Algebraic Geometry I (Math 6130)

Utah/Fall 2020

7. Local Properties.

A point $x \in X$ of a variety over k is **xgular** if $\dim_k(m_x/m_x^2) = \dim(X)$, i.e. if the stalk $\mathcal{O}_{X,x}$ of the sheaf of regular functions is a regular local ring. The locus of non-singular points of X is open, and X itself is said to be nonsingular if every point $x \in X$ is non-singular. Given a singular projective variety X, one seeks a **desingularization**, i.e. a birational regular map $f : X_{ns} \to X$ such that X_{ns} is non-singular and projective. These can be difficult to find, but there is an intermediate notion of *normality* that does give rise to a canonical *normalization* $f : X_{nor} \to X$ that is a *finite* birational morphism.

Definition 7.1. The **Zariski cotangent space** to X at $p \in X$ is the vector space:

$$T_p^*(X) = m_p/m_p^2$$

Proposition 7.2. (a) The function $e(p) = \dim(T_p^*(X))$ is upper-semicontinuous.

(b) $e(p) \geq \dim(X)$, and $e(p) = \dim(X)$ on a non-empty open subset $U \subset X$.

Proof. We may prove (a) and (b) on each open subset of an open cover of X, so we may assume X is affine with $k[X] = k[x_1, ..., x_n]/\langle f_1, ..., f_m\rangle$. In that case,

$$\dim(T_p^*(X)) = n - \mathrm{rk}(\mathrm{Jac}(f_1, ..., f_m))(p)$$

where $\mathrm{Jac}(f_1, ..., f_m)$ is the Jacobian matrix of partial derivatives:

$$\left(\frac{\partial f_i}{\partial x_j}\right)$$

We see this using $X \subset \mathbb{A}_k^n$ and the fact that $dx_i(p) = x_i - p_i \pmod{m_p^2}$ are a basis for $T_p^*(\mathbb{A}_k^n)$ while the kernel of the surjective restriction map $T_p^*(\mathbb{A}_k^n) \to T_p^*(X)$ is generated by:

$$df_i(p) = f_i(x) - f_i(p) = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(p) dx_i(p) \pmod{m_p^2}$$

Since the dimension of the rank of a matrix with polynomial entries is lower-semicontinuous, it follows that $e(p)$ is upper-semicontinuous. This gives (a).

Let $r = \dim(X)$. Then by Noether Normalization there are linear combinations $y_1, ..., y_r \in k[X]$ of the x_i such that $k[y_1, ..., y_r] \subset k[x_1, ..., x_n]/\langle f_1, ..., f_m\rangle$ is a finite module and $k(y_1, ..., y_r) \subset k(X)$ is a finite field extension. With some care this can be done so that the field extension is separable, and then by the theorem of the primitive element,

$$k(y_1, ..., y_r)[\alpha] = k(X) \text{ with minimal polynomial } g \in k(y_1, ..., y_r)[y_{r+1}]$$

and we may assume the coefficients of g are polynomials in $y_1, ..., y_r$ and that g is an irreducible polynomial in $k[y_1, ..., y_{n+1}]$. This determines a birational map to a hypersurface $f : X - - > X(g) \subset \mathbb{A}_k^{n+1}$, which we saw in §6 induces an isomorphism between an open subset $V \subset X$ and $U \subset X(g)$. But it is clear that every hypersurface $X(g) \subset \mathbb{A}_k^{r+1}$ contains an open subset $U = X(g) - X(\nabla g)$ of points for which $\dim(T_p^*(X(g)) = r$, and therefore X does as well. □

Remark. If $f : X \to Y$ is a regular map of varieties and $f(x) = y$, then:

$$f^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} \text{ maps } m_y \text{ to } m_x \text{ and } m_y^2 \text{ to } m_x^2$$

so it induces a pull-back $f^* : T_y^*Y \to T_x^*X$ on Zariski cotangent spaces and, dually, a push-forward *derivative* map $df(x) : T_{X,x} \to T_{Y,f(x)}$ on Zariski tangent spaces.

