
We study some basic properties of rational maps and regular maps of varieties. Upper semi-continuity of discrete invariants is a prominent feature in algebraic geometry, and we will see that the fiber dimension of a regular map \( f : X \to Y \) of varieties is an upper semi-continuous function. As an application, we prove Chevalley’s Theorem that proper algebraic groups are abelian.

A regular map \( f : X \to Y \) of varieties is a morphism of sheaved spaces, i.e. \( f \) is a continuous map with the property that:

\[
f^* (\mathcal{O}_Y(U)) \subset \mathcal{O}_X(f^{-1}(U))
\]

for all open subsets \( U \subset Y \). In particular, if the image of \( f \) is dense in \( Y \), then passing to direct limits, we get:

\[
f^* : k(Y) \hookrightarrow k(X)
\]

an injection of fields of rational functions. Such a regular map is called dominant.

Conversely, an inclusion of fields of rational functions:

\[
i : k(Y) \hookrightarrow k(X)
\]

determines a dominant rational map \( f : X \to Y \) (partially) defined as a map to each open affine subset \( U = \text{maxspec}(k[y_1, ..., y_n]/P) \subset Y \) by:

\[
f(x) = (i(y_1)(x), ..., i(y_n)(x)) \in U \subset \mathbb{A}^n_k
\]

when each \( i(y_j) \in \mathcal{O}_{X,x}. \) Intrinsically, if \( x \in X \), then \( f(x) = y \) provided that:

\[
\mathcal{O}_{X,x} \cap i(k(Y)) = i(\mathcal{O}_{Y,y})
\]

and \( f \) is undefined at \( x \) if there is no such \( y \in Y \). A general rational map from \( X \) to \( Y \) is a dominant rational map to a closed subvariety \( Z \subset Y \).

A rational map \( f \) is regular when restricted to a well-defined (maximal) open subset \( V \subset X \), which is its domain. Alternatively, let:

\[
Z = \Gamma_f \subset X \times Y
\]

be the closure of the graph of \( f \) (restricted to \( V \)). Via the projection to \( Y \), we get: \( \overline{f} := \pi_Y : Z \to Y \) which is a regular extension of the regular map \( f|_V : V \to Y \).

We say that \( X \xleftarrow{\pi_X} Z \xrightarrow{\overline{f}} Y \) resolves the indeterminacy of the rational map \( f \).

Example. The map \( f : \mathbb{A}^{n+1}_k \to \mathbb{P}^n_k, f(a_0, a_1, ..., a_n) = (a_0 : a_1 : ... : a_n) \) is defined away from the origin. The closure of the graph of \( f \) is:

\[
Z = X(\langle x_i y_j - x_j y_i \rangle) \subset \mathbb{A}^{n+1}_k \times \mathbb{P}^n_k
\]

If \( (b_0 : ... : b_n) \in \mathbb{P}^n_k \), then:

\[
\overline{f}^{-1}(b_0 : ... : b_n) = \{((\lambda b_0, ..., \lambda b_n), (b_0 : ... : b_n)) | \lambda \in k\} \subset Z
\]

and the first projection \( \pi : Z \to \mathbb{A}^{n+1}_k \) satisfies:

\[
\pi^{-1}(0, ..., 0) = \{(0, ..., 0)\} \times \mathbb{P}^n_k \subset Z \text{ and } \pi : (\pi^{-1}(V)) \to V \text{ is an isomorphism for } V = \mathbb{A}^{n+1}_k - 0.
\]
This generalizes to affine cones. If \( P \subset k[x_0, \ldots, x_n] \) is a homogeneous prime ideal and \( X = X(P) \subset \mathbb{P}^n_k \), then the rational map \( f : C(X) \dashrightarrow X \) defined as above is defined away from the origin, and

\[
Z = \overline{f \circ \gamma}_f = X(\langle P, x_i y_j - x_j y_i \rangle) \subset \mathbb{A}^{n+1}_k \times \mathbb{P}^n_k
\]

for the homogeneous ideal in \( A = k[x_0, \ldots, x_n] \otimes_k S_* \) generated by \( P \) (in \( A_0 \)) and the \( x_i y_j - x_j y_i \) (in \( A_1 \)). The fibers of the map \( \overline{f} : Z \to X \) are lines and the map \( \pi : Z \to C(X) \) is an isomorphism away from \( 0 \in C(X) \) with fiber \( \{0\} \times X \) over 0.

