We prove that projective varieties are proper and also discuss various equivalent formulations of the **dimension** of a variety, one of which is the degree of the Hilbert polynomial, when \( X \) is projective.

**Theorem 5.1.** \( \mathbb{P}^n_k \) is a proper variety for all \( n \).

**Proof.** Let \( X \) be a prevariety with affine open cover \( \{ Y_i \} \). Then

\[
\pi_X : X \times \mathbb{P}^n_k \to X
\]

is a closed map if each projection \( \pi_{Y_i} : Y_i \times \mathbb{P}^n_k \to Y_i \) is a closed map, and if \( Y \subset \mathbb{A}^m_k \) is a closed subvariety, then \( \pi_Y \) is a closed map if \( \pi_{\mathbb{A}^m_k} : \mathbb{A}^m_k \times \mathbb{P}^n_k \to \mathbb{A}^m_k \) is closed.

So we are reduced to showing the projections:

\[ \pi : \mathbb{A}^m_k \times \mathbb{P}^n_k \to \mathbb{A}^m_k \]

are closed maps for all \( m \) and \( n \)

In other words, we need to show that if \( Z \subset \mathbb{A}^m_k \times \mathbb{P}^n_k \) is a closed subset, then:

\[ X(I(\pi(Z))) = \pi(Z) \subset \mathbb{A}^m_k \]

since this is the defining property of a closed subset of \( \mathbb{A}^m_k \).

**Step 1.** Consider the \( k[y_1, \ldots, y_m] \)-algebra:

\[ A_* = k[y_1, \ldots, y_m] \otimes_k S_* \]

of polynomials in \( x_0, \ldots, x_n \) (graded by degree) with coefficients in \( k[y_1, \ldots, y_m] \).

Then a homogeneous polynomial \( F \subset A_d \) determines a well-defined subset:

\[ X(F) = \{ (a_1, \ldots, a_n), (b_0 : \ldots : b_n) \mid F(a_1, \ldots, a_n, b_0, \ldots, b_n) = 0 \} \subset \mathbb{A}^m_k \times \mathbb{P}^n_k \]

that is a closed subset of the product prevariety, since for each \( U_i \subset \mathbb{P}^n_k \),

\[ X(F) \cap (\mathbb{A}^m_k \times U_i) = X(f) \subset \mathbb{A}^m_k \times U_i = \mathbb{A}^{m+n}_k \]

for \( f = F/x_i^d = F(y_1, \ldots, y_n, \frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}) \). Thus as in the case of \( \mathbb{P}^n_k \), homogeneous ideals \( I \subset A_* \) determine closed subsets \( X(I) \subset \mathbb{A}^m_k \times \mathbb{P}^n_k \).

In fact, we claim that this property characterizes closed subsets \( Z \subset \mathbb{A}^m_k \times \mathbb{P}^n_k \). Indeed, given \( Z \), we define a homogeneous ideal \( I \) with:

\[ I_d = \{ F \in A_d \mid \frac{F}{x_i^d} \in I(Z \cap (\mathbb{A}^m_k \times U_i)) \text{ for all } i \} \]

Then \( Z \subset X(I) \) since every such homogeneous polynomial \( F \) vanishes of \( Z \), by construction. On the other hand, if \( p = ((a_1, \ldots, a_m), (b_0 : \ldots : b_n)) \not\in Z \) but \( p \in \mathbb{A}^m_k \times U_i \), then there is an \( f \in I(Z \cap (\mathbb{A}^m_k \times U_i)) \) such that \( f(p) \neq 0 \), and:

\[ x_i^d f = F \in I_d \]

for all sufficiently large values of \( d \)

so \( p \not\in X(F) \), and we conclude that \( Z = X(I) \) for this homogeneous ideal \( I \).