Proposition 7.3. For all $x \in X$, a set of generators of the vector space m_x/m_x^2 always lifts to a set of generators of the maximal ideal m_x.

Proof. Let $g_1, ..., g_s \in m_x$ be a lift of generators of the vector space m_x/m_x^2. Then the cokernel of the $\mathcal{O}_{X,x}$-module homomorphism

$$\overset{s}{\oplus}\, \mathcal{O}_{X,x} \to m_x; \quad (f_1, ..., f_s) \mapsto \sum f_i g_i$$

is a module N satisfying $m_x N = N$. By Nakayama's Lemma (see §5), there is an element $a = 1 + b \in \mathcal{O}_{X,x}$ with $b \in m_x$ such that $aN = 0$. But a is a unit in this local ring, so $N = 0$, as desired. □

Definition 7.4. If $p \in X$ is non-singular, then a set of generators $g_1, ..., g_r \in m_p$ (reducing to a basis of T_p^*X) is called a *system of local parameters* for X near p.

A system of local parameters near x determines a rational map:

$$f : X - - > \mathbb{A}_k^r; \quad f(p) = 0$$

that is regular near p and induces isomorphisms $df(q) : T_qX \to T_{f(q)}\mathbb{A}_k^r$ on Zariski tangent spaces for all points q in a neighborhood of p. This is not a system of local *coordinates* near p (in the sense of differentiable manifolds), since the map f is finite-to-one, but it is the best we can do with rational functions.

Example. Looking at the factored (affine) cubic curve in Weierstrass form again:

$$C = X(f) \text{ for } f = y^2 - (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$$

we see that when $y_0 \neq 0$, then $x - x_0 \in m_{(x_0,y_0)}$ is a generator, since

$$df(x_0, y_0) = 2y_0dy - (x_0-\lambda_1)(x_0-\lambda_2)dx - (x_0-\lambda_1)(x_0-\lambda_3)dx - (x_0-\lambda_2)(x_0-\lambda_3)dx$$

and this parameter corresponds to the projection to the x-axis, while when $y_0 = 0$, then $x_0 = \lambda_i$ for some i and $y - y_0$ is a local parameter (projecting to the y-axis).

The following Theorem captures an important feature of regular local rings.

Theorem 7.4. If $p \in X$ is a non-singular point, then the local rings $\mathcal{O}_{X,Z}$ are unique factorization domains for all closed subvarieties $Z \subset X$ passing through p.

Proof. It suffices to prove that $\mathcal{O}_{X,p}$ is a UFD, since each $\mathcal{O}_{X,Z}$ is the localization of $\mathcal{O}_{X,p}$ at the prime ideal corresponding to Z, and a unique factorization domain localizes to a unique factorization domain. Next, complete the local ring $\mathcal{O}_{X,p}$ to:

$$\widehat{\mathcal{O}}_{X,p} = \lim_{\leftarrow} \mathcal{O}_{X,p}/m_p^n$$

and we appeal to the following commutative algebra facts:

(a) $\widehat{\mathcal{O}}_{X,p}$ is isomorphic to the power series ring $k[[g_1, ..., g_r]]$, which is a UFD.

(b) The map $i : \mathcal{O}_{X,p} \to \widehat{\mathcal{O}}_{X,p}$ is injective and flat; i.e. any injective map:

$$M \to N \text{ of finitely generated } \mathcal{O}_{X,p}\text{-modules}$$

remains injective after tensoring by $\widehat{\mathcal{O}}_{X,p}$.