Next we ask “What do dominant regular maps look like?”

**Definition 6.1.** A dominant regular map \( f : X \to Y \) is birational if the associated injective map on fields \( f^* : k(Y) \to k(X) \) is an isomorphism. Equivalently, \( f \) is birational if it has a rational inverse (corresponding to the isomorphism \( (f^*)^{-1} \)).

**Proposition 6.2.** If \( f : X \to Y \) is a birational regular map, then there is a nonempty open subset \( U \subset Y \) such that the induced regular map:

\[
f|_{f^{-1}(U)} : f^{-1}(U) \to U
\]

is an isomorphism of varieties (i.e. a biregular map).

**Proof.** First, we show that there are open subsets of \( X \) and \( Y \) such that:

\[
f|_V : V \to U
\]

Let \( U_0 \subset Y \) and \( V_0 \subset f^{-1}(U_0) \) be open affine subsets. Then \( f|_{V_0} : V_0 \to U_0 \) is a birational regular map of affine varieties, so \( f^* : k[V_0] \to k[U_0] \) is an injective map of \( k \)-algebras that is an isomorphism on fields of fractions. If \( k[V_0] = k[x_1, \ldots, x_n]/P \) and \( (f^*)^{-1}(x_i) = g_i/h_i \in k(U_0) \), let \( h = \prod h_i \). Then:

\[
f^* : k[U_0]/h \to k[V_0]_{f^*(h)}
\]

is an isomorphism of \( k \)-algebras, and so \( f : V_0 - X(h) \to U_0 - X(f^* h) \) is the desired isomorphism. Next, let \( U_1 = U_0 - X(f^* h) \) and \( Z = f^{-1}(U_1) - (V_0 - X(h)) \). Then \( Z \) is an algebraic set, which is a union of irreducible closed sets \( Z_i \) of dimensions \( \dim(Z_i) < \dim(X) \). The closures of their images \( \overline{f(Z_i)} \subset U_1 \) are also irreducible, of dimension at most \( \dim(Z_i) \), and so:

\[
U = U_1 - \overline{f(Z)}
\]

is nonempty, and satisfies the requirement of the Proposition. \( \square \)

**Example.** Apply the proof of the Proposition to the map:

\[
f : \mathbb{A}^2_k \to \mathbb{A}^2_k; \quad f(x_1, x_2) = (x_1, x_1x_2)
\]

This map collapses the \( y \)-axis to the origin and does not map to any points of the form \((0, y)\) with \( y \neq 0 \). The map \( f^* : k[y_1, y_2] \to k[x_1, x_2] \) takes \( y_1 \mapsto x_1 \) and \( y_2 \mapsto x_1x_2 \) so \( (f^*)^{-1}(x_1) = y_1 \) and \( (f^*)^{-1}(x_2) = y_2/y_1 \). Thus:

\[
f^* : k[y_1, y_2]/y_1 \to k[x_1, x_2]/x_1 = f^* y_1
\]

is an isomorphism, which corresponds to the isomorphism:

\[
f : \mathbb{A}^2_k - X(x_1) \to \mathbb{A}^2_k - X(y_1) \quad \text{of planes minus y-axes}
\]

**Definition 6.3.** A dominant rational map of varieties is birational if it has an inverse rational map, or equivalently if the map on fields is an isomorphism.
By restricting to the open domain of a birational map \( f : X \to Y \) and applying the Proposition, we obtain an open subset \( U \subset Y \) such that \( f : f^{-1}(U) \to U \) is an isomorphism. In other words, inside the image of every birational map of varieties there is an open set that is isomorphic to its preimage. Notice that it isn’t even a priori clear that the image of a dominant rational map must contain an open set (all we know from the definition is that the image is dense!). On the other hand, a birational regular map \( f : X \to Y \) of projective varieties is surjective because projective varieties are proper, but such a map need not be an isomorphism.

For general dominant regular maps, we have the following:

**Theorem 6.4.** If \( f : X \to Y \) is a dominant map, let \( r = \dim(X) - \dim(Y) \). Then:

(a) (i) Every irreducible component of every fiber \( f^{-1}(y) \) has dimension \( \geq r \).

More generally,

(ii) If \( Z \subset Y \) is a closed subvariety and \( W \subset f^{-1}(Z) \) is an irreducible component that dominates \( Z \), then \( \dim(W) \geq r + \dim(Z) \).

