Algebraic Geometry I (Math 6130)

Utah/Fall 2020

π. An Interlude. What Are We Doing?

Three reasons to study Algebraic Geometry.

1. Representation Theory.

Let G be a semi-simple Lie group over $\mathbb{C}$. For example,

$$G = \mathrm{SL}(n,\mathbb{C}), \mathrm{SO}(n,\mathbb{C}), \mathrm{Sp}(2n,\mathbb{C})$$

Every complex representation of the groups above is recovered from the action of G on G/B for a Borel (generalized upper-triangular matrix) subgroup $B \subset G$. (The Borel-Weil-Bott Theorem). The homogeneous spaces G/B are non-singular complex projective varieties whose points correspond to flags of subspaces of $\mathbb{C}^n$, and the representations are obtained from the **algebraic geometry** of G/B.

For example, $SL(2,\mathbb{C})/B = \mathbb{P}^1_{\mathbb{C}}$ is the complex projective line

$$\mathbb{P}^1_{\mathbb{C}} = \mathrm{maxproj}(\mathbb{C}[x_0,x_1]_\bullet)$$

and the graded summands $\mathbb{C}[x_0,x_1]_d = \langle x_0^d, x_0^{d-1}x_1, ..., x_1^d\rangle = \mathrm{Sym