Algebraic Geometry I (Math 6130)

Utah/Fall 2020

3. Abstract Varieties.

The categories of affine and quasi-affine varieties over k are (by definition) full subcategories of the category of sheaved spaces $(X, \mathcal{O}_X)$ over k. We now enlarge the category just enough within the category of sheaved spaces to define a category of abstract varieties analogous to the categories of differentiable or analytic manifolds. Varieties are Noetherian and locally affine and separated (analogous to Hausdorff), the latter defined via products, which are also used to define the notion of a proper (analogous to compact or complete) variety.

Definition 3.1. A topology on a set X is **Noetherian** if every descending chain:

$$X \supseteq X_1 \supseteq X_2 \supseteq \cdots$$

of closed sets eventually stabilizes (or every ascending chain of open sets stabilizes).

Remarks. (a) A Noetherian topological space X is *quasi-compact*, i.e. every open cover of X has a finite subcover, but a quasi-compact space need not be Noetherian.

(b) The Zariski topology on a quasi-affine variety is Noetherian.

(c) It is possible for a topological space X to fail to be Noetherian even if it has a cover by open sets, each of which is Noetherian with the induced topology. If the open cover is finite, however, then X is Noetherian.

Definition 3.2. A Noetherian topological space X is **reducible** if $X = X_1 \cup X_2$ for closed sets X_1, X_2 properly contained in X. Otherwise it is **irreducible**.

Remarks. (a) As we saw in §1, every Noetherian topological space X is a finite union of irreducible closed subsets, accounting for the *irreducible components* of X.

(b) The underlying topological space of a quasi-affine variety is irreducible.

Definition 3.3. A sheaved space $(X, \mathcal{O}_X)$ over k is a **prevariety** if:

- X is an irreducible Noetherian topological space, and
- $(X, \mathcal{O}_X)$ is **locally affine**, i.e. there is a (finite) open cover $X = U_1 \cup \cdots \cup U_n$ such that each sheaved space $(U_i, \mathcal{O}_X|_{U_i})$ is an affine variety over k.

Example. There are two ways to add a point to $\mathbb{A}^1_k$ to enlarge it as a prevariety.

(The Projective Line) We may glue two copies of $\mathbb{A}^1_k$ along the common open set $\mathbb{A}^1_k - \{0\} \subset \mathbb{A}^1_k$ via the isomorphism $y = x^{-1}$ of associated k-algebras:

$$k[x, x^{-1}] \text{ and } k[y, y^{-1}]$$

This allows us to define the prevariety:

$$\mathbb{P}^1_k = \mathbb{A}^1_k \sqcup \mathbb{A}^1_k/(x^{-1} \sim y)$$

(The Doubled Origin) When one applies instead the identity $y = x$ above, to glue:

$$\mathbb{A}^1_k \sqcup \mathbb{A}^1_k/(x \sim y)$$

along the common open subset $\mathbb{A}^1 - \{0\}$, the result is the affine line with two origins.

To define a variety we need to explore the categorical definition of a product.

Definition 3.4. A categorical product of objects X, Y of a category $\mathcal{C}$ is a triple:

$$(X \times Y, \pi_X : X \times Y \to X, \pi_Y : X \times Y \to Y)$$

consisting of an object $X \times Y$ and "projection" morphisms to X and Y respectively. The triple is required to be *universal* in the sense that all other such triples

$$(Z, p : Z \to X, q : Z \to Y)$$

are obtained via a unique morphism $Z \stackrel{u}{\to} X \times Y$ with $p = \pi_X \circ u$ and $\pi_Y = q \circ u$.

Note. The product (when it exists) is unique up to a unique isomorphism.

Examples. (a) The Cartesian product is the product in the category of sets.

(b) The intersection is the product in the category of subsets of a fixed set U.

Proposition 3.6. Products exist in the category of affine varieties over k.

Proof. Every pair of commutative k-algebras with 1 has the **tensor product**:

$$A \otimes_k B \text{ with } \alpha : A \to A \otimes_k B \text{ and } \beta : B \to A \otimes_k B$$

which is a **co**product with the arrow-reversed universal property. To apply the functor maxspec and obtain a product of affine varieties, we need to show:

(a) If A and B are finitely generated k-algebras, then $A \otimes_k B$ is finitely generated.

