Algebraic Curves/Fall 2015

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6. Differentials and Genus. To each a non-singular curve:

$$C = V(F_1) \cap \cdots \cap V(F_m) \subset \mathbb{CP}^n$$

we associate the following information:

(i) The homogeneous prime ideal defined by C:

$$I(C) = \{F \in \mathbb{C}[x_0, ..., x_n]_d \mid F(C) \equiv 0\} \subset \mathbb{C}[x_0, ..., x_n],$$

(ii) The homogeneous coordinate ring (graded integral domain):

$$R_C = \mathbb{C}[x_0, ..., x_n]/I(C),$$

(iii) The field of rational functions of C:

$$K(C) = \left\{ \frac{F}{G} \mid F, G \in (R_C)_d, G \neq 0 \right\},$$

(iv) The local rings of (germs of) regular functions at each $p \in C$:

$$\mathcal{O}_{C,p} = \left\{ \phi \in K(C) \mid \phi = \frac{F}{G}, G(p) \neq 0 \right\} \subset K(C)$$

(v) The maximal ideals:

$$m_{C,p} = \{\phi \in \mathcal{O}_{C,p} \mid \phi(p) = 0\} \subset \mathcal{O}_{C,p}$$

Remark. A rational function F/G on $\mathbb{CP}^n$ restricts to a meromorphic function on C which therefore has finitely many zeroes and poles. This explains why $I(C)$ is a prime ideal. As usual, if $x_0 \notin I(C)$, then the polynomials $f(y_1, ..., y_n)$ that vanish on the affine curve $C \cap U_0$ are all of the form F/x_0^d for $F \in I(C)$, and an affine (integral domain) coordinate ring for $C \cap U_0$ results, with ordinary field of fractions equal to $K(C)$.

Proposition 6.1. Each of the maximal ideals $m_{C,p}$ is principal.
In other words, the local rings $\mathcal{O}_{C,p}$ are discrete valuation rings.

Proof. This is a generalization of Proposition 4.2(a). We may choose coordinates so that $p = (1 : 0 : ... : 0)$ and moreover so that the nonsingular hypersurfaces $F_{i_1}, ..., F_{i_{n-1}}$ cutting out C near p (with linearly independent Zariski tangent spaces) have affine equations:

$$f_{i_k}(y_1, ..., y_n) = y_k g_k + h_k$$

where g_k are units in $\mathcal{O}_{\mathbb{C}^n,0}$ and h_k are polynomials in all the variables other than y_k with leading term of degree ≥ 2. It follows by induction that $\overline{y}_n \in m_{C,p}$ generates the ideal. $\square$

Remark. It follows from the proof that if $y = \sum_{i=1}^n k_i y_i$ is not in the linear span of the tangent hyperplanes to the equations cutting out the affine curve $C \cap U_0$ near $p = (0, ..., 0) \in \mathbb{C}^n$, then $\overline{y}$ generates $m_{C,p}$.

We now revisit the divisor associated to $\phi \in K(C)$ in algebraic terms. For each $p \in C$, let $y_p \in m_p$ be a choice of generator. Then:

Definition 6.1. (a) For each $\phi \in K(C) - \{0\}$ and $p \in C$, we have:

$$\phi = uy_p^d \ \text{ for a } \textbf{unique } d \text{ and unit } u \in \mathcal{O}_{C,p}$$

and then $d =: \mathrm{ord}_p(\phi)$ is the order of vanishing (or pole) of ϕ at p.

(b) The divisor of zeroes (and poles) of $\phi \in K(C) - \{0\}$ is:

$$\mathrm{div}(\phi) = \sum_{p \in C} \mathrm{ord}_p(\phi) \cdot p$$

(c) A divisor on C that is $\mathrm{div}(\phi)$ of some ϕ is a **principal divisor**.

Proposition 6.2. Principal divisors all have degree zero.

There are several ways to see this.

Analytic Argument. Let $p_1, ..., p_n$ be the points in the support of $\mathrm{div}(\phi)$, so ϕ is a meromorphic function on C with zeroes (or poles) of order d_i at p_i and no other zeroes or poles. Then $\frac{1}{2\pi i}\frac{d\phi}{\phi}$ is a meromorphic (analytic) differential on C that, when integrated around small closed loops about each p_i computes the order of ϕ at p_i as the residue of the differential form. Integrating around all the closed loops gives the degree of $\mathrm{div}(\phi)$, which is zero because the union of the closed loops bounds a complementary region in the curve where the differential has no zeroes and no poles.