(b) There is an open subset \( U \subset f(X) \) of the image of \( f \) with the following “transverse” property:

(i) If \( g \in U \) then every component of \( f^{-1}(y) \) has dimension equal to \( r \).

More generally,

(ii) If \( Z \subset U \) is a closed subvariety then every irreducible component of \( f^{-1}(Z) \) has dimension \( r + \dim(Z) \).

**Proof.** It suffices to prove (a) replacing \( Y \) by an affine open subset \( U \subset Y \) intersecting \( Z \), and \( Z \) by \( Z \cap U \) and \( W \) by \( W \cap f^{-1}(U) \). Then as in \( \S 5 \), \( Z \cap U \subset U \) is an irreducible component of \( X(g_1, \ldots, g_c) \) for \( c = \text{codim}_Z(Y) \) and \( g_1, \ldots, g_c \in k[U] \). Shrinking \( U \) (and \( Z \cap U \)) further to exclude the other components, we may assume that \( Z = X(g_1, \ldots, g_c) \subset U \) is a set-theoretic complete intersection. Then \( W \subset f^{-1}(Z) = X(f^*g_1, \ldots, f^*g_c) \) is an irreducible component, and by Krull’s Theorem, it follows that \( \text{codim}_X(W) \leq c \), which gives (a).

For (b), we may also assume \( Y \) is affine (replacing it with an open affine subset), and we may also assume \( X \) is affine, since if (b) holds for each \( f : V_i \to Y \) for an open cover \( \{V_i\} \) of \( X \), then it holds for \( X \) itself. So we consider:

\[
f : X \to Y \text{ and } f^* : k[Y] \to k[X] \subset k(X)
\]

with \( k(Y) \subset k(X) \) a field extension of transcendence degree \( r \). Consider:

\[
k(Y) \subset A = k(Y) \otimes_{k[Y]} k[X]
\]

This is a finitely generated domain over the field \( k(Y) \) with fraction field \( k(X) \), so by the Noether Normalization Theorem,

\[(** \quad k(Y)[x_1, \ldots, x_r] \subset A\]

is a finitely generated module for some \( x_1, \ldots, x_r \in A \), which we may take to be elements of \( k[X] \) (clearing denominators). Now consider:

\[( \star \quad k[Y][x_1, \ldots, x_r] \subset k[X]\]

If this were a finitely generated module, we’d have:

\[
f : X \xrightarrow{g} Y \times A^r_k \xrightarrow{\pi_Y} Y
\]
a composition of a finite map and a projection, which is a surjective dominant map satisfying (b) with \( U = Y \). We can’t conclude (*) is a finite module, but if \( \phi \in k[X] \subset A \), then \( \phi \) satisfies a monic polynomial relation:

\[
\phi^n + p_1 \phi^{n-1} + \cdots + p_n = 0 \quad \text{for} \quad p_i \in k(Y)[x_1, \ldots, x_r]
\]

since \( \phi \in A \) is a finite module over \( k(Y)[x_1, \ldots, x_n] \), and if \( h \in k[Y] \) is chosen so that \( hp_i \in k[Y][x_1, \ldots, x_r] \) for all \( i \), then \( \phi \) is finite over \( k[Y]_h[x_1, \ldots, x_r] \). Since \( k[X] \) is finitely generated as an algebra over \( k[Y][x_1, \ldots, x_r] \), it follows by induction that:

\[
(\ast \ast \ast) \quad k[Y]_h[x_1, \ldots, x_r] \subset k[X]_h
\]

is a finitely generated module for some \( h \in k[Y] \), giving (b) for \( U = U_h \subset Y \).

\[\square\]

**Definition 6.5.** An integer-valued function:

\[ e : X \to \mathbb{Z} \]

is upper-semi continuous if \( X_n = \{ x \in X \mid e(x) \geq n \} \subset X \) are closed for all \( n \).

**Remark.** If \( X \) is an irreducible Noetherian space, then the nested closed sets:

\[
\cdots \supset X_{n-1} \supset X_n \supset \cdots
\]

stabilize to the empty set, and \( e \) is bounded above. If \( e \) has a minimum value \( r \), then \( U = X - X_{r+1} \) is a dense open subset on which \( e(x) \equiv r \) for all \( x \in U \).