**Step 2.** Notice that \( I_0 = I(\pi(Z)) \), by definition. We claim that \( \pi(Z) = X(I_0) \).

To prove this, we will use the full homogeneous ideal \( I \) of \( Z \) and:

**Nakayama's Lemma.** If \( A \) is a commutative ring with 1, \( M \) is a finitely generated \( A \)-module, and \( I \subseteq A \) is an ideal such that \( IM = M \) then \( aM = 0 \) for some \( a \in 1+I \).
Proof. Let $m_1, \ldots, m_n$ generate $M$. By assumption, we can solve:

$$m_i = \sum_{j=1}^{n} b_{ij}m_j$$

for a matrix $B = (b_{ij})$ of elements $b_{ij} \in I$

Then the matrix $I_n - B$ annihilates all elements $m = \sum a_im_i \in M$ and then by Cramer’s rule, $a = \det(I_n - B)$ also annihilates $M$, and has the desired form. \hfill $\Box$

Proof of Step 2. Let $p \notin \pi(Z)$. Our goal is to find an $f \in I_0$ so that $f(p) \neq 0$.

Let $m_p \subset k[y_1, \ldots, y_m]$ be the associated maximal ideal, and consider the pair of homogeneous ideals $I$ (from Step 1), and $m_p \otimes S_\bullet = m_pA_\bullet$. Since:

$$X(I) = Z$$

and then it follows that $f \in I_0$ and $f(p) \neq 0$, as desired. \hfill $\Box$

Corollary 5.2. All projective varieties are proper.

Example. Consider the universal hypersurface degree $d$ hypersurface in $\mathbb{P}^n_k$ defined by the bihomogeneous polynomial inserting variables in place of the coefficients:

$$F = \sum_{|I|=d} y_I x_0^{i_1} \cdots x_n^{i_n}$$

defining a closed (projective) subvariety $X(F) \subset \mathbb{P}^{(n+d)-1}_k \times \mathbb{P}^n_k$.

This is the “universal family” of hypersurfaces of $\mathbb{P}^n_k$ in the sense that:

$$\pi^{-1}(\cdots : p_I : \cdots) = X(\sum_I p_I x_I) \subset \{p\} \times \mathbb{P}^n_k$$

is the hypersurface with coefficients $p_I$. Recalling that $(c_0 : \ldots : c_n) \in X(\sum_I p_I x_I)$ is a singular point if the gradient $\nabla(\sum_I p_I x_I)(c_0 : \ldots : c_n) = 0$ we can define the relative singular locus of the projection $\pi$ as the closed subset:

$$Z = X(\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n}) \subset \mathbb{P}^{(n+d)-1}_k \times \mathbb{P}^n_k$$

and then conclude that:

(*) The locus of non-singular hypersurfaces $X(F) \subset \mathbb{P}^n_k$ is open in $\mathbb{P}^{(n+d)-1}_k$.

The process of deriving equations for $\pi(Z) \subset X$ from equations for $Z \subset X \times \mathbb{P}^n_k$ is elimination theory, but merely just that the image is closed gives us information. For example, the Fermat hypersurface:

$$X(x_0^d + \cdots + x_n^d) \subset \mathbb{P}^n_k$$

is non-singular (if char($k$) does not divide $d$), from which we conclude that the locus of non-singular hypersurfaces is not just non-empty, but also open (and dense!).
Consider now a graded module $M_\bullet$ over the polynomial ring $S$. That is,

$$M_\bullet = \bigoplus_{d \in \mathbb{Z}} M_d \text{ with } S_d \cdot M_e \subset M_{d+e}$$

The finitely generated graded modules over $S_\bullet$ determine coherent sheaves on $\mathbb{P}^n_k$. This will be another conversion of an algebraic object to a geometric structure (coherent sheaves on $\mathbb{P}^n_k$ include vector bundles on closed subvarieties $X \subset \mathbb{P}^n_k$).

The twisted modules:

$$S_\bullet(k) := \bigoplus_{d \geq -k} S_d + k$$

are freely generated by $1 \in S(k)_{-k}$, and any homogeneous polynomial $F \in S_k$ determines a graded “multiplication by $F$” homomorphism:

$$S \xrightarrow{F} S(k); \ G \mapsto FG$$

More generally, multiplication by $F$ is a graded homomorphism:

$$S \otimes_S M_\bullet = S_\bullet F \rightarrow S(k) \otimes S M_\bullet = M_\bullet(k)$$

for any graded module $M_\bullet$, and its twist $M_\bullet(k) = \oplus M_{d+k}$.

**Theorem 5.3.** If $M_\bullet$ is a finitely generated graded module over $S_\bullet = k[x_0,...,x_n]$, then the Hilbert function

$$h_M(d) = \dim_k M_d$$

agrees with the Hilbert polynomial $H_M(d)$, which is a polynomial of degree $\leq n$, for all sufficiently large values of $d$.