Field Theoretic Argument. The rational function ϕ defines a map $\phi : C \to \mathbb{CP}^1$ that corresponds to the inclusion of fields $\mathbb{C}(\phi) \subset K(C)$. This is a finite field extension of degree $d = [K(C) : K(\phi)]$, and d is **both** the degree of the divisor of zeroes of ϕ and the divisor of zeroes of ϕ^{-1}, which is the divisor of poles of ϕ.

Definition 6.2. Let $\phi \in \mathcal{O}_{C,p}$. Then:

$$d\phi := \phi - \phi(p) \in m_p/m_p^2 =: \mathrm{T}_p^*C (= \mathbb{C})$$

Since $\phi \in K(C)$ belongs to $\mathcal{O}_{C,p}$ for all but finitely many $p \in C$, we may a priori define $d\phi$ for all but finitely many points, though the values of $d\phi$ land in the one-dimensional vector spaces m_p/m_p^2 that **vary** with the point p. We can extend the definition “rationally” across all points of C with the help of the following observations:

Definition 6.3. A non-zero (rational) **differential** on C is:

$$\omega = fd\phi \text{ defined by } \omega(p) = f(p)d\phi(p) \in m_p/m_p^2$$

for some $f, \phi \in \mathcal{O}_{C,p}$.

Proposition 6.3. Differentials satisfy the following, wherever defined:

(a) $dc = 0$ for constant functions $c \in \mathbb{C}$.

(b) $d(\phi + \psi) = d\phi + d\psi$

(c) $d(\phi \cdot \psi) = \phi d\psi + \psi d\phi$

(e) $d(f(\phi)) = f'(\phi)d\phi$ for $f \in \mathbb{C}(z)$, wherever both sides are defined.

Proof. Exercise!

The idea behind extending the differential is to use the properties in Proposition 6.3 to push all the poles into the leading rational function f in $\omega = f\phi$ by choosing $\phi \in m_p - m_p^2$ when evaluating near p.

Example. (a) Consider the differential dy_1 (for $y_1 = x_1/x_0$) on $\mathbb{CP}^1$.

(i) $dy_1(c) = y_1 - c \neq 0 \in m_p/m_p^2$ for all $c \in U_0$.

(ii) $y_1 = y_0^{-1}$, so $dy_1 = -y_0^{-2}dy_0$ has a pole of order 2 at $y_0 = 0$.

(b) A more "homogeneous" differential is:

$$\frac{dy_1}{y_1} = -\frac{dy_0}{y_0}$$

which has poles of order one at 0 and ∞ (with opposite residues!)

Definition 6.4. The *order of vanishing* of $\omega = fd\phi$ at p is obtained from the local expansions in terms of the generator $y_p \in m_p$:

$$\phi = uy_p^n \text{ and } f = vy_p^m$$

to rewrite:

$$fd\phi = vy_p^m(u(ny_p^{n-1})dy_p + y_p^n du)$$

and conclude that $y_p^{(-m-n+1)}\omega(p) \neq 0$. Hence we (well-)define:

$$\text{ord}_p(\omega) = m + n - 1$$

unless $n = 0$ or $f = 0$, in which case $\omega = 0$. And then:

$$\deg(\omega) = \sum_{p \in C} \text{ord}_p(\omega)p$$

Proposition 6.4. Any two differentials $\omega_1 = f_1 d\phi_1$ and $\omega_2 = f_2 d\phi_2$ are related by multiplication by a single rational function:

$$\omega_1 = g \cdot \omega_2$$

Proof. $\phi_1, \phi_2 \in K(C)$ are related by a polynomial: $P(\phi_1, \phi_2) = 0$ since the transcendence degree of $K(C)$ over $\mathbb{C}$ is one. Then by implicit differentiation, we get:

$$g_1 d\phi_1 = g_2 d\phi_2$$

for rational functions g_1, g_2, and the result follows. □

Corollary 6.1. The degree $\deg(\mathrm{div}(\omega))$ is independent of ω.

Proof. Since all other differentials are of the form $g\omega$, we have:

$$\deg(\mathrm{div}(g\omega)) = \deg(\mathrm{div}(g)) + \deg(\mathrm{div}(\omega)) = \deg(\mathrm{div}(\omega))$$

by Proposition 6.2. □

Definition 6.5. (a) $\Omega(C)$ is the space of rational differentials, and:

$$\Omega(C) = \{g\omega \mid g \in K(C)\} = K(C)$$

identifies $\Omega(C)$ with $K(C)$ (given a choice of ω).

