Algebraic Curves/Fall 2015

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3. Projective Hypersurfaces. A *homogeneous polynomial* of degree d in variables $x_0, ..., x_n$ is a nonzero linear combination:

$$F \in \mathbb{C}[x_0, ..., x_n]_d$$

of monomials $x_0^{d_1} \cdots x_n^{d_n}$ with $d_1 + \cdots + d_n = d$.

Homogeneneity refers to fact that:

$$F(\lambda \underline{b}) = \lambda^d F(\underline{b}) \text{ for } \underline{b} = (b_0, ..., b_n) \text{ and } \lambda \in \mathbb{C}$$

and therefore, in particular, F determines a well-defined *hypersurface*:

$$V(F) = \{(b_0 : ... : b_n) \mid F(b_0, ..., b_n) = 0\} \subset \mathbb{CP}^n$$

The action of $\mathrm{PGL}(n+1, \mathbb{C})$ on $\mathbb{CP}^n$:

$$A(b_0 : ... : b_n) = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{n1} \\ \vdots & & & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_{00}b_0 + \cdots + a_{0n}b_n \\ a_{10}b_0 + \cdots + a_{1n}b_n \\ \vdots \\ a_{n0}b_0 + \cdots a_{nn}b_n \end{bmatrix}$$

induces an linear action on homogeneous polynomials via:

$$(AF)(x_0 : ... : x_n) = F(A^{-1}(x_0 : ... : x_n))$$

so that the hypersurfaces transform consistently: $A(V(F)) = V(AF)$.

Definition 3.1. Two hypersurfaces in $\mathbb{CP}^n$ are *projectively equivalent* if they are in the same orbit under the action of $\mathrm{PGL}(n+1, \mathbb{C})$.

Two central problems immediate present themselves:

(i) Understand/describe the hypersurfaces of a given degree in $\mathbb{CP}^n$.

(ii) Understand the intersections of hypersurfaces (in $\mathbb{CP}^n$).

focusing on properties that are invariant under projective equivalence (which is, after all, "just" changing the choice of basis of $\mathbb{C}^{n+1}$).

As a first example:

Definition 3.2. A hypersurface associated to a prime F is *irreducible.*

Remark. By unique factorization, every polynomial F factors:

$$F = \prod_{i=1}^{m} F_i$$

as a product of primes.

Thus every hypersurface $V(F)$ is a finite union:

$$V(F) = V(F_1) \cup \cdots \cup V(F_m)$$

so we will generally restrict our attention to irreducible hypersurfaces, and reducibility is evidently invariant under projective equivalence.

Example 3.1. By the fundamental theorem of algebra, every homogeneous polynomial in two variables is a product of **linear** polynomials:

$$F = \prod_{i=1}^{d} (\mu_i x_1 - \lambda_i x_0); \quad p_i := (\lambda_i : \mu_i)$$

and so the only irreducible hypersurfaces in $\mathbb{CP}^1$ are points, and a general hypersurface is a finite union of points (but *multiplicities* would be required to recover repeated terms in the factorization of F).

Example 3.2. The degree one hypersurfaces are hyperplanes:

$$H = \{(b_0 : ... : b_n) \mid L(\underline{b}) = a_0 b_0 + \cdots + a_n b_n = 0\} \subset \mathbb{CP}^n$$

(i) All hyperplanes are irreducible and projectively equivalent.

(ii) An intersection of r hyperplanes is a projective linear subspace:

$$\mathbb{P}(V) \subset \mathbb{CP}^n$$

where $V \subset \mathbb{C}^{n+1}$ is the kernel of the $r \times (n+1)$ matrix:

$$A = \begin{bmatrix} a_{10} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{r0} & \cdots & a_{rn} \end{bmatrix}$$

whose rows represent the r hyperplanes and $\mathbb{P}(V)$ is the space of lines (through the origin) in V. In particular, if A has full rank r, then $\mathbb{P}(V) = \mathbb{CP}^{n-r}$ (via a choice of basis of V), in which case we say that the hyperplanes intersect *transversely*. In particular, n transverse hyperplanes intersect in a single point. Notice that to intersect non-transversely, the coefficients of the hyperplanes must satisfy a collection of polynomial relations.