**Proof.** The polynomials $H : \mathbb{Z} \rightarrow \mathbb{Z}$ of degree $\leq n$ have a $\mathbb{Z}$-basis:

$$1 = \begin{pmatrix} d \\ 0 \end{pmatrix}, d = \begin{pmatrix} d \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} d \\ n \end{pmatrix}$$

and if

$$H(d) = \sum_{i=0}^{n} a_i \begin{pmatrix} d \\ i \end{pmatrix}, \text{ then } H(d+1) - H(d) = \sum_{i=0}^{n-1} a_{i+1} \begin{pmatrix} d \\ i \end{pmatrix}.$$ 

Consider the graded homomorphism $\cdot x_n$ with kernel and cokernel $N$ and $L$:

$$0 \rightarrow N \rightarrow M \xrightarrow{x_n} M(1) \rightarrow L \rightarrow 0$$

Then $N$ and $L$ are graded $k[x_0, ..., x_{n-1}]$ modules, and:

$$h_M(d+1) - h_M(d) = h_L(d) - h_N(d)$$

By induction, we may assume that $h_L(d)$ and $h_N(d)$ are polynomial functions of degree $\leq n - 1$ for large values of $d$, and then:

$$h_M(d+1) - h_M(d) = \sum_{i=0}^{n-1} b_i \begin{pmatrix} d \\ n-1 \end{pmatrix}$$

for large $d$, and it follows that:

$$h_M(d) = \text{constant} + \sum_{i=1}^{n} b_{i-1} \begin{pmatrix} d \\ i \end{pmatrix}$$

i.e. $h_M$ is a polynomial function of degree $\leq n$ for large $d$. \qed
Examples. (a) The Hilbert polynomial of $S(k)$ is:

$$H_{S(k)} = \left(\frac{d + n + k}{n}\right)$$

and the Hilbert function is:

$$h_{S(k)}(d) = \begin{cases} 0 & \text{for } d \leq -k \\ H_{S(k)}(d) & \text{for } d \geq -k - n \end{cases}$$

(b) Let $F \in S_k$, and let $A_* = S/(F)$. From the short exact sequence:

$$0 \to S(-k)^F \to S \to A_* \to 0$$

we see that $H_{A_*}(d) = H_S(d) - H_{S(-k)}(d)$, which is a polynomial of degree $n - 1$.

(c) If $A_* = k[x_0, \ldots, x_n]_*/P$, then the Hilbert polynomial of $X(P) \subset \mathbb{P}^n_k$ is $H_A(d)$. It is not an invariant of the isomorphism class of variety $X$ itself. E.g. $A_*$ and $A_m*$ yield isomorphic projective varieties with Hilbert polynomials $H_A(d)$ and $H_A(md)$.

Dimension. The dimension of a variety is its most basic invariant.

Definition 5.4. Let $X$ be a variety over $k$. Then the dimension of $X$ is:

$$\dim(X) = \tr \deg_k k(X)$$

where $k(X)$ is the field of rational functions on $X$.

Topology Detects Dimension. $\dim(X)$ is the length $n$ of the longest chain:

$$X_0 \subset X_1 \subset \cdots \subset X_n = X$$

of closed irreducible subsets of $X$. Thus it is a topological invariant of $X$.

Proof. Given a chain as above, choose an open affine $U \subset X$ with $U \cap X_0 \neq \emptyset$. Then $Y_i = X_i \cap U$ are a chain of irreducible closed subsets of $U$, and $\bigcap Y_i = X_i$. Since $k(X) = k(U)$, it suffices to prove this for $X = U$, an affine variety. If:

$$Z \subset X = \text{maxspec}(A)$$

is a closed subvariety, let $f \in A$ be a regular function with $Z \subset X(f)$. Then $Z$ is contained in one of the irreducible components $Y \subset X(f)$. On the other hand, by the Krull Principal Ideal Theorem, $\tr \deg_k k(Y) = \tr \deg_k k(X) - 1$.