We prove that $\mathcal{O}_{X,p}$ is a UFD via the following:

UFD Criterion. A Noetherian domain A is a UFD if and only if each of the ideals

$$(f:g)_A = \{h \in A \mid f \text{ divides } hg\} \text{ is principal}$$

Proof. If A is a UFD and $f = u\prod p_i^{n_i}$ and $g = v\prod p_i^{m_i}$ with units u, v and primes p_i, then $(f:g) = (e)$ for $e = \prod p_i^{\max\{n_i - m_i, 0\}}$. Conversely, suppose $f \in A$ is irreducible and $f|gh$. If $(f:g) = (e)$, then $e, f, h \in (f:g) = (e)$, so:

$$eg = fa_1, f = a_2 e \text{ and } h = a_3 e \text{ for } a_1, a_2, a_3 \in A$$

and then because f is irreducible, either a_2 is a unit and $h = a_3 e = a_3 a_2^{-1} f$, or else e is a unit and $g = e^{-1} a_1 f$, so f divides g or h, i.e. f is prime. □

Returning to the proof of the Theorem, suppose $f, g \in \mathcal{O}_{X,p}$. Then:

$$0 \to (f:g) \to \mathcal{O}_{X,p} \stackrel{\cdot g}{\to} \mathcal{O}_{X,p}/(f)$$

is exact, and it follows from (b) above that:

$$(f:g)_{\widehat{\mathcal{O}}_{X,p}} = (f:g)_{\mathcal{O}_{X,p}} \otimes \widehat{\mathcal{O}}_{X,p}$$

But the former is a principal ideal by (a) (and the Criterion), and so:

$$(f:g)_{\widehat{\mathcal{O}}_{X,p}} / \widehat{m}(f:g)_{\widehat{\mathcal{O}}_{X,p}} \text{ has dimension one}$$

On the other hand, if $I \subset \mathcal{O}_{X,p}$ is an ideal, then $I \otimes \widehat{\mathcal{O}}_{X,p}/\widehat{m} \cdot I \otimes \widehat{\mathcal{O}}_{X,p} = I/mI$ and so letting $I = (f:g)$, we see that $\dim(I/mI) = 1$, and then by Nakayama's Lemma (as in Proposition 7.3) we conclude that $(f:g)$ is a principal ideal.

Remark. A point $p \in X$ is called *locally factorial* if $\mathcal{O}_{X,p}$ is a UFD. We have proved above that non-singular points are locally factorial, but the reverse is not true.

Definition 7.5. A birational regular map $f : Y \to X$ of projective varieties is a **desingularization** of X if Y is non-singular.

It can be challenging to find desingularizations of projective varieties, and there is, in general, no canonical "smallest" desingularization of a singular variety X. There is, however, an important less-demanding property of a variety Y that does "partially desingularize" a projective variety with a canonical birational map.

Definition 7.6. A point $p \in X$ is **normal** if $\mathcal{O}_{X,p}$ is integrally closed in $k(X)$. The variety X is itself normal if every point $p \in X$ is normal.

Remark. Recall that a subring $A \subset K$ of a field is *integrally closed* in K if each $\phi \in K$ that satisfies a monic polynomial equation $\phi^n + a_{n-1}\phi^{n-1} + \cdots + a_0 = 0$ with coefficients in A is itself an element of A. It is straightforward to see that a unique factorization domain is integrally closed in its field of fractions, and therefore that a non-singular (or locally factorial) point of a variety is a normal point. In dimension one, the reverse is also true, and a normal curve is a non-singular curve. However the two notions diverge in dimensions two and more.

Recall also that $\phi \in K$ is integral over A (satisfying a monic polynomial) if and only if $A[\phi]$ is a finite A-module, from which it follows that the set of integral elements over A, i.e. the **integral closure** $\overline{A}$ of A, is a subring of K. Moreover, the ring $\overline{A}$ is integrally closed in K, and if K is a finite extension of the fraction field $k(A)$ of A, then the fraction field $k(\overline{A})$ of $\overline{A}$ is equal to K.

Theorem 7.7. Let X be an affine variety, and let $k(X) \subset k(Y)$ be a finite separable field extension. Then the integral closure $\overline{k[X]} \subset k(Y)$ of $k[X]$ in $k(Y)$ is:

(a) Finitely generated as an algebra over k, and

(b) A finite module over $k[X]$ (with fraction field $k(Y)$).