Example 3.3. Writing a degree two homogeneous polynomial F as:

$$F = \sum_{i,j=0}^{n} a_{ij} x_i x_j \text{ with } a_{ij} = a_{ji}$$

has the result that we may interpret points of the quadric hypersurface:

$$Q = V(F) = \{(b_0 : ... : b_n) \mid \underline{b} A \underline{b}^T = 0\} \subset \mathbb{CP}^n$$

as vectors of length zero for the symmetric bilinear form $A = (a_{ij})$.

The Gram-Schmidt process shows that every quadric hypersurface in $\mathbb{CP}^n$ is projectively equivalent to exactly one of the following:

$$Q_i = V(x_0^2 + ... + x_i^2); \quad i = 0, ..., n$$

with $Q \sim Q_i \Leftrightarrow \mathrm{rk}(A) = i+1$ for the associated matrix A.

Exercise 3.1. (a) Explain the previous paragraph in detail.

(b) Show that Q_i is irreducible for each $i > 1$ but Q_1 is reducible.

(c) Find the maximal dimension of an "isotropic" projective linear subspace $\mathbb{P}(V) \subset Q_i \subset \mathbb{CP}^n$ for each $i \geq 0$.

We've already seen some interesting examples of **intersections** of quadrics.

(i) The rational normal curves are intersections of quadrics.

(ii) The elliptic curve E_Λ embedded in $\mathbb{CP}^3$ by $(1 : \mathcal{P} : \mathcal{P}' : \mathcal{P}'')$ is an intersection of two quadrics, but so is the twisted cubic union a line.

In fact, **every projective variety** is isomorphic to an intersection of quadrics in some projective space. Even if we stick to transverse intersections of quadrics, the list is very large. For example, curves of arbitrarily high genus can occur as transverse intersections of quadrics.

When we pass to cubics and higher degree hypersurfaces in $\mathbb{CP}^n$, (with one exception) then there are "moduli" to the hypersurfaces, i.e. the dimension of the (projective) space of homogeneous polynomials exceeds the dimension of the group of projective transformations:

$$\binom{n+d}{d} = \dim(\mathbb{C}[x_0, ..., x_n]_d) > (n+1)^2 = \dim(GL(n+1, \mathbb{C}))$$

and so the projective equivalence classes vary in continuous families.

Remark. The exception is the cubic hypersurface in $\mathbb{CP}^1$, which consists of three points (if the linear factors are distinct). By Exercise 1.1, these are taken to $\{0, 1, \infty\}$ by a unique projective linear transformation.

Definition 3.3. (a) A hypersurface $X = V(F)$ is *nonsingular* at $p \in X$ if:

$$\nabla F(p) = \left(\frac{\partial F}{\partial x_0}(p), ..., \frac{\partial F}{\partial x_n}(p)\right) \neq (0, ..., 0)$$

and in that case,

$$T_pX = V\left(\frac{\partial F}{\partial x_0}(p)x_0 + \cdots + \frac{\partial F}{\partial x_n}(p)x_n\right) \subset \mathbb{CP}^n$$

is the *embedded Zariski tangent space* of X at p.

(b) X is *nonsingular* if it is nonsingular at every $p \in X$.

Remarks. (i) Singularity is invariant under projective equivalence.

(ii) If $p \in V(F) \cap V(G)$, then $V(FG)$ is singular at p (Leibniz rule!).

(iii) By Euler's relation if F is homogeneous of degree d, then:

$$\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} = d \cdot F$$

It follows that the Zariski tangent space actually contains the point p, and that the condition $\nabla F(p) = \underline{0}$ for $p \in \mathbb{CP}^n$ also implies that $p \in X$. To see that it merits the title of "tangent space," we need to pass to affine coordinates.

Suppose $p = (1 : 0 : \cdots : 0)$ and $p \in V(F)$. Then:

$$V(F) = 0 \cdot x_0^d + \cdots \quad \text{and} \quad \frac{\partial F}{\partial x_0}(p) = 0$$

and we may *dehomogenize* the polynomial F to get a Taylor polynomial:

$$f\left(\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}\right) = \frac{F(x_0, ..., x_n)}{x_0^d} = a_0 \frac{x_1}{x_0} + \cdots + a_n \frac{x_n}{x_0} + \text{higher order}$$

and then $a_i = \frac{\partial f}{\partial y_i}(0, ..., 0)$ where $y_i = x_i/x_0$ is the ith coordinate, but we also have:

$$\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial y_i} \cdot \frac{1}{x_0} = \frac{\partial F}{\partial x_i} \cdot \frac{1}{x_0^d} \quad \text{so} \quad \frac{\partial f}{\partial y_i} = \frac{\partial F/\partial x_i}{x_0^{d-1}}$$

is the dehomogenized partial derivative and the ordinary tangent space at the origin agrees with the Zariski tangent space. We can bring any point p to $(1 : 0... : 0)$ by a projective transformation and conclude:

Implicit Function Theorem. If a hypersurface $X \subset \mathbb{CP}^n$ is nonsingular at $p \in X$, then X is a complex analytic manifold of (complex) dimension $n-1$ near $p \in X$, i.e. it has $n-1$ local **analytic** coordinates.

