Algebraic Geometry (Math 6130)

Utah/Fall 2016.

3. The Category of Quasiprojective Varieties over a Field. We need next to understand affine and projective varieties *categorically* without referencing their particular realizations as subsets of k^n or $\mathbb{P}^n_k$. This is analogous to considering a finitely generated algebra without specifying a choice of generators. We accomplish this by specifying a topology and a sheaf of regular functions. Once this is done, we will be able to define the category of **quasi-projective varieties**, which contains both the projective and affine varieties as special cases.

Let $X \subset k^n$ be an affine variety with coordinate ring and field:

$$k[X] \subset k(X)$$

Definition 3.1. A subset $Z \subset X$ is **Zariski closed** if:

$$Z = V(I) = \{x \in X \mid f(x) = 0 \text{ for all } f \in I\}$$

for some ideal $I \subseteq k[X]$.

Remark. It follows from the correspondences of §1 that Zariski closed subsets of X are in bijection with radical ideals in $k[X]$, via:

$$I(Z) = \{f \in k[X] \mid f(x) = 0 \text{ for all } x \in Z\}$$

and that in particular the maximal ideals in $k[X]$ are in bijection with the points and prime ideals are in bijection with the varieties $Y \subset X$.

Proposition 3.1. The Zariski closed subsets $Z \subset X$ form a **topology**.

Proof. A designation of closed sets of X forms a topology if:

(i) $\emptyset$ and X are closed sets.

(ii) The union of finitely many closed sets is closed, and

(iii) The intersection of an arbitrary collection of closed sets is closed.

We get (i) from the ideals $k[X]$ and $\langle 0 \rangle$, respectively.

If $Z_1 = V(I_1)$ and $Z_2 = V(I_2)$, then $Z_1 \cup Z_2 = V(I_1 I_2)$, giving (ii) by induction, and if $Z_\lambda = V(I_\lambda); \lambda \in \Lambda$ for any index set Λ, then $\bigcap_{\lambda\in\Lambda} Z_\lambda = V(\sum_{\lambda\in\Lambda} I_\lambda)$. □

Warning. For any finite set of ideals,

$$V(I_1 \cdots I_m) = V(I_1) \cup \cdots \cup V(I_m) = V(I_1 \cap \cdots \cap I_m)$$

but there is no product of infinitely many ideals and the equality on the right generally fails for infinite sets of ideals.

Definition 3.2. A topology on X is **Noetherian** if every chain:

$$X \supset Z_1 \supset Z_2 \supset \cdots$$

of closed sets is eventually stationary, i.e. $Z_n = Z_{n+1} = ..$ for some n. In a Noetherian topology, a closed set $Z \subset X$ is **irreducible** if:

$Z_1 \cup Z_2 = Z$ for closed subsets Z_1, Z_2 if and only if $Z_1 = Z$ or $Z_2 = Z$

i.e. Z irreducible when every nonempty open subset $U \subset Z$ is dense.

Proposition 3.2. The Zariski topology on an affine variety $X \subset k^n$ is Noetherian and the irreducible closed subsets of X are the varieties.

Proof. The correspondence between closed sets and radical ideals:

$$Z \subset X \leftrightarrow I \subset k[X]$$

is *inclusion reversing*. Thus, a chain of closed sets corresponds to:

$$0 \subset I_1 \subset I_2 \subset \cdots ; \quad I_i = I(Z_i)$$

and since $k[x_1, .., x_n]$ is a Noetherian commutative ring, it follows that $k[X]$ is Noetherian, and, in particular, the ideal $I = \cup I_i$ is finitely generated, so the chain of ideals (hence also the chain of closed sets) is eventually stationary.

Now suppose $Y \subset X$ is a reducible closed set with $Y = Y_1 \cup Y_2$ and $Y \neq Y_1, Y_2$. Then there are points $y_1 \in Y_1 - Y_2$ and $y_2 \in Y_2 - Y_1$ with $f_1 \in I(Y_2)$ and $f_2 \in I(Y_1)$ satisfying $f_1(y_1) \neq 0$ and $f_2(y_2) \neq 0$, so $f_1 f_2 \in I(Y)$ and $f_1, f_2 \notin I(Y)$. That is, $I(Y)$ is not a prime ideal. Conversely, if $I = I(Y)$ is not prime, let $f_1 f_2 \in I(Y)$ with $f_1, f_2 \notin I(Y)$. Then $Y_1 = V(\langle I, f_1 \rangle)$ and $Y_2 = V(\langle I, f_2 \rangle)$ exhibit Y as a reducible closed set. □

Corollary 3.1. Every closed subset $Z \subset X$ of an affine variety is a union of finitely many irreducible *distinct* closed sets $Z = Z_1 \cup \cdots \cup Z_m$, which are the **irreducible components** of Z.

Proof. This is an exercise, true of any Noetherian topology.