Thus the integral closure yields a finite and surjective regular map $f : Y \to X$ of affine varieties. Moreover, Y is a normal variety.

Proof. Of course (b) implies (a), and it suffices to prove the Theorem when $k[X]$ is a polynomial ring, since by Noether Normalization there is a finite (separable) module extension:

$$k[y_1, ..., y_r] \subset k[X]$$

and so $\overline{k[y_1, ..., y_r]} = \overline{k[X]}$ in the field $k(Y)$. Note that $A = k[y_1, ..., y_r]$ is integrally closed in its field of fractions $k(A) = k(y_1, ..., y_r)$, and $k(Y)$ is a separable field extension of $k(A)$, by assumption.

Consider now the trace pairing on $k(Y)$, viewed as a vector space over $k(A)$:

$$(a, b) = tr_{k(Y)/k(A)}(ab)$$

Separability says this is non-degenerate. Since integral closure commutes with localizing, we have:

$$(\overline{A})_S = \overline{A_S} = k(Y), \text{ for the multiplicative set } S = A - \{0\}$$

and so every element $\phi \in k(Y)$ is of the form $\phi = \alpha \cdot \psi$ for $\alpha \in \overline{A}$ and $\psi \in k(A)$. Thus there is a basis for $k(Y)$ of vectors:

$$\alpha_1, ..., \alpha_n \in \overline{A}, \text{ with dual basis } \beta_1, ..., \beta_n \in k(Y)$$

with respect to the trace. The claim is that $\overline{A}$ is a submodule of the free A-module:

$$\beta_1 A + \cdots + \beta_n A$$

and therefore a finite A-module. To see this, expand $\alpha \in \overline{A}$ in terms of the β basis $\alpha = \sum \phi_i \beta_i \in \overline{A}$ and solve for the coefficients ϕ_i via:

$$\phi_i = (\alpha_i, \alpha) = tr(\alpha_i \cdot \alpha)$$

But $\alpha_i \cdot \alpha \in \overline{A}$ satisfies a monic polynomial with coefficients in A, so all the roots of the minimal polynomial are also integral over A, and the *coefficients* of the minimal polynomial are both integral over A and elements of the field $k(A)$. Since A is integrally closed in $k(A)$, it follows that $\phi_i = tr(\alpha_i \cdot \alpha) \in A$, as desired. □

Corollary 7.8. For any affine variety X, the integral closure $\overline{k[X]} \subset k(X)$ of X in its own field of fractions (canonically) defines a normal affine variety X_{nor} with $k[X_{nor}] = \overline{k[X]}$ and a birational finite regular map:

$$f : X_{nor} \to X$$

Corollary 7.9. The normal points of a variety X are an open subset of X.

Proof. From Corollary 7.8, the normal points *contain* an open subset of X. Suppose $p \in X$ is normal and $p \in U$ is an open neighborhood. Consider the normalization map $f : U_{norm} \to U$, Then because $\mathcal{O}_{X,p} = \mathcal{O}_{U,p}$ is integrally closed in $k(X)$, it follows that the generators $b_1, ..., b_n$ of the module $k[U_{norm}]$ over $k[U]$ are elements of $\mathcal{O}_{U,p}$, and then it follows that $b_1, ..., b_n \in k[U]_f$ for some $f \notin m_p$, and $\overline{k[U]_f} = k[U_{norm}]_f = k[U]_f$, so $U - X(f) \subset U$ is a normal nbhd of $p \in X$. □

Let X be an affine variety. Then the normalization $f : X_{norm} \to X$ satisfies the:

Universal Property. Every finite dominant regular map $g : Y \to X$ from a normal affine variety to X factors uniquely through a finite map to the normalization:

$$g_{norm} : Y \to X_{norm} \to X$$

Indeed, if $k[X] \subset k[Y]$ and $k[Y] \subset k(Y)$ is integrally closed, then integral elements over $k[X]$ in $k(X)$ are all contained in $k[Y]$. So $k[X] \subset \overline{k[X]} \subset k[Y]$.