(b) If A and B are integral domains, then $A \otimes_k B$ is an integral domain.

Given generators $A = k[x_1, ..., x_n]/\langle f_1, ..., f_m\rangle$ and $B = k[y_1, ..., y_q]/\langle g_1, ..., g_p\rangle$ then

$$A \otimes_k B = k[x_1, ..., x_n, y_1, ..., y_q]/\langle f_1, ..., f_m, g_1, ..., g_p\rangle$$

so $A \otimes_k B$ is finitely generated. Moreover, the algebraic set:

$$X \times Y = X(f_1, ..., f_m, g_1, ..., g_p) \subset k^{n+q}$$

is the Cartesian product of $X \subset k^n$ and $Y \subset k^q$. To prove (b), we will show:

(i) $X \times Y$ is an *irreducible* algebraic subset of k^{n+q}, so $k[X \times Y]$ is a domain.

(ii) $A \otimes_k B$ has no nilpotent elements (other than zero).

Suppose $X \times Y = Z_1 \cup Z_2$ is a union of closed subsets. Then for all $y \in Y$,

$$(Z_1 \cap (X \times \{y\}) \cup (Z_2 \cap (X \times \{y\}) = X \times \{y\} \subset k^{n+q}, \text{ which is irreducible}$$

so for each $y \in Y$, either $Z_1 \cap (X \times \{y\}) = X \times \{y\}$ or $Z_2 \cap (X \times \{y\}) = X \times \{y\}$. Now let:

$$Y_i = \{y \in Y \mid Z_i \cap (X \times \{y\})\} = X \times \{y\}$$

Each $\pi_Y : \{x\} \times Y \to Y$ is an isomorphism, so the $Y_i = \cap_{x \in X} \pi_Y (\{x\} \times Y \cap Z_i)$ are closed sets in Y, and $Y_1 \cup Y_2 = Y$, so $Y_i = Y$ for some i and then $Z_i = X \times Y$. So $X \times Y$ is irreducible.

It follows from the Nullstellensatz that $\mathrm{rad}(\langle f_1, ..., f_n, g_1, ..., g_m\rangle) = I(X \times Y)$ is a prime ideal, so the proof is finished if we show (ii). Suppose $\sum c_{ij} a_i \otimes b_j \in A \otimes_k B$ is nilpotent and that $\{a_i(x)\}$ and $\{b_j(y)\}$ are linearly independent in A and B. Then

$$0 = \sum c_{ij} a_i(x) b_j(y) \in k[X \times Y] \text{ (evaluating at the points of } X \times Y)$$

and so for each $y_0 \in Y$, we have $\sum c_{ij} a_i(x) b_j(y_0) = 0$. By the linear independence of the $a_i(x)$, we get $\sum c_{ij} b_j(y) = 0$ for all $y \in Y$ and then by the linear independence of the $b_j(y)$, we have $c_{ij} = 0$ for all i and j. $\square$

Note. (a) The condition $k = \overline{k}$ is necessary for (b) above. For example,

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{R}[x,y]\langle x^2+1, y^2+1\rangle = \mathbb{C}[y]/\langle y^2+1\rangle$$

is not a domain, since $y^2 + 1 = (y-i)(y+i)$ factors in $\mathbb{C}[y]$.

(b) The proof of Proposition 3.6 is a curious blend of geometry and algebra.

Proposition 3.7. The product in Proposition 3.6 is also the product of the affine varieties in the category of prevarieties.

Proof. The universal property requires more argument in this larger category. Namely, if $(Z, p : Z \to \text{maxspec}(A), q : Z \to \text{maxspec}(B))$ is a triple in the category of prevarieties, we need to find the (unique) lift

$$u : Z \to \text{maxspec}(A \otimes_k B) \text{ to the product affine variety}$$

Let $U_i = \text{maxspec}(C_i)$ be an affine open cover of Z. Then we lift:

$U_i \subset Z \to \text{maxspec}(A)$ and $U_i \subset Z \to \text{maxspec}(B)$ to $u_i : U_i \to \text{maxspec}(A \otimes_k B)$

using Proposition 3.6. It remains to show that these morphisms *patch*, i.e. that:

$$u_i|_{U_i \cap U_j} = u_j|_{U_i \cap U_j}$$

But if $z \in U_i \cap U_j$, then the two k-algebra homomorphisms to the stalk $\mathcal{O}_{Z,z}$:

$$A \otimes_k B \to C_i \to \mathcal{O}_{Z,z} = (C_i)_{m_z} \text{ and } A \otimes_k B \to C_j \to \mathcal{O}_{Z,z} = (C_j)_{m_z}$$

coincide, by the universal property of the tensor product, so $u_i(z) = u_j(z)$. $\square$

This is the foundation for the following:

Proposition 3.8. Products of prevarieties exist in the category of prevarieties.