Remark. This is a strange topology when compared, for example, to metric space topologies. In particular, every nonempty open subset $U \subset X$ is dense, so any finite collection of nonempty open sets has a nonempty intersection, and thus the Zariski topology on X fails, spectacularly, to be Hausdorff. As a special case of Corollary 3.1., notice that any hypersurface $V(f) \subset k^n$ is a union of finitely many irreucible hypersurfaces, corresponding to the irreducible factors of f. Notice also the implication of the Corollary in commutative algebra: Every radical ideal is a finite intersection of "minimal" prime ideals.

The Zariski topology also leads to a (third!) notion of dimension:

Definition 3.3. The **Krull dimension** of an irreducible Noetherian topological space X is the supremum of the lengths of all chains of proper inclusions of irreducible closed subsets:

$$\emptyset \subset Z_1 \subset Z_2 \subset \cdots \subset Z_n \subset X$$

Example. A point has Krull dimension zero and the affine line $k = k^1$ has Krull dimension one, but note that a Noetherian topological space need not, in general, have a finite Krull dimension.

There is one other important feature of the Zariski topology on an affine variety X, namely the existence of **basic open sets**

$$U_f := X - V(f) \text{ for } f \in k[X]$$

Proposition 3.3. Every open subset $U \subset X$ of an affine variety is a union of finitely many basic open sets.

Proof. If $U \subset X$ is an open subset and $X - U = Z = V(I)$ with $I = \langle f_1, ..., f_m \rangle$ then $Z = V(f_1) \cap \cdots \cap V(f_m)$ and $U = U_{f_1} \cup \cdots \cup U_{f_m}$.

Example. The open set $U = k^2 - \{0\}$ is not a basic open set in k^2. The origin is not the zero locus of an irreducible polynomial $f \in k[x, y]$ since all non-trivial coordinate rings $k[X] = k[x, y]/\langle f \rangle$ have transcendence degree (at least) one over k.

Remark: In this Example it is crucial that k be algebraically closed. The polynomial $f(x, y) = x^2 + y^2$ satisfies $V(f) = \{0\} \in \mathbb{R}^2$.

Next, we turn to the field of rational functions $k(X)$.

Definition 3.4. The *ring of germs* of regular functions at $x \in X$ is:

$$\mathcal{O}_{X,x} := \left\{ \phi \in k(X) \mid \phi = \frac{f}{g} \text{ with } g(x) \neq 0 \right\} \subset k(X)$$

In other words, a rational function $\phi \in k(X)$ belongs to $\mathcal{O}_{X,x}$ if there is an expression for ϕ so that $\phi(x)$ well-defined. In commutative algebra, this ring is obtained from the coordinate ring $k[X]$ by the process of *localization.* If $P \subset k[X]$ is a prime ideal, then:

$$k[X]_P := \left\{ \frac{f}{g} \mid g \in k[X] - P \right\} \subset k(X)$$

is the localization of $k[X]$ at P. These rings are **local**, i.e.

$$m_P = P \cdot k[X]_P = \left\{ \frac{f}{g} \mid g \in k[X] - P, f \in P \right\} \subset k[X]_P$$

is the unique maximal ideal in $k[x]_P$.

Note that if m_x is the maximal ideal associated to x, then:

$$k[X]_{m_x} = \mathcal{O}_{X,x}$$

whereas the local rings $k[X]_P$ for sub-maximal prime ideals are comprised of rational functions that are defined at *some point* of the variety $Y = V(P) \subset X$ associated to the prime ideal P. One of the properties of Grothendieck's schemes is the existence of a "generic" point attached to each prime ideal, so that each such local ring is a ring of germs of functions at a point.

Focusing on the functions instead of the points, we have:

Definition 3.5. The *regular locus* of $\phi \in k(X)$ is the open set:

$$\mathrm{dom}(\phi) := \{x \in X \mid \phi \in \mathcal{O}_{X,x}\}$$

i.e. the set of points of X for which $\phi(x)$ is defined.

Remark. When the ring $k[X]$ is a UFD, it is easy to find $\mathrm{dom}(\phi)$. Putting $\phi = f/g \in k(X)$ in lowest terms, then $\mathrm{dom}(\phi) = U_g$.

However, $k[X]$ is frequently not a UFD. For example, in the ring:

$$k[X] = k[x_0, x_1, x_2, x_3]/\langle x_0x_3 - x_1x_2\rangle \text{ for } X \subset k^4$$

the rational function:

$$\phi = \frac{x_0}{x_1} = \frac{x_2}{x_3}$$

satisfies $\mathrm{dom}(\phi) = U_{x_1} \cup U_{x_3}$ which is not a basic open subset of X.

Here is another important consequence of the Nullstellensatz.

Proposition 3.4. Let $U_h \subset X$ be a basic open subset. Then:

$$\{\phi \mid U_h \subset \mathrm{dom}(\phi)\} = \left\{\frac{f}{h^n} \mid f \in k[X], n \geq 0\right\} = k[X, h^{-1}]$$

which is, in particular, a finitely generated k-algebra domain.