(b) The *canonical linear series* $L(K_C)$ is abstractly:

$$\Omega[C] = \{\omega \in \Omega(C) \mid \mathrm{div}(\omega) \geq 0\} \subset \Omega(C)$$

which embeds as a linear series in $K(C)$ given a choice of ω:

$$L(K_C) = \{g \in K(C) \mid \mathrm{div}(g\omega) \geq 0\} \subset K(C)$$

as a finite-dimensional vector space with:

$$|K_C| = \{\text{divisors of differentials in } \Omega[C]\} \cong \mathbb{P}(L(K_C))$$

It is customary to let K_C stand for any of these effective divisors that are divisors of zeroes of a *regular* differential in $\Omega[C]$.

Theorem 6.1. Let $\chi(C)$ be the topological Euler characteristic of C, i.e. $\chi(C) = 2 - 2g$ where g is the genus (number of holes) of C. Then:

$$\deg(K_C) = -\chi(C) = 2g - 2$$

Proof. Let $\phi \in K(C) - \mathbb{C}$ be a non-constant function, determining:

$$\phi : C \to \mathbb{CP}^1$$

Then $\mathrm{div}(d\phi)$ can be seen in terms of the ramification of the map ϕ. Namely, if $p \in C$ has the property that $\phi - \phi(p) \in m_p^e - m_p^{e-1}$ and $e > 1$, then we say that p is a ramification point of order e of the map. Similarly, ϕ^{-1} may define ramification via $\phi^{-1} - \phi^{-1}(p) \in m_p^e - m_p^{e-1}$ and wherever $\phi(p) \neq 0, \infty$, they define the **same** notion of ramification:

$$\phi - \phi(p) = -(\phi \cdot \phi(p))(\phi^{-1} - \phi^{-1}(p))$$

so the ramification e_p of the map ϕ at p is well-defined.

We conclude the Theorem from the Rieman–Hurwitz formula:

$$\mathrm{div}(d\phi) = -2d + \sum_{p\in C}(e_p - 1)p$$

and the Euler characteristic formula:

$$\chi(C) = 2d - \sum_{p\in C}(e_p - 1)p$$

where d is the degree of the divisor of zeroes (or poles) of ϕ.

Consider the points $p \in C$ where ϕ is defined (i.e. p is not a pole). Then $\mathrm{ord}_p(d\phi) = e_p - 1$ by definition since $\phi = uy_p^{e_p}$. Next, using $d\phi = -\phi^{-2}d\phi^{-1}$ we see that at a pole of ϕ (= zero of ϕ^{-1}), we have:

$$\mathrm{ord}_p(d\phi) = 2\mathrm{ord}_p(\phi^{-1}) + \mathrm{ord}_p(\phi^{-1}) = 2\mathrm{ord}_p(\phi^{-1}) + (e_p - 1)$$

and the Riemann-Hurwitz formula follows by summing.

Next, we turn to analysis and topology to get the second formula. Near a point $p \in C$ with ramification e_p, we can choose an analytic coordinate so that $\phi(z) = (z-p)^{e_p}$. One may then triangulate $\mathbb{CP}^1$ with vertices including all images of points of ramification $e_p > 1$ in such a way that it lifts to a triangulation of C. By Euler, any triangulation of $\mathbb{CP}^1$ with v vertices, e edges and f faces satisfies:

$$v - e + f = 2 = \chi(\mathbb{CP}^1)$$

and the vertices, edges and faces of the lift to C satisfy:

$$v_C = d\cdot v - \sum_{p\in C}(e_p - 1), e_C = d\cdot e \text{ and } f_C = d\cdot f$$

from which we obtain the second formula and the Theorem. □

Example. The projection of the affine elliptic curve:

$$E_{\mathrm{aff}} = \{(y_1, y_2) \in U_0 \mid y_2^2 = y_1(y_1 - 1)(y_1 - \lambda)\} \subset U_0 = \mathbb{C}^2$$

onto the y_1 axis ramifies at the three points of the y_1-axis, so dy_1 would pick up zeroes at these points. But by implicit differentiation:

$$\omega = \frac{dy_1}{y_2} = \frac{dy_2}{2[y_1(y_1 - 1) + y_1(y_1 - \lambda) + (y_1 - 1)(y_1 - \lambda)]}$$

and it follows that ω has no zeroes or poles on E_{aff}. But now:

$$E = V(x_0x_2^2 - x_1(x_1 - x_0)(x_1 - \lambda x_0)) = E_{\mathrm{aff}} \cup (0:0:1) \subset \mathbb{CP}^2$$

and since ω has degree zero, it follows (and can be seen explicitly) that ω is also regular at the additional point "at infinity." If we recall that E_{aff} is the image of the pair of meromorphic functions $(\mathcal{P}(z), \mathcal{P}'(z))$ on E_Λ (from §2) then $\omega = d(\mathcal{P}(z))/\mathcal{P}'(z) = dz$ as a differential on E_Λ!