mathcal{O}_{\mathbb{CP}^2, p} / \langle \alpha, \alpha', m_p^N \rangle = \mathbb{C}[y_1, y_2] / \langle \alpha, \alpha', I^N \rangle$$

is injective for sufficiently large values of N . Thus, it suffices to show:

$$\mathbb{C}[y_1, y_2] / \langle \alpha, \alpha', I^N \rangle \rightarrow \mathbb{C}[y_1, y_2] / \langle \alpha, \alpha', I^{m+m'} \rangle$$

is injective for all $N \geq m + m'$. But this follows from:

Exercise 4.4. If $\alpha_m, \alpha'_{m'} \in \mathbb{C}[y_0, y_1]$ are homogeneous forms of degrees $m, m' > 0$ with no common factors, then:

$$\mathbb{C}[y_0, y_1]_{m'-1} \cdot \alpha_m + \mathbb{C}[y_0, y_1]_{m-1} \cdot \alpha'_{m'} = \mathbb{C}[y_0, y_1]_{m+m'-1}$$

As an example of how to use the Proposition, consider:

Exercise 4.5. Find the intersection points of the singular cubic curves:

$$E_0 \cap E_{\infty} \text{ (from Exercise 4.1),}$$

compute each multiplicity $I_p(E_0 \cap E_{\infty})$ and check that they sum to 9.

Bézout's Theorem extends as stated to divisors:

Bézout's Theorem (Divisor Version) Let $D \in \mathbb{P}(d), D' \in \mathbb{P}(d')$ be effective divisors with associated homogeneous polynomials F and F' . Then either the supports $|D|$ and $|D'|$ share an irreducible curve or else their intersection is finite, and:

$$dd' = \sum_{p \in |D| \cap |D'|} \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{CP}^2, p} / \langle f, f' \rangle$$

where $f = F/x_i^d, f' = F'/x_i^{d'}$ for any coordinate x_i not vanishing at p . Moreover,

$$dd' = \dim_{\mathbb{C}} R_n = \dim_{\mathbb{C}} (\mathbb{C}[x_0, x_1, x_2] / \langle F, F' \rangle)_n \text{ for } n \geq d + d'$$

the multiplication maps $\mu_{x_i} : R_n \rightarrow R_{n+1}$ are all injective and all of the maps $R_n \rightarrow \mathbb{C}[y_1, y_2] / \langle f, f' \rangle \rightarrow \times \mathcal{O}_{\mathbb{CP}^2, p} / \langle f, f' \rangle$ are isomorphisms when $n \geq d + d'$ as in the proof of Theorem 4.1 (provided $|D| \cap |D'| \subset U_i = \mathbb{C}^2$)

Remark. The support of a divisor $D = \sum d_i C_i$ is the union:

$$|D| = \bigcup_{i|d_i \neq 0} C_i$$

of the irreducible curves appearing with nonzero coefficient in D .

Proof. The equality of numbers follows from Theorem 4.1 and:

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{CP}^2, p} / \langle f, gh \rangle = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{CP}^2, p} / \langle f, g \rangle + \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{CP}^2, p} / \langle f, h \rangle$$

The rest is proved simply by replacing prime polynomials A and A' with relatively prime polynomials F and F' in all the earlier proofs. $\square$

$$\text{Let } D = \sum d_i C_i \text{ and } D' = \sum d'_j C'_j.$$

Corollary 4.2. Suppose $|D| \cap |D'|$ consists of dd' distinct points. Then every intersection point is a non-singular point of both D and D' , i.e.

- (a) Every intersection point belongs to a unique pair C_i and C'_j .
- (b) Each intersection is a nonsingular point of C_i and C'_j .
- (c) Each of the local intersection rings at $p \in C_i \cap C'_j$ satisfies:

$$\langle f, f' \rangle = m_p \subset \mathcal{O}_{\mathbb{CP}^2}$$

(i.e. the intersections of C_i and C'_j are transverse).

- (d) Each $d_i = d'_j = 1$ for all i, j (because each C_i and C'_j intersects!)

Proof. If any of these conditions were violated, then there would be an intersection point with intersection number > 1 . $\square$

Noether's Theorem. Let $F, F' \in \mathbb{C}[x_0, x_1, x_2]$ be homogeneous and relatively prime and suppose $G \in \mathbb{C}[x_0, x_1, x_2]_d$ maps to zero under:

$$R_d \rightarrow \mathbb{C}[y_1, y_2] / \langle f, f' \rangle \cong \times_{p \in V(F) \cap V(F')} \mathcal{O}_{\mathbb{CP}^2, p} / \langle f, f' \rangle$$

Then $G = HF + H'F'$ for homogeneous polynomials H, H' .