Observation. Every variety X admits a birational finite map $f : X_{norm} \to X$ from a normal variety that is uniquely determined by the universal property.

"Proof". Glue the universal affine normalizations of an open affine cover along the normalizations of their intersections, using the universal property.

Proposition 7.10. The normalization of a projective variety is projective.

Proof. Given $X = \text{maxproj}(A_\bullet)$, note that if we choose $l \in A_1 - \{0\}$, then:

$$k(X) = k(A_{\bullet(l)}) \text{ and } k(X) \subset k(X)[l] \subset k(A_\bullet) = k(C(X))$$

is an intermediate graded polynomial ring in one variable between the field of rational functions on X and on the affine cone $C(X)$. Since the polynomial ring $k(X)[l]$ is integrally closed and graded, the integral closure of $k[C(X)] = A_\bullet$ is a finitely generated graded subring:

$$B_\bullet = \overline{A_\bullet} \subset k(X)[l]$$

with $B_0 = A_0 = k$. The catch is that $B_\bullet$ need not be generated by B_1. If it is, then $B_\bullet$ is said to be **projectively normal**, and $X_{norm} = \text{maxproj}(B_\bullet)$ is the (projective) normalization of X. In general, however, $B_\bullet$ is generated by B_m for some $m \geq 1$, and then recalling that $X = \text{maxproj}(A_\bullet) = \text{maxproj}(A_{m\bullet})$, we have $B_{m\bullet} = \overline{A_{m\bullet}}$ and $X_{norm} = \text{maxproj}(B_{m\bullet})$. Geometrically, this corresponds to reembedding $X \subset \mathbb{P}^n$ in $\mathbb{P}^{\binom{n+m}{m}-1)}$ via the m-uple embedding, and then normalizing the affine cone over the reembedded X to obtain a cone whose coordinate ring is generated in degree one.

Finally, we look at integrally closed local rings in dimension one.

Proposition 7.11. Suppose $(A, \mathfrak{m})$ is an integrally closed local Noetherian domain of dimension one with residue field $k = A/\mathfrak{m}$ (not necessarily algebraically closed). Then the maximal ideal $\mathfrak{m}$ is a principal ideal, i.e. $(A, \mathfrak{m})$ is a DVR.

Proof. Choose an element $a \in \mathfrak{m} - \mathfrak{m}^2$. Because $(A, \mathfrak{m})$ has dimension one,

$$A/(a) \text{ is a finite-dimensional } k\text{-vector space}$$

and so in particular, $\mathfrak{m}^n \subset (a)$ for some n. Let n be minimal. If $n = 1$, we're done. Otherwise, choose $b \in \mathfrak{m}^{n-1}$ such that $b \notin (a)$, and consider the element:

$$x = b/a \in k(A)$$

Then $x \notin A$, so x is not integral over A (since we assumed A was integrally closed). But $\mathfrak{m}x \subset A$ since $\mathfrak{m}b \subset \mathfrak{m}^n \subset (a)$.

If $\mathfrak{m}x \subset \mathfrak{m}$, then by the same argument as in the proof of Nakayama's Lemma, we let $m_1, ..., m_r$ be generators of $\mathfrak{m}$ and solve: $xm_i = \sum_{j=1}^r a_{ij} m_j$ to get an operator $(xI_r - A)$ that annihilates $\mathfrak{m}$. It follows that $0 = \det(xI_r - A) \in k(A)$. But this is a monic polynomial in x, which is not allowed. So $1 \in \mathfrak{m}x$ and x^{-1} generates $\mathfrak{m}$. $\square$

Corollary 7.12. If X is a normal variety and $Z \subset X$ is a closed irreducible subvariety of codimension one, then $\mathcal{O}_{X,Z}$ is a DVR with fraction field $k(X)$.