Proof. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be prevarieties. Then the Cartesian product $X \times Y$ is the underlying set of the product (though not with the product topology!). If $U_1, ..., U_n \subset X$ and $V_1, ..., V_m \subset Y$ are open covers by affine varieties, then:

$$U_i \times V_j \text{ cover } X \times Y \text{ as a set}$$

The realizations of $U_j \times V_j$ as affine varieties (topology and sheaf of functions) uniquely patch together to define on $X \times Y$ a topology and sheaf of functions for which the inclusions $U_i \times V_j \subset X \times Y$ are open immersions of sheaved spaces. This is done with a lot of appeals to the uniqueness of lifts in the universal property. $\square$?!

Important Remark. Products in the category of topological spaces are Cartesian products of sets with the product topology. There is no contradiction in the fact that in the category of prevarieties, the topology on the product fails to be the product topology, as is seen in the very first example:

$$\mathbb{A}^1_k \times \mathbb{A}^1_k = \mathbb{A}^2_k$$

in which the **diagonal** $X(x-y)$ in $\mathbb{A}^2_k$ is a closed subset of the affine plane, but is not closed in the product of the cofinite topologies on the set $\mathbb{A}^1_k \times \mathbb{A}^1_k$.

Let $\mathcal{C}$ be a category of topological spaces (with additional structure) in which products exist and the underlying set of the product is the Cartesian product.

Definition 3.9. For each object X, the diagonal $\delta : X \to X \times X$ is the lift of:

$$p = 1_X : X \to X \text{ and } q = 1_X : X \to X$$

and X is **separated** if the image of δ is a closed subset of $X \times X$.

Example. In the category of topological spaces, X is separated if and only if X is Hausdorff, i.e. pairs of points $x, y \in X$ are contained in disjoint neighborhoods.

Proof. In the category of topological spaces products have the product topology, meaning that the products $U \times V$ of open sets $U \subset X$ and $V \subset Y$ are a basis for the topology on $X \times Y$. Then $\delta(X) \subset X \times X$ is closed if and only if each $(x, y) \notin \delta(X)$ is contained in a product open set $U_x \times U_y$ not intersecting the diagonal, if and only if $x \in U_x$ and $y \in U_y$ are contained in disjoint neighborhoods for all $x \neq y$. $\square$

Proposition 3.10. X is separated in a category with products if and only if

$$\{y \in Y \mid p(y) = q(y)\} \subset Y \text{ is closed for every } Y \text{ and pair } p, q : Y \to X$$

Proof. Each pair of morphisms $p, q : Y \to X$ lifts to:

$$(p, q) : Y \to X \times X \text{ and } \{y \in Y \mid p(y) = q(y)\} = (p, q)^{-1}\delta(X)$$

so if $\delta(X)$ is closed, then so too are all the sets $\{y \in Y \mid p(y) = q(y)\}$.

Conversely, $\delta(X)$ itself is $\{(x_1, x_2) \in X \times X \mid \pi_1(x_1, x_2) = \pi_2(x_1, x_2)\}$. $\square$

Example. The doubled origin in $\mathbb{A}^1_k$ is not separated. The two open inclusions:

$$i, j : \mathbb{A}^1_k \subset \mathbb{A}^1_k \sqcup_{x=y} \mathbb{A}^1_k$$

agree on the subset $U_y \subset \mathbb{A}^1_k$, which is not closed. In contrast, the open inclusions

$$i, j : \mathbb{A}^1_k \subset \mathbb{A}^1_k \sqcup_{x^{-1}=y} \mathbb{A}^1_k = \mathbb{P}^1_k$$

agree at $y^{-1} = y$, i.e. $y = \pm 1$, which is closed. We will see $\mathbb{P}^1_k$ is separated in §4.

Definition 3.11. A separated prevariety over k is called a **variety**.