Proof. If $\phi = fh^{-n}$ and $x \in U_h$, then evidently $x \in \mathrm{dom}(\phi)$. In the other direction, we need to be aware of the different expressions for ϕ. Indeed, given $\phi \in k(X)$, consider the *ideal of denominators* of ϕ:

$$I_\phi = \left\{g \mid \phi = \frac{f}{g}\right\} \cup \{0\} = \{g \in k[X] \mid g\phi \in k[X]\}$$

If $g\phi = f \in k[X]$ and $a \in k[X]$, then $ag\phi = af \in k[X]$, and if $g_1\phi = f_1$ and $g_2\phi = f_2$, then $(g_1 + g_2)\phi = f_1 + f_2$. Thus I_ϕ is an ideal, which in particular shows that $\mathrm{dom}(\phi) = X - V(I_\phi)$. But if $V(I_\phi) \subset V(h)$, then $h^n \in I_\phi$ for some n by the Nullstellensatz, so $h^n\phi \in k[X]$ and $\phi = f/h^n$ for some $f \in k[X]$. □

In particular, the coordinate ring itself can be recovered:

$$\{\phi \in k(X) \mid \mathrm{dom}(\phi) = X\} = k[X]$$

We are ready to define the **presheaf of regular functions** on X. This is a contravariant functor from the category of (Zariski) open subsets of X to the category of k-algebra domains given by:

Definition 3.6. The presheaf $\mathcal{O}_X$ **of regular functions** is given by:

$$\mathcal{O}_X(U) = \{\phi \in k(X) \mid U \subseteq \mathrm{dom}(\phi)\} \subset k(X)$$

i.e. $\mathcal{O}_X(U)$ is the ring of rational functions that are defined on U and

$$\rho_{U_2,U_1} : \mathcal{O}_X(U_2) \to \mathcal{O}_X(U_1)$$

is the natural **restriction map** of functions whenever $U_1 \subset U_2$.

Note that each $\phi \in \mathcal{O}_X(U)$ is a continuous function $\phi : U \to k$ for the Zariski topologies on U and k. We call ϕ a **section** of $\mathcal{O}_X$ over U. The assignment of the ring of regular functions $\mathcal{O}_X(U)$ to each open set $U \subset X$ clearly defines a functor, with $\rho_{U_3,U_1} = \rho_{U_2,U_1} \circ \rho_{U_3,U_2}$ whenever $U_1 \subset U_2 \subset U_3$, but it is also a **sheaf** of k-valued continuous functions on X, by virtue of satisfying two additional properties:

(i) (The only section that is locally zero is the zero section). I.e.

For all open sets $U \subset X$ and open covers $U = \bigcup_{\lambda \in \Lambda} U_\lambda$, if $\phi \in \mathcal{O}_X(U)$ and if $\rho_{U,U_\lambda}(\phi) = 0$ for every $\lambda \in \Lambda$, then $\phi = 0$.

(ii) (Locally defined sections patch to define a global section). I.e.

If $U = \bigcup_{\lambda \in \Lambda} U_\lambda$ and sections $\phi_\lambda \in \mathcal{O}_X(U_\lambda)$ are given, satisfying

$$\rho_{U_\lambda,U_\lambda \cap U_\mu}(\phi_\lambda) = \rho_{U_\mu,U_\lambda \cap U_\mu}(\phi_\mu) \text{ for all pairs } \lambda, \mu \in \Lambda$$

then there is a (unique!) $\phi \in \mathcal{O}_X(U)$ with $\rho_{U,U_\lambda}(\phi) = \phi_\lambda$ for all $\lambda \in \Lambda$.

In fact, stronger versions of (i) and (ii) hold because of the nature of the Zariski topology and the irreducibility of X. For example, in (i), if $U_\lambda \subset U$ is a **single** nonempty open subset such that $\rho_{U,U_\lambda}(\phi) = 0$, then $\phi = 0 \in k(X)$, hence $\phi = 0$ as a function on U. This is because $U_\lambda \subset X$ is dense and the locus of points satisfying $\phi(x) \neq 0$ is open. Similarly, if a single $\phi \in \mathcal{O}_X(U_\lambda)$ is given, with the property that $U \subset \mathrm{dom}(\phi)$ (which is certainly implied by (ii)), then $\phi \in \mathcal{O}_X(U)$.

Remark. (a) We will later define a presheaf of coherent $\mathcal{O}_X$-modules, for which the sheaf properties will not be so trivial precisely because of the absence of an analogue of the field $k(X)$ containing all the sections.

(b) In general, it is not easy to determine the ring $\mathcal{O}_X(U)$, and also somewhat irrelevant for most purposes. The important exception to this rule are the basic open sets U_h. In that case, Proposition 3.4 gives:

$$\mathcal{O}_X(U_h) = k[X, h^{-1}]$$

Sheaves of differentiable or analytic functions on a manifold allow one to define differentiable or analytic maps of manifolds, and hence the **categories** of differentiable or analytic manifolds. Similarly, our definition of the sheaf of regular functions will enable us to define the categories of quasi-affine and quasi-projective varieties.

