What is Algebraic Geometry?

The quick answer is that algebraic geometry is the study of the geometry of the loci of solutions of systems of polynomial equations:

\[ f_1(x_1, \ldots, x_n) = \cdots = f_m(x_1, \ldots, x_n) = 0 \]

with coefficients in a field \( k \) (or commutative ring \( A \)). These loci, denoted by:

\[ X = X(f_1, \ldots, f_m) \subset k^n \]

are the (affine) algebraic sets associated to the polynomials \( f_1, \ldots, f_m \in k[x_1, \ldots, x_n] \).

It should be noted that the set \( X = X(I) \) only depends on the ideal \( I = \langle f_1, \ldots, f_m \rangle \) generated by \( f_1, \ldots, f_m \) in the polynomial ring \( k[x_1, \ldots, x_n] \). Algebraic sets are the closed sets of the Zariski topology on \( k^n \), used to define the sheaf of regular functions on \( k^n \). This information transforms the vector space \( k^n \) into the affine variety \( A^n_k \).

Until we develop the theory of schemes, we will restrict ourselves to fields \( k \) that are algebraically closed, in which the closed sets \( X \subset A^n_k \) are in bijection with the radical ideals \( I \subset k[x_1, \ldots, x_n] \) (which are always finitely generated), and in particular the points of \( A^n_k \) are in bijection with the maximal ideals. Each closed set is the union of finitely many irreducible closed sets \( X(P) \), for prime ideals \( P \subset k[x_1, \ldots, x_n] \) (in the theory of schemes, these also define points of \( A^n_k \).)

An affine variety is a set with a topology and sheaf of regular functions that is isomorphic to some irreducible closed subset \( V = X(P) \subset A^n_k \) with the Zariski topology and sheaf of regular functions inherited from \( A^n_k \). Although they are defined via irreducible closed sets in \( A^n_k \), the affine varieties are a “good” basis of open sets of an abstract variety over \( k \), in the same way that open balls are a “good” basis of open sets in a differentiable or analytic manifold. The sheaf of regular functions allows us to use algebra to define dimension, non-singularity, normality and other important geometric characteristics of a variety. A non-singular abstract variety over the field \( \mathbb{C} \) of complex numbers is a complex analytic manifold, but one of the virtues of algebraic geometry is that abstract varieties can be defined over (algebraically closed) fields of any characteristic.

A projective variety is a particular example of an abstract variety, defined by systems of homogeneous polynomial equations. Projective varieties are proper, which makes them analogues of compact manifolds. This means that, as is the case with compact manifolds, we can use auxiliary geometric constructions (vector bundles and cohomology theories) to define numerical invariants that quantify geometric characteristics of \( X \). The algebraic geometry analogues are (coherent) sheaves of modules over the sheaf of regular functions and their coherent sheaf cohomology (computed with Čech cohomology on open affine acyclic covers of \( X \)). The first concrete example of this is the genus of a one-dimension non-singular projective variety, defined via either the sheaf of differential one-forms on \( X \) (analogous to the cotangent bundle) or the Euler characteristic of the sheaf of regular functions. This gives an algebraic computation of the number of holes in \( X \) when \( k = \mathbb{C} \), and \( X \) is a compact Riemann surface.
The Plan.

- §1. Algebraic sets.
- §2. Affine and quasi-affine varieties.
- §3. Abstract varieties.
- §5. Geometric features.
- §6. Divisors and Line Bundles
- §7. Differentials
- §9. Cohomology of coherent sheaves
- §10. Serre Duality