We can similarly extract information from singular points.

Definition 3.4. (a) A point $p \in X = V(F)$ *has multiplicity* m if:

$$0 = F(p) = ... = \frac{\partial^d F}{\partial x_0^{d_0} \cdots \partial x_n^{d_n}}(p)$$

for all $d < m$, but some partial of order m does not vanish at p, and in that case the *embedded tangent cone* to X at $p \in X$ is:

$$C_p X = V\left(\sum_{|\underline{m}|=m} \frac{\partial^m F}{\partial x_0^{m_0} \cdots \partial x_n^{m_n}}(p) \cdot x_0^{m_0} \cdots x_n^{m_n}\right)$$

Exercise 3.2. (a) Check as above that the equation defining C_pX agrees with the lowest order term in the Taylor series for $f(y_1, ..., y_n)$ at $(0, ..., 0)$ when $p = (1 : 0 : ... : 0)$ and $f = F/x_0^d$.

(b) Find the singular points $p \in Q_i \subset \mathbb{CP}^n$ and their tangent cones.

(c) Use Euler's equation to conclude that if all the partial derivatives of F of order $m - 1$ vanish at p, then all the partial derivatives of all smaller orders also vanish at p.

While we are on the subject of cones, let $V \subset \mathbb{C}^{n+1}$ be a subspace of dimension d, and consider:

Definition 3.5. The "rational" projection map:

$$\pi_V : \mathbb{CP}^n - -- > \mathbb{CP}(\mathbb{C}^{n+1}/V) = \mathbb{CP}^{n-d}$$

defined by:

$$\pi(l) = \begin{cases} l \ (\text{mod } V) \text{ if } l \notin \mathbb{P}(V) \\ \text{undefined otherwise} \end{cases}$$

(the last identification with $\mathbb{CP}^{n-d}$ requires a choice of basis for $\mathbb{C}^{n+1}/V$).

Definition 3.6. A hypersurface $X \subset \mathbb{CP}^n$ is a **cone** if:

$$X = \pi_V^{-1}(Y) \cup \mathbb{P}(V)$$

for some non-zero $V \subset \mathbb{C}^{n+1}$ and hypersurface $Y \subset \mathbb{CP}^{n-d}$, and in that case we say $X = C(Y)$ is the *cone over* Y *with vertex* $\mathbb{P}(V)$.

Remark. Given $V \subset \mathbb{C}^{n+1}$, choose projective coordinates so that $\mathbb{P}(V)$ is given by equations $x_0 = ... = x_{n-d} = 0$. Then with respect to these coordinates, $X = V(F) \subset \mathbb{CP}^n$ is a cone with vertex $\mathbb{P}(V)$ if F is a polynomial in only the variables $x_0, ..., x_{n-d}$.

Examples. (a) Each $Q_i \subset \mathbb{CP}^n$ is a cone over $Q_i \subset \mathbb{CP}^i$, and the latter is non-singular, as you discovered in Exercise 3.2 (b).

(b) The tangent cones C_pX are cones with vertex $p = \mathbb{P}(V)$.

We have been defining properties of a hypersurface X in terms of the defining equation F. This is a sensible thing to do because of the following special case of the Hilbert Nullstellensatz:

Theorem 3.1. Suppose $X = V(F) \subset \mathbb{CP}^n$ is a hypersurface.

(i) If $F = 0$, then $X = \mathbb{CP}^n$.

(ii) If $F \in \mathbb{C} - \{0\}$, then $X = \emptyset$.

(iii) Otherwise, $X \subset \mathbb{CP}^n$ is proper and non-empty, and projection from any $p \notin X$ is a mapping that is finite-to-one and onto:

$$\pi_p : X \to \mathbb{CP}^{n-1}$$

(iv) A prime $F \subset \mathbb{C}[x_0, ..., x_n]_d$ can be recovered (up to a nonzero scalar multiple) from the hypersurface $X = V(F)$ that it defines.