Fix an algebraically closed field k with the Zariski topology.

Definition 3.7. $\mathcal{N}_k$ is the category of all pairs $(X, \mathcal{C}_X)$ consisting of:

(i) An irreducible Noetherian topological space X and

(ii) A sheaf $\mathcal{C}_X$ of k-algebras of continuous functions $f : U \to k$ that always include the constant functions and for which the maps $\rho_{U_2,U_1} : \mathcal{C}_X(U_2) \to \mathcal{C}_X(U_1)$ are the restriction of functions.

A morphism of pairs $\Phi : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ in $\mathcal{N}_k$ consists of

(i) A continuous map $\Phi : X \to Y$ that also

(ii) Pulls back sections of the sheaf $\mathcal{C}_Y$ to sections of the sheaf $\mathcal{C}_X$:

$$\Phi^* : \mathcal{C}_Y(U) \to \mathcal{C}_X(\Phi^{-1}(U)) \text{ under } \Phi^*(\phi)(x) := \phi(\Phi(x))$$

for all open subsets $U \subset Y$.

This is larger than the category that we will eventually settle upon as the category of varieties, but it serves as an ambient category in which to locate affine and projective varieties, just as the category of topological manifolds is an ambient category for the categories of differentiable and analytic manifolds. To get used to the category $\mathcal{N}_k$ consider two objects of $\mathcal{N}_k$ that are **not** varieties. Let X be a fixed irreducible Noetherian topological space (other than a point).

(a) The sheaf $\mathbf{k}$ of **constant** fuctions on X is defined by:

$$\mathbf{k}(U) = k \text{ for all nonempty open sets } U \subset X$$

This is evidently a presheaf that also satisfies property (i) of a sheaf. It satisfies property (ii) as well because all open subsets in X are dense! (In other topologies, presheaves of constant functions are **not** sheaves!) This is the bare minimum sheaf of k-algebras of continuous functions, and indeed the identity map defines a morphism from any other pair:

$$\text{id} : (X, \mathcal{C}_X) \to (X, \mathbf{k})$$

because constant functions pull back to constant functions, which are in $\mathcal{C}_k(U)$ for any of the sheaves in $\mathcal{N}_k$. But the identity is **not** a morphism in the opposite direction if $\mathcal{C}_k \neq \mathbf{k}$.

(b) The sheaf $\mathcal{F}$ of **all** continuous functions $f : U \to k$ lies at the other extreme. Because it contains all continuous functions,

$$\mathrm{id} : (X, \mathcal{F}) \to (X, \mathcal{C}_X)$$

is a morphism, regardless of the sheaf $\mathcal{C}_X$, while it is only a morphism in the reverse direction if $\mathcal{C}_X = \mathcal{F}$.

Affine Varieties. Each variety $X \subset k^n$ with the Zariski topology and sheaf $\mathcal{O}_X$ of regular functions is an object of $\mathcal{N}_k$. An isomorphic object of $\mathcal{N}_k$ may be obtained **directly** from $k[X]$ itself, by setting:

$$X = \{\text{maximal ideals } m_x \in k[X]\} \text{ with closed sets}$$

$$Z = Z(I) = \{x \in X \mid m_x \supset I \text{ for some } I \subset k[X]\},$$

$$\mathcal{O}_{X,x} := k[X]_{m_x} \subset k(X) \text{ and } \quad \mathcal{O}_X(U) := \bigcap_{x \in U} \mathcal{O}_{X,x}$$

for open sets $U \subset X$. Each $\phi \in \mathcal{O}_X(U)$ is a function from U to k via:

$$\phi(m_x) = \overline{\phi} \in k[X]_{m_x}/m_x k[X]_{m_x} = k \text{ for each } m_x \in U$$

by passing to the quotient of the local ring $\mathcal{O}_{X,x}$ by its maximal ideal.

Definition 3.8. An object $(X, \mathcal{C}_X)$ of $\mathcal{N}_k$ is an **affine variety** if it is isomorphic to $(X, \mathcal{O}_X)$ for some variety $X \subset k^n$. The full subcategory of affine varieties (in other words, all morphisms in $\mathcal{N}_k$ permitted) will be designated as $\mathcal{A}_k \subset \mathcal{N}_k$.

Quasi-Affine Varieties. Let $(X, \mathcal{C}_X)$ be an object of $\mathcal{N}_k$, and $U \subset X$ be an open subset with the induced topology. Then we may **restrict** the sheaf $\mathcal{C}_X$ to a sheaf $\mathcal{C}_X|_U$ on U by setting $\mathcal{C}_X|_U(V) := \mathcal{C}_X(V)$ for all open sets $V \subset U$. Then $(U, \mathcal{C}_X|_U)$ is another object of category $\mathcal{N}_k$ and by construction the inclusion map $i : U \hookrightarrow X$ defines a morphism.