Affine varieties are varieties. If $A = k[x_1, ..., x_n]/P$ for $P = \langle f_1, ..., f_m\rangle$, then

$$\operatorname{maxspec}(A) = X = X(P) \subset \mathbb{A}^n_k$$

and we've seen that:

$$\operatorname{maxspec}(A \otimes_k A) = X \times X \subset \mathbb{A}^{2n}_k$$

where $A \otimes_k A = k[x_1, ..., x_n, y_1, ..., y_n]/\langle f_1(x), ..., f_m(x), f_1(y), ..., f_m(y)\rangle$. But then:

$$\delta(X) = X(I) \subset X \times X \text{ for the ideal } I = \langle x_1 - y_1, ..., x_n - y_n\rangle$$

Quasi-affine varieties are varieties. If $(X, \mathcal{O}_X)$ is a variety and $j : U \hookrightarrow X$ is an open subset, then $(U, \mathcal{O}_U)$ is a variety, applying Proposition 3.10 to:

$$\delta(U) = \{(u_1, u_2) \in U \times U \mid j \circ \pi_1(u_1, u_2) = j \circ \pi_2(u_1, u_2)\}$$

for the two morphisms $j \circ \pi_i : U \times U \to X$.

Definition 3.12. A separated object X of a category of topological spaces with Cartesian products is **proper** if the projections:

$$\pi_2 : X \times Y \to Y$$

are closed maps for all objects Y of the category.

Remarks. (i) Every morphism $f : X \to Y$ from a proper object to a separated object is a closed map since $f(Z) = \pi_2(\pi_1^{-1}(Z) \cap \Gamma)$ where $\Gamma \subset X \times Y$ is the **graph**. The graph of a morphism is closed when Y is separated, since it is the inverse image of the diagonal under the map:

$$(f, \mathrm{id}_Y) : X \times Y \to Y \times Y$$

(ii) The prevarieties over k are irreducible topological spaces X, so in particular a variety X is **not** proper if it is an open subset of another variety Y, since the inclusion $i : X \to Y$ is not a closed map. For this reason, we call proper varieties **complete**; there is no way to enlarge them as varieties by adding points. When we pass from affine varieties to their *projective closures* in §4, we will be "completing" the affine variety in this sense. It is important to keep in mind, however, that in general the completion of a given affine variety is not unique. The first example of a completion is the inclusion $\mathbb{A}^1_k \subset \mathbb{P}^1_k$.

Example. In the category of topological spaces, proper coincides with compact.

Proposition 3.13. The only proper affine variety over k is the point maxspec(k).

Proof. Let $x \in X$ and choose a non-constant $f \in k[X]$, which we regard as:

$$f : X \to k = \mathbb{A}^1_k = \text{maxspec}(k[y])$$

The image $f(X)$ is dense in $\mathbb{A}^1_k$ (Exercise 2.6), since $f^* : k[y] \to k[X]$ is injective.

Then the hyperbola over U_f is the closed subset

$$X(fy - 1) \subset X \times \mathbb{A}^1_k$$

projecting onto a dense subset of $\mathbb{A}^1_k - \{0\}$, which is not closed in $\mathbb{A}^1_k$. $\square$

Remark. If f is not surjective, then f is not a closed map and it is immediate that X is not proper. But even when f is surjective (e.g. $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$ is the identity), then image of the projection of the hyperbola is not closed.

Assignment 3.

1. (a) Find the ring $\mathcal{O}_X(X)$ of regular functions on the projective line $X = \mathbb{P}^1_k$.

(b) Do the same for the doubled affine line.

2. Glue two copies of $\mathbb{A}^2_k$ to get a variety that is neither quasi-affine nor proper.

Hint: The ring of regular functions will be the key.

3. Prove that the intersection of two open affine subsets of a variety X is affine. (An open subset $U \subset X$ is affine if the variety $(U, \mathcal{O}_U)$ is an affine variety).

Hint: Given $U, V \subset X$, realize $U \cap V$ as a *closed* subset of the affine variety $U \times V$. Closed irreducible subsets of an affine variety are affine varieties!

Problem 3 is very significant! It implies that any open affine **cover** of a variety has the property that all (multiple) intersections of open subsets in the cover are affine.

4. Show with an example that separatedness is necessary in Problem 3.

5. Prove that compact spaces are proper in the category of topological spaces.