Proof. If $d \geq n = \deg(F) + \deg(F')$, then the map in question is an isomorphism, and we are done by definition. Otherwise, by injectivity of the maps μ_{x_i} , we see that $x_i^{n-d}H = 0 \in R_n$, so $H = 0$ in R_d . $\square$

Corollary 4.3. Suppose D and D' intersect in dd' points, and D'' passes through all of them. Then the associated polynomials satisfy:

$$F'' = HF + H'F' \text{ for some homogeneous } H, H'$$

Proof. Using Corollary 4.2(c), Noether's Theorem applies!

As an application, we study the following:

Proposition 4.4. Let D, D' be cubic divisors on $\mathbb{CP}^2$ intersecting in 9 distinct points $p_1, \dots, p_9$. Then:

- (a) $\dim \mathbb{P}(3; p_1, \dots, p_9) = 1$
- (b) $\mathbb{P}(3; p_1, \dots, p_8) = \mathbb{P}(3; p_1, \dots, p_9)$.

Recall that the expected dimension of $\mathbb{P}(3; p_1, \dots, p_n) = 9 - n$, so (b) says that $p_1, \dots, p_8$ “impose independent conditions” on cubic divisors.

Proof. (a) is immediate from the Corollary. If $D'' \in \mathbb{P}(3; p_1, \dots, p_9)$, then the associated cubic polynomial F'' vanishes at all the points, and $F'' = aF + a'F'$ for constants a and b . Thus $\mathbb{P}(3; p_1, \dots, p_9) = \mathbb{P}(V)$ where V is the two-dimensional vector space with basis F and F' .

For (b), suppose $D'' \in \mathbb{P}(3; p_1, \dots, p_8) - \mathbb{P}(3; p_1, \dots, p_9)$. Let q be the ninth point of the intersection $D'' \cap D$ (possibly coming from a point with intersection number 2). If $q \neq p_9$, let L be a line through p_9 but not q , and let $L \cap D = p_9 + r + s$ with $r, s \notin |D'|$. Then:

$$lF'' = l_1F + l_2F'$$

by the Corollary for the associated cubic polynomials F, F', F'' and linear forms l (defining L) and some forms L_1, L_2 since $L + D''$ contains all the points $p_1, \dots, p_9 = D \cap D'$. Evaluating at the points, we see that $l_2(r) = l_2(s) = 0$, so $l_2 = l = l_1$ (up to scalar multiples), and $F'' = aF + a'F'$, which is a contradiction!

Example 4.3. (Pascal's Mystic Hexagon) If a hexagon is inscribed in an irreducible conic, then the three sets of opposite sides meet in three collinear points.

Proof. Let C be the conic, L_1, L_2, L_3 be one set of opposite sides and M_1, M_2, M_3 the other. Then:

$$(L_1 + L_2 + L_3) \cap (M_1 + M_2 + M_3) = p_1 + \dots + p_6 + q_1 + q_2 + q_3$$

where $p_1, \dots, p_6$ are the vertices of the hexagon and q_1, q_2, q_3 are the other points. But a third cubic divisor

$$(L_1 + L_2 + L_3) \cap (C + \overline{q_1q_2})$$

contains $p_1, \dots, p_6, q_1, q_2$, so it contains q_3 as well, and $q_3 \notin C$, so it must be on the line $\overline{q_1q_2}$.

Exercise 4.6. (a) Let $E \subset \mathbb{CP}^2$ be a nonsingular cubic curve, choose an arbitrary point $0 \in E$. If L is a line in $\mathbb{CP}^2$, define:

$$L \cdot E = p_1 + p_2 + p_3 := \sum I_p(L, E)p$$

as a divisor on E and the sum $p_1 \oplus p_2$ of points $p_1, p_2 \in E$ by:

(i) $\phi(p_1, p_2) = p_3$ if $L \cdot E = p_1 + p_2 + p_3$ and:

(ii) $p_1 \oplus p_2 = \phi(0, \phi(p_1, p_2))$.

This is evidently commutative. Show that it is associative with 0 as an additive identity and that it agrees with the addition on the curves E_Λ coming from a lattice $\Lambda \subset \mathbb{C}$ from §2.