Definition 3.9. An object $(Y, \mathcal{C}_Y)$ of $\mathcal{N}_k$ is a **quasi-affine variety** if it is isomorphic to $(U, \mathcal{O}_X|_U)$ for some open subset $U \subset X$ of an affine variety with the induced topology and restricted sheaf. This defines a full subcategory $\mathcal{A}_k \subset \mathcal{Q}A_k \subset \mathcal{N}_k$ of quasi-affine varieties.

A Simple Example. Consider the quasi-affine variety $k^* \subset k$ with $\mathcal{O}_k(k^*) = k[x, x^{-1}]$. Then the map $\Phi : k^* \to Y \subset k^2$; $\Phi(x) = (x, \frac{1}{x})$ to the *hyperbola* $Y = V(xy - 1)$ is an **isomorphism** in $\mathcal{Q}A_k$, with inverse morphism $\Phi^{-1}(x, y) = x$. Thus $k^* \cong Y$ is an affine variety!

Theorem 3.1. The *global section* functor from the category $\mathcal{A}_k$ of affine varieties to the category of finitely generated k-algebra domains:

$$\Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X) \text{ with}$$
$$\Gamma\left((X, \mathcal{O}_X) \stackrel{\Phi}{\to} (Y, \mathcal{O}_Y)\right) = \left(\Gamma(Y, \mathcal{O}_Y) \stackrel{\Phi^*}{\to} \Gamma(X, \mathcal{O}_X)\right)$$

is a contravariant **equivalence of categories**.

In other words, there is *no qualitative difference* between the category of affine varieties and the category of finitely generated k-algebra domains. We will prove this by finishing the construction of the inverse functor that we unknowingly started to define earlier.

Proof. Consider the "maximum spectrum" object of $\mathcal{N}_k$ associated to a finitely generated k-algebra domain A taking:

$$X := \text{mspec}(A) = \{\text{maximal ideals } m_x \in A\}$$

with Zariski topology and sheaf $\mathcal{O}_X$ as defined for $k[X]$ above. If we start with an affine variety $(X, \mathcal{O}_X)$ for some $X \subset k^n$, then

$$\text{mspec}(\Gamma(X, \mathcal{O}_X)) = \text{mspec}(k[X]) = (X, \mathcal{O}_X)$$

by Proposition 3.4. Conversely, if $A = k[X]$ for some $X \subset k^n$, then:

$$\Gamma(\text{mspec}(A)) = \Gamma(X, \mathcal{O}_X) = A$$

It remains to define mspec as a **functor**, i.e. to specify the morphism: $\Phi = \text{mspec}(f : A \to B)$ associated to a k-algebra homomorphsm from $\text{mspec}(B) = (Y, \mathcal{O}_Y)$ to $\text{mspec}(A) = (X, \mathcal{O}_X)$. As a map of sets,

$$\Phi(m_y) = f^{-1}(m_y) \text{ for maximal ideals } m_y \subset B$$

and this is continuous since $\Phi(Z(I)) = Z(f^{-1}(I))$ for all ideals $I \subset B$. Under this map, if $h \in A$, then

$$\Phi^{-1}(U_h) = \{m_y \in B \mid f(h) \notin m_y\} = U_{f(h)}$$

is non-empty if and only if $h \notin \ker(f)$ and in that case the pull-back map of regular functions is the natural map of rings (Proposition 3.4):

$$f : A[h^{-1}] = \mathcal{O}_X(U_h) \to B[f(h)^{-1}] = \mathcal{O}_Y(\Phi^{-1}(U_h))$$

extending $f : A \to B$, from which we may conclude that Φ pulls back regular functions to regular functions on all open subsets of Y. □

Corollary 3.2. Every basic open set $U_h \subset X$ of an affine variety is itself an affine variety.

Proof. By Proposition 3.4, we have $\Gamma(U_h, \mathcal{O}_X|_{U_h}) = k[X][h^{-1}]$, and the inclusion of k-algebras $k[X] \subset k[X][h^{-1}]$ corresponds to $U_h \subset U$ (with sheaves of regular functions) in the category $\mathcal{A}_k$.

This generalizes the simple example and has the following important:

Corollary 3.3. Every quasi-affine variety is covered by affine varieties.

Proof. The inclusions $U_h \subset U$ of a basic open set in an arbitrary open set of an affine variety X are morphisms in the category $\mathcal{Q}A_k$ for the induced topologies and sheaves from X. But U_h, as an object of $\mathcal{Q}A_k$, is an affine variety by Corollary 3.2.

Important Example. The complement of the origin $k^n - \{0\} \subset k^n$ is not an affine variety when $n \geq 2$. This follows from the observation:

$$\mathcal{O}_{k^n}(k^n - \{0\}) = k[x_1, ..., x_n], \text{ which is easily checked}$$

(Contrast this with the simple example when $n = 1$).

Next, we turn to **graded rings** for an analogous construction of an object of $\mathcal{N}_k$ intrinsically attached to a graded ring.