By induction, then, for any closed, irreducible $Z \subset X$, there is a chain:

$$Z = Y^c \subset Y^{c-1} \subset \cdots \subset Y^1 \subset Y^0 = X$$

of irreducible closed subsets and regular functions $f_i \in k[Y^{i-1}]$ such that:

$$Y^i$$

is a component of $X(f_i) \subset Y^{i-1}$ and $\dim(Y^i) = \dim(X) - i$

In other words the codimension $\text{codim}_X(Y^i)$ of $Y^i$ in $X$ is $i$ and in particular, every (zero-dimensional) point $x \in X$ has codimension equal to $\dim(X)$.

Example. (a) The dimension of $X \times Y$ is $\dim(X) + \dim(Y)$. If:

$$X_0 \subset X_1 \subset \cdots \subset X_n = X$$

and $Y_0 \subset Y_1 \subset \cdots \subset Y_m = Y$

are maximal chains in $X$ and $Y$, respectively, then:

$$X_0 \times Y_0 \subset X_0 \times Y_1 \subset \cdots \subset X_0 \times Y_m \subset X_1 \times Y_m \subset \cdots \subset X_n \times Y_m = X \times Y$$

is a maximal chain of closed irreducible subsets of the product.
(b) If $X$ is affine and $Z \subset X$ is a closed subvariety of codimension $c$, then the
regular functions $f_1, \ldots, f_c$ in the proof above lift to regular functions in $k[X]$ with
the property that $Z \subset X$ is an irreducible component of $X(f_1, \ldots, f_c)$. It is tempting
to conclude that if $Z_1, Z_2$ have codimension $c_1, c_2$ in $X$, then every component of
$Z_1 \cap Z_2$ has codimension $\leq c_1 + c_2$. But this is false:

(!) Consider the three-dimensional variety $X = X(x_0 x_3 - x_1 x_2) \subset \mathbb{A}^4_k$. Then:

$$Z_1 = X(x_0, x_1) \subset X \text{ and } Z_2 = X(x_2, x_3) \subset X$$

are two-dimensional closed subvarieties (planes) in $X$, and:

$$Z_1 \text{ is a component of } X(x_0) \subset X \text{ and } Z_2 \text{ is a component of } X(x_3) \subset X$$

These two planes intersect only at the origin, which has codimension three in $X$ and in particular is not a component of:

$$X(x_0, x_3) \subset X, \text{ which consists of two lines!}$$

**Definition 5.5.** If $X$ is a variety and $Z = X(f_1, \ldots, f_c) \subset X$ is irreducible of
codimension $c$, then $Z$ is a (set-theoretic) **complete intersection** in $X$.

**Example.** The planes $Z_1, Z_2$ above are not complete intersections. There is no single function $g \in k[X]$ such that $Z_1 = X(g)$.

**Proposition 5.6.** If $X$ is an affine variety, $Z = X(f_1, \ldots, f_c) \subset X$ is a complete
intersection, and $Z' \subset X$ is a closed subvariety of codimension $c'$, then

$$\text{codim}_Y(X) \leq c + c' \text{ for all irreducible components } Y \subset Z \cap Z'$$

**Proof.** As remarked above, we may conclude that $Z'$ is an irreducible component of $X(g_1, \ldots, g_{c'})$ for regular functions $g_i \in k[X]$. Unlike example (!) we may now also conclude from Krull’s Theorem that the irreducible components of $Z \cap Z' = X(f_1, \ldots, f_c) \cap Z'$ are irreducible components of $X(g_1, \ldots, g_{c'}, f_1, \ldots, f_c) \subset X$, and therefore have codimension $\leq c + c'$ in $X$. \hfill \Box

**Corollary 5.7.** If $Z, Z' \subset \mathbb{A}^n_k$ are closed subvarieties of codimension $c$ and $c'$, then
every component of $Z \cap Z'$ has codimension $\leq c + c'$ in $\mathbb{A}^n_k$.

**Proof.** We use the fact that:

$$Z \cap Z' = (Z \times Z') \cap \Delta \subset \mathbb{A}^n_k \times \mathbb{A}^n_k$$

The codimension of $Z \times Z'$ in $\mathbb{A}^{2n}_k$ is $c + c'$, and the codimension of $\Delta$ is $n$. Moreover, $\Delta$ is a complete intersection:

$$\Delta = X(y_1 - x_1, \ldots, y_n - x_n)$$

so the Proposition applies to the components of $Z \cap Z' = (Z \times Z') \cap \Delta$. \hfill \Box

**Corollary 5.8.** Closed subvarieties $Z_1, Z_2 \subset \mathbb{P}^n_k$ have a non-empty intersection
when their codimensions satisfy $c_1 + c_2 \leq n$.