Proof. (i) and (ii) are obvious.

Given $F \in \mathbb{C}[x_0, ..., x_n]_d - \{0\}$, then for points $(a_0, ..., a_n) \in \mathbb{CP}^n$ with $|a_0| >> \cdots >> |a_n| >> 0$, the absolute value of the monomial in F that is first lexicographically will dominate the others, and $F(\underline{a}) \neq 0$ for all such points.

For the rest of (iii), take $p \notin X$ and consider the projection map $\pi_p : \mathbb{CP}^n - - > \mathbb{CP}^{n-1}$. If $\overline{pq} - \{p\}$ is a fiber of this map, then after a suitable projective transformation, $\overline{pq}$ is defined by $x_0 = ... = x_{n-2} = 0$, and $p = (0 : ... : 0 : 1)$. Then the intersection $X \cap \overline{pq}$ is the set of zeroes of the homogeneous polynomial $F|_{x_0=...=x_{n-2}=0}$, which is **nonzero** (otherwise $p \in X = V(F)$), and in fact consists of exactly d points (counting multiplicity) so it is finite-to-one and onto.

As for (iv), suppose F is prime and $G \in \mathbb{C}[x_0, ..., x_n]_e - \{0\}$ satisfies $G(X) \equiv 0$ for $X = V(F)$. We may assume that $p = (1 : 0 : ... : 0) \notin X$ with a suitable choice of coordinates, and that $G(p) \neq 0$, as well. Then:

$$f(y_0, ..., y_{n-1}) = ay_0^d + ... \text{ and } g(y_0, ..., y_{n-1}) = by_0^e + ...$$

where $f = F/x_n^d$ and $g = G/x_n^e$. Regard these as polynomials:

$$f, g \in \mathbb{C}(y_1, ..., y_{n-1})[y_0]$$

of degrees d and e respectively. Since f is prime, then in this Euclidean domain, we can find polynomials $h, k \in \mathbb{C}(y_1, ..., y_{n-1})[y_0]$ so that:

$$hf - kg \in \mathbb{C}(y_1, ..., y_{n-1})$$

and we may additionally assume that $\deg(k) < d$.

Clearing denominators of $hf - kg$ as a polynomial in $y_1, ..., y_{n-1}$ and then multiplying through by a suitable power of x_n, we then have:

$$HF - KG \in \mathbb{C}[x_1, ..., x_n]$$

In other words, $V(HF - KG) = C(Y)$ is a cone over a hypersurface $Y \subset \mathbb{CP}^{n-1}$. But $X \subset V(HF - KG)$, and H projects **onto** $\mathbb{CP}^{n-1}$ by (iii), so $Y = \mathbb{CP}^{n-1}$ and $HF - KG = 0$, also by (iii). But then F divides KG and because the degree of x_0 in K is less than d, it follows that F divides G. Thus the **ideal** $I(X)$ of homogeneous polynomials G that vanish on X is the principal ideal $\langle F \rangle$, and so F is recovered (up to a scalar) from X.

Note. The only property of $\mathbb{C}$ that we used here was the fact that it is algebraically closed. This argument (and the Nullstellensatz itself) only requires this property. Contrast this with the use of the Implicit Function Theorem earlier to conclude that a hypersurface is a manifold near each non-singular point $p \in X$.

Final Remark. The Theorem allows us to interpret irreducible hypersurfaces $X = V(F)$ as **points** of the projective space of polynomials:

$$X \leftrightarrow [F] \in \mathbb{P}(\mathbb{C}[x_0, ..., x_n]_d)$$

where $[F]$ is the equivalence class (line through the origin) of the polynomial F. As we did with curves, we can extend this identification to all points of the projective space by means of *divisors*, where an effective divisor on $\mathbb{CP}^n$ is a non-negative integer linear combination:

$$D = \sum_{i=1}^{m} d_i X_i$$

of **irreducible** hypersurfaces X_i, with associated homogeneous polynomials F_i, and the identification is:

$$D \leftrightarrow [\prod F_i^{d_i}] \in \mathbb{P}(\mathbb{C}[x_0, ..., x_n]_d)$$

with

$$d = \deg(D) = \sum_{i=1}^{m} d_i \deg(F_i)$$

as the natural generalization of the degree of a divisor on a curve.