Let $R_\bullet$ be a graded k-algebra, finitely generated in degree one, i.e.

$$R_0 = k \text{ and } R_\bullet \text{ is generated by } R_1$$

(as is the case for the homogeneous coordinate rings $k[X]_\bullet$). Then:

$$R_+ := \bigoplus_{d>0} R_d$$

is the unique maximal homogeneous ideal in R ideals, and we define:

$$X = \text{mproj}(R_\bullet) := \{\text{maximal homogeneous prime ideals } m_x \subset R_+\}$$

i.e. m_x is homogeneous, prime, and maximal among all such ideals properly contained in R_+. This has a Zariski topology:

$$Z(I) = \{\text{maximal ideals } m_x \text{ that contain a homogeneous } I\}. \text{ Let}$$

$$k(X) = \left\{ \frac{F}{G} \mid F, G \in R_d \text{ and } G \neq 0 \right\} \subset k(R)$$

and define the sheaf of regular functions on $X = \text{mproj}(R_\bullet)$ via:

$$\mathcal{O}_{X,x} = \left\{ \frac{F}{G} \mid F \in R_d, G \in R_d - m_x \right\} \subset k(X) \text{ and}$$

$$\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x} \subset k(X)$$

for all Zariski open sets $U \subset X = \text{mproj}(R_\bullet)$.

Remark. Unlike the case with a k-algebra A without grading, there is nearly a preferred choice of generators for the graded ring $R_\bullet$. A choice of basis $x_0, ..., x_n \in R_1$ for the vector space R_1 determines a surjection: $S \to R$ with homogeneous kernel $P \subset S$ and two such are related by a change of basis of R_1.

By identifying $R_\bullet = S/P$, and appealing to the correspondence there between points and ideals, we see that the maximal prime homogeneous ideals $m_x \in \text{mproj}(R_\bullet)$ correspond to hyperplanes $H \subset R_1$ and that conversely, given a hyperplane $H \subset R_1$, then **either**

(i) H generates a maximal ideal $m_x \in \text{mproj}(R_\bullet)$, with $R_\bullet/m_x \cong k[x]$

or else (by the projective Nullstellensatz)

(ii) For some $N > 0$, the ideal $\langle H \rangle$ contains $x_0^N, ..., x_n^N$, which implies that $R_d \subset \langle H \rangle$ for all $d > (n+1)(N-1)$.

In particular, this tells us the sense in which an element $\phi \in \mathcal{O}_X(U)$ is a (continuous) function from $\phi : U \to k$. Namely, if $\phi = F/G$, then:

$$F, G \text{ are proportional in } (R_\bullet/m_x)_d = k[x]_d = k$$

and $\phi(x)$ is this proportion. Continuity follows from the observation that each level set $\phi^{-1}(c) \subset U$ of $\phi : U \to k$ is a union of components of the closed subset $V(F - cG) \cap U \subset U \subset X$.

Once again, we have the (homogeneous) ideal of denominators:

$$I_\phi := \{G \in R_\bullet \mid \; G\phi \in R_\bullet\}$$

generated by the homogeneous denominators that we use to conclude:

$$\text{dom}(\phi) = X - V(I_\phi) \text{ is an open subset of } X$$

But now the analogue of Proposition 3.4 contains a surprise:

Proposition 3.5. If $G \in R_d - \{0\}$, let $U_G = X - V(G)$. Then:

$$\mathcal{O}_X(U_G) = (R_\bullet[G^{-1}])_0 \subset k(X)$$

In particular, taking $G = 1$, we have:

$$\mathcal{O}_X(X) = k$$

i.e. the only regular functions on $X = \text{mproj}(R_\bullet)$ are the constants.

Proof. When $G \neq 1$, the proof is essentially same as Proposition 3.4, using the Projective Nullstellensatz in place of the Nullstellensatz.

When $G = 1$, there is wrinkle due to the wrinkle in the Projective Nullstellensatz. If $\phi \in \mathcal{O}_X(X)$, we may only initially conclude that:

$$\phi = \frac{F_0}{x_0^N} = \frac{F_1}{x_1^N} = \cdots = \frac{F_n}{x_n^N}$$

for some $F_0, ..., F_n \in R_N$, from which it follows that $R_d \cdot \phi \in R_d$ for all $d > (n+1)(N-1)$. Now choose any $G \in R_d$ for some such d. Then:

$$G\phi^n \in R_d \text{ for } \textbf{all } n \geq 0$$

from which it follows that the sequence of modules:

$$R_\bullet \subset R_\bullet + \phi R_\bullet \subset R_\bullet + \phi R_\bullet + \phi^2 R_\bullet \subset \cdots$$

are all contained in the single principal graded module $G^{-1}R_\bullet$ and therefore since $R_\bullet$ is Noetherian, the union of these modules is finitely generated, and in particular for some m, there is an identity:

$$\phi^m = f_{m-1}\phi^{m-1} + \cdots + f_0 \in G^{-1}R_\bullet$$

for (not necessarily graded) elements $f_i \in R_\bullet$.