**Proof.** The affine cones $C(Z_1), C(Z_2) \subset \mathbb{A}^{n+1}_k$ over $Z_1$ and $Z_2$ have the same
codimensions $c_1$ and $c_2$ in $\mathbb{A}^{n+1}_k$ and $0 \in C(Z_1) \cap C(Z_2)$. But by Corollary 5.7, each component of $C(Z_1) \cap C(Z_2)$ (which is necessarily itself an affine cone), has codimension $\leq c_1 + c_2 \leq n$ in $\mathbb{A}^{n+1}_k$. Thus $0 \in C(Z_1) \cap C(Z_2)$ is contained in a component of positive dimension, and $Z_1 \cap Z_2 \subset \mathbb{P}^n_k$ is not empty. \hfill \Box

**Exercise.** The affine cone $C(Z)$ over $Z \subset \mathbb{P}^n$ has dimension $\dim(Z) + 1$. 

Examples. (a) Curves in $\mathbb{P}_k^2$ always intersect! (E.g. parallel lines meet at infinity). When counted with correct multiplicities, the number of points of $X(F) \cap X(G)$ for homogeneous polynomials $F$ and $G$ not sharing a common factor is:

$$\deg(F) \cdot \deg(G)$$

This is Bézout’s Theorem. We will discuss it later.

(b) The non-empty intersection property of Corollary 5.8 is topological. Thus:

$$\mathbb{P}_k^{n+m}$$

is not homeomorphic to $\mathbb{P}_k^n \times \mathbb{P}_k^m$ because the latter fails Corollary 5.8. If $n \geq m$, then:

$$\mathbb{P}_k^n \times \{x_1\} \cap \mathbb{P}_k^n \times \{x_2\} = \emptyset$$

for $x_1 \neq x_2 \in \mathbb{P}_k^n$ but $\text{codim}_{\mathbb{P}_k^n \times \mathbb{P}_k^n}(\mathbb{P}_k^n \times \{x\}) = m$ and $2m \leq n + m$.

(c) If $X$ is affine and $0 \neq f \in k[X]$, then every irreducible component of $X(f)$ has codimension one. This is not true of two or more functions. All we can say is that every component of $X(f_1, \ldots, f_c) \subset X$ has codimension $c$ or less.

Hilbert Polynomials and Dimension. If $X = \text{maxproj}(A_\bullet)$ is projective, then:

$$\dim(X)$$

is the degree of the Hilbert polynomial $H_{A_\bullet}(d)$

Proof. We will use the closed embedding:

$$X \subset \mathbb{P}_k^n$$

with $A_\bullet = S_\bullet / P$

(*) Projecting $\pi_p : \mathbb{P} \rightarrow \mathbb{P}^{n-1}$ from a point $p \notin X$ restricts to a morphism:

$$\pi_p : X \rightarrow \mathbb{P}^{n-1}$$

that is finite onto its image. If $r = \dim(X)$, then projecting from $n - r$ (successive) points defines a morphism: $\pi : X \rightarrow \mathbb{P}^r$ which we can interpret as the projection from a linear projective subspace $\Lambda \subset \mathbb{P}_k^n$ of dimension $n - r - 1$ that does not intersect $X$. With a change of basis (of $A_1$), we may assume that the projection is:

$$\pi_A(a_0 : \ldots : a_n) = (a_0 : \ldots : a_r : 0 : \ldots : 0)$$

and when restricted to any of the affine open subsets $U_i$, $i = 0, \ldots, r$,

$$k[x_1, \ldots, x_d] \subset A_{(x_i)}$$

is a polynomial subring over which $A_{(x_i)}$ is a finite module (Noether Normalization).