**Corollary 6.6.** If \( f : X \to Y \) is a dominant regular map, then the function:

\[ e : X \to \mathbb{Z}; \quad e(x) = \max\{\dim(Z) \mid Z \subset f^{-1}(x) \text{ and } x \in Z\} \]

is an upper-semi continuous function with minimum value \( r = \dim(X) - \dim(Y) \).

**Proof.** This follows from the Theorem and induction on the dimension of \( Y \). The minimum value \( e(x) = r \) is taken on the open set \( f^{-1}(U) \subset X \) and \( X - f^{-1}(U) \) is a union of irreducible components \( Z_i \) dominating \( \overline{f(Z_i)} = W_i \subset Y - U \). Then \( X_k = \cup(Z_i)_k \) for all \( k > r \) is a union of closed sets by the inductive assumption, since \( \dim(W_i) < \dim(Y) \).

\[\square\]

**An Application.** An **algebraic group** \( G \) over \( k \) is a pointed variety \( e \in G \) together with regular “multiplication” and “inverse” maps:

\[ m : G \times G \to G \quad \text{and} \quad i : G \to G \]

that make the points of \( G \) into a group with identity element \( e \).

Examples of **affine** algebraic groups include:

\[ G_m := \text{maxspec}(k[x, x^{-1}]) = k^* \text{ with multiplication and} \]

\[ G_n := \text{maxspec}(k[x]) = k \text{ with addition} \]

and the matrix groups:

\[ \text{GL}(n, k), \quad \text{SL}(n, k), \quad \text{PGL}(n, k), \quad \text{SO}(n, k), \quad \text{Sp}(2n, k) \]

that are not abelian groups when \( n > 1 \).

**Theorem (Chevalley).** If \( e \in G \) is a proper algebraic group, then \( G \) is abelian.

**Proof.** Consider the regular conjugation map:

\[ c : G \times G \to G; \quad c(g, h) = ghg^{-1} \]

We may think of this as a **family** of regular maps indexed by \( h \in G \):

\[ c_h : G \to G; \quad c_h(g) = ghg^{-1} \]
Note that $c_e : G \to G$ is the **constant** map to $e \in G$ and that in general $c_h$ is the constant map (to $h$) if and only if $h$ is in the center of $G$.

The Theorem follows from:

**Proposition 6.7.** If $f : X \times Y \to Z$ is a regular map of varieties and $X$ is proper, then if $f_y : X \to Z$ is constant for some $y_0 \in Y$, it is constant for all $y \in Y$.

**Proof.** Consider the graph $\Gamma_f \subset X \times Y \times Z$ and the projection:

$$\pi : \Gamma_f \to Y \times Z \text{ with image } \pi(\Gamma_f) = W$$

Then $W$ is a closed subvariety of $Y \times Z$ since $X$ is proper, and the fibers of $\pi$ are:

$$\pi^{-1}(y, z) = \{x \in X \mid f_y(x) = z\} \subset X \times \{y\} \times \{z\}$$

Thus to show that $f_y$ is the constant map it suffices to prove that the dimension of (each component of) $\pi^{-1}(y, z) \subset \Gamma_f$ is greater than or equal to $\dim(X)$.

Since $\Gamma_f$ is the graph of $f$, we have $\dim(\Gamma_f) = \dim(X) + \dim(Y)$, and then the desired inequality follows from Theorem 6.4(a) for all $y$ if we can show that $\dim(W) \leq \dim(Y)$.

**Example.** Start with an affine cubic curve in Weierstrass form:

$$X = X(y^2 - x^3 - Ax - B) \subset \mathbb{A}^2_k$$

with nonzero discriminant and take the closure:

$$E = X \cup \{e\} = X(y^2 z - x^3 - Axz^2 - Bz^2) \subset \mathbb{P}^2_k$$

where $e \in E$ is the single extra “point at infinity” $e = (0 : 1 : 0)$. Define the inverse:

$$i : E \to E \text{ by } i(a : b : c) = (a : -b : c)$$

(noticing that the inverse fixes $e$), and then:

$$m = i \circ t : E \times E \to E$$

where $t(p, q) \in E$ is the “third point of intersection” of $E$ with the line $l = \overline{pq}$. There are two important points to verify:

a) Why is the map $t$ a regular map?

b) Why is the group law defined this way **associative**?