But $\phi^m \in (G^{-1}R_\bullet)_0$, so restricting to the zero-graded part of $G^{-1}R_\bullet$:

$$\phi^m = c_{m-1}\phi^{m-1} + \cdots + c_0 \text{ for } \textbf{constants } c_i \in k$$

and it follows immediately that ϕ itself is a constant. □

The other open sets $U_G \neq X$ have lots of regular functions. In fact:

Proposition 3.6. Each open subset $U_G \subset X = \text{mproj}(R_\bullet)$ for $G \notin R_0$ is an **affine variety** (with the induced sheaf of regular functions).

Proof. You will do this in the Exercises.

Definition 3.10. (a) The category $\mathcal{P}_k$ of **projective varieties** is the full subcategory of pairs $(X, \mathcal{O}_X) \in \mathcal{N}_k$ that are isomorphic to mproj$(R_\bullet)$ for some graded k-algebra domain $R_\bullet$ that is generated by R_1 (a finite dimensional vector space over k) with $R_0 = k$.

(b) The category $\mathcal{QP}_k$ of **quasi-projective varieties** is the full subcategory of pairs $(U, \mathcal{O}_U)$ that are isomorphic to an open subset $U \subset X$ of a projective variety, with the induced sheaf.

Important Remark. The assignment $R_\bullet \mapsto \text{mspec}(R_\bullet)$ is pseudo-functorial, in the sense that given $f_\bullet : R_\bullet \to Q_\bullet$, we "define:"

$$\Phi(m_y) = f^{-1}(m_x)$$

and note that, **if this is well-defined** (which, for example, is the case when $f_\bullet$ is surjective), then Φ is continuous and pulls back regular functions to regular functions as in the case of mspec. **However** $f^{-1}(m_y)$ is not, in general, a maximal ideal in $R^\bullet$!

Even in the simplest conceivable example, this fails. Namely, let:

$$f : k[x] \to R_\bullet;\ x \mapsto x_0 \in R_1$$

Then mproj$(k[x]) = p$ is a point, and the associated map $\Phi : X \to p$ should be the constant map. Indeed, for maximal ideals $m_x = \langle H \rangle$ for which $x_0 \notin H$, we have $f^{-1}(m_x) = 0$ is the maximal ideal, but the closed subset $Z(x_0) \subset X$ consists of maximal ideals whose inverse image is the **irrelevant** ideal $\langle x \rangle \subset k[x]$, and Φ fails to be well-defined.

We finish with a reality check, that will continue into the exercises.

Reality Check 1. What is a morphism $\Phi : X \to k^n$?

If $(X, \mathcal{O}_X)$ is an arbitrary element of $\mathcal{N}_k$, then the pull-back map on global sections is a homomorphism of k-algebras:

$$\Phi^* : k[y_1, ..., y_n] \to \Gamma(X, \mathcal{O}_X)$$

and the images $f_i = \Phi^*(y_i)$ of the coordinate functions are continuous functions on X that recover $\Phi(x) = (f_1, ..., f_n)$. Thus, for example, if X is a **projective** variety, then the fact that $\Gamma(X, \mathcal{O}_X) = k$ means that the only maps from X to k^n are the constant maps. In particular, the point is the only variety that is simultaneously affine and projective.

If X is an **affine variety**, then the converse is also true; any collection of n regular functions on X defines a map $\Phi = (f_1, ..., f_n)$ from X to k^n. This is because of the equivalence of categories. The map on global sections $k[y_1, ..., y_n] \to k[X], y_i \mapsto f_i$ maps via the functor mspec to the desired morphism Φ.

A morphism $\Phi : X \to Y$ of affine varieties $X \subset k^m$ and $Y \subset k^n$ is determined by its global section map $\Phi^* : k[Y] \to k[X]$, which can be lifted (in many ways) to a map of polynomial rings $F : k[y_1, ..., y_n] \to k[x_1, ..., x_n]$. Applying the equivalence of categories, this means that Φ can be realized as the restriction of a polynomial map $\widetilde{\Phi} : k^m \to k^n$.

Reality Check 2. What is a morphism $\Phi : X \to \mathbb{P}^n_k$?

Assume that the open subset $U_0 = \mathbb{P}^n_k - V(x_0)$ intersects the image of Φ, changing coordinates on k^{n+1} if necessary. Then Φ restricts to:

$$\Phi|_W : W = \Phi^{-1}(U_0) \to k^n = U_0$$

which is therefore given by coordinate functions:

$$\Phi_W(x) = (\phi_1(x), \cdots, \phi_n(x)) \text{ for } \phi_i = \Phi|_W^*(y_i) \in \mathcal{O}_X(U) \subset k(Y)$$

and dehomogenizing, we get:

$$\Phi|_W(x) = (1 : \phi_1(x) : \cdots : \phi_n(x)) \in \mathbb{P}^n_k$$

Conversely, a morphism from an open subset of X to $U_0 = k^n$ may (or may not) extend to a morphism from X to $\mathbb{P}^n_k$. There is some issue about uniqueness, which we will address in the next section.