It follows that $\pi$ is surjective and finite-to-one and that the Hilbert polynomial of:

$$A_\bullet / \langle x_1, \ldots, x_d \rangle = S_\bullet / \langle P, x_1, \ldots, x_d \rangle$$

is a constant $\delta > 0$, equal to the dimension of the $k$-algebra:

$$A_{(x_0)} / \langle x_1, \ldots, x_d \rangle$$

But it follows that:

$$H_{A_\bullet}(d) = \delta \cdot \binom{d}{r} + \text{lower order}$$

and in particular that the Hilbert polynomial has degree $r$. □

Remark. The constant $\delta$ is the degree of the projective variety $X \subset \mathbb{P}_k^n$. Unlike the dimension, this is not an isomorphism invariant of the variety $X$, since, for example, the degree of $\mathbb{P}_k^1$ is one, but the degree of the conic $C \subset \mathbb{P}_k^2$ is two. Interestingly, though, we will see that the constant term of the Hilbert polynomial is an invariant.
Assignment 5.

1. Prove that the the affine cone $C(X)$ over a projective variety $X \subset \mathbb{P}^n$ satisfies:
   \[ \dim(C(X)) = \dim(X) + 1 \]

2. A subvariety $X \subset \mathbb{P}_k^n$ is an ideal-theoretic complete intersection if the ideal:
   \[ I(X) = (F_1, ..., F_c) \]
   is generated by $c = \text{codim}_{\mathbb{P}^n}(X)$ homogeneous polynomials.

   (a) Find homogeneous polynomials $F, G$ of degrees two and three so that the twisted cubic $C \subset \mathbb{P}^3$ is the set-theoretic complete intersection $C = X(F) \cap X(G)$.

   (b) Find the leading coefficient of the Hilbert polynomial:
   \[ H_X(d) \]
   of an ideal-theoretic complete intersection of polynomials of degrees $d_1, ..., d_c$.

3. (a) Prove that if $f: \mathbb{P}^n \to \mathbb{P}^m$ is a morphism then there are $m + 1$ homogeneous polynomials $F_0, ..., F_m \in S_d$ for some $d$ such that:
   \[ f(x) = (F_0(x) : ... : F_m(x)) \]
   "on the nose." i.e. $F_i(x) \neq 0$ for some $i$ for each $x \in \mathbb{P}^n$.

   (b) Conclude that there are no morphisms $f: \mathbb{P}^n \to \mathbb{P}^m$ when $m < n$.

4. A nonsingular curve is a variety $C$ such that:
   \[ \dim(C) = 1 \text{ and } \mathcal{O}_{C,p} \text{ is a discrete valuation ring} \]
   for all points $p \in C$. Prove that every rational map:
   \[ f: C \dashrightarrow \mathbb{P}^n \]
   is a morphism, defined at all points of $C$.

5. (a) Compute the dimension of the Grassmannian $\text{Gr}(m, n)$ of $m$ planes in $k^n$.

   The degeneracy loci:
   \[ D_r = \{ A: k^m \to k^n \mid \text{rk}(A) \leq r \} \subset \mathbb{A}_k^{mn} \]
   are algebraic subsets. There is an "incidence correspondence" between $D_r$ and $\text{Gr}(m - r, m)$ defined by:
   \[ I = \{ (A, \Lambda) \mid \Lambda \subset \ker(A) \} \subset \mathbb{A}_k^{mn} \times \text{Gr}(m - r, r) \]
   This is an algebraic subset of the product,

   (b) Analyze the fibers $\pi_2^{-1}(A) \subset I$ of the projection $\pi_2|_I: I \to \text{Gr}(m - r, r)$ and use your analysis to argue that $I$ is a variety and find its dimension.

   (c) Analyze the morphism $\pi_1: I \to D_r$ and conclude that $\dim(I) = \dim(D_r)$. Verify that the codimension of $D_r$ in $\mathbb{A}_k^{mn}$ is $(n - r)(m - r)$.

6. Comment on the following. If $X \subset \mathbb{P}^n$, then any rational map:
   \[ f: X \dashrightarrow \mathbb{P}^m \]
   is given by $F_0, ..., F_m \in S_d/P_d$ ($P$ is the homogeneous ideal of $X$). Reembedding $X$ in $\mathbb{P}^{(n+1)}$ (via the $d$-uple embedding), the forms $F_0, ..., F_m$ become linear, and $f$ is the restriction of a projection (rational map) from $\mathbb{P}^{(n+1)}$ to $\mathbb{P}^m$. 