This inspires the following:

Definition 3.11. A *rational map* $\Phi : X - - > \mathbb{P}^n_k$ is defined by:

$$\Phi = (\phi_0 : \cdots : \phi_n) \text{ for } \phi_0, ..., \phi_n \in k(X) \text{ not all zero}$$

and Φ is a regular map if Φ extends to a morphism on X.

Two Examples. (a) Consider the "projection" rational map:

$$\Phi : \mathbb{P}^2_k - - > \mathbb{P}^1_k; \quad \Phi(x_0 : x_1 : x_2) = (1 : \frac{x_1}{x_0})$$

which is exhibited in this way as a rational map defined on the open set $U_0 = \mathbb{P}^2_k - V(x_0)$. But wherever they are both defined, we have:

$$(1 : \frac{x_1}{x_0}) = (\frac{x_0}{x_1} : 1)$$

and the latter is defined on the set $U_1 = \mathbb{P}^2_k - V(x_1)$, so we may use the latter expression to extend the definition of Φ to $U_0 \cup U_1$. This leaves only the point $(0 : 0 : 1)$, and you should check that Φ cannot be extended across this remaining point. Using projective coordinates, we may realize Φ as the linear projection:

$$(x_0 : x_1 : x_2) \mapsto (x_0 : x_1)$$

though for now this is only a convenient notation.

(b) Now define Φ in the same way as a rational map:

$$\Phi : X = V(x_0x_2 - x_1^2) \to \mathbb{P}^1_k$$

This is a smooth conic that contains the point $(0 : 0 : 1)$ across which Φ could not be extended in (a). In this case, however,

$$\frac{x_0}{x_1} = \frac{x_1}{x_2} \in k(X)$$

and so there is a further extension:

$$(1 : \frac{x_1}{x_0}) = (\frac{x_0}{x_1} : 1) = (\frac{x_1}{x_2} : 1)$$

that enables the map Φ to be defined on all of X. In fact, this projection is an isomorphism, with inverse:

$$v_2(y_0 : y_1) = (y_0^2 : y_0y_1 : y_1^2)$$

This discussion motivates the following:

Definition 3.12. Let X, Y be quasi-projective varieties. Then every morphism $\Phi : U \to Y$ from a non-empty open subset $U \subset X$ to Y is said to define a **rational map** $\Phi : X - - > Y$.

Exercises 3.

1. Prove that in a Noetherian topology, every closed set is a union of finitely many (distinct) irreducible closed sets.

2. Krull's Principal Ideal Theorem states that if

$$Y_1 \cup \cdots \cup Y_m = V(f) \subset X$$

are the irreducible components of a (nonempty) hypersurface $V(f) \subset X$ of an affine variety over an algebraically closed field, then for all i,

$$\text{tr deg}(k(Y_i)/k) = \text{tr deg}(k(X)/k) - 1.$$

Use this to show that the two definitions of dimension (transcendence degree of $k(X)$ and Krull dimension of the space X) agree. Moreover, use this to show that every chain of closed irreducible subsets of X can be "lengthened" to a chain of length $\dim(X)$, so in particular every maximal chain of irreducible subsets realizes the dimension of X.

3. Prove that the open set $U \subset X = V(x_0x_3 - x_1x_2) \subset k^4$ defined by:

$$U = \text{dom}(\phi) \ \text{ for } \phi = \frac{x_0}{x_1} = \frac{x_2}{x_3}$$

is not a basic open subset of X.

4. Let the first $n+1$ coordinates of the d-uple embedding be:

$$\nu_{n,d}(x_0 : \cdots : x_n) = (x_0^d : x_0^{d-1}x_1 : \cdots : x_0^{d-1}x_n : \cdots)$$

Show that the projection map onto the first $n+1$ coordinates:

$$\Phi : \mathbb{P}_k^{\binom{n+d}{d}-1} - - > \mathbb{P}_k^n$$

inverts the map $\nu_{n,d}$, when restricted to the image of the d-uple embedding. Conclude that the two projective varieties:

$$\text{mproj}(S_\bullet) \text{ and } \text{mproj}(S_{d\bullet})$$

are isomorphic. Generalize the result to any graded ring $R_\bullet$.

5. Prove that a basic open subset $U_G \subset X$ of a projective variety $X = \text{mproj}(R_\bullet)$ is an affine variety. In fact, you need only to do this for $G = x_0 \in R_1$ and then appeal to the isomorphism of Problem 4.

6. Show that the projection rational maps:

$$\pi : \mathbb{P}_k^n - - > \mathbb{P}_k^m; \ \pi(x_0 : \cdots : x_n) = (x_0 : \cdots : x_m)$$

extend to a morphism defined on the open set $U = U_{x_0} \cup \cdots \cup U_{x_m} \subset \mathbb{P}_k^n$ but that they cannot be extended any further.

7. Let $x \in \mathbb{P}_k^n$. Show that the quasi-projective variety $Y = \mathbb{P}_k^n - \{x\}$ is neither an affine nor a projective variety if $n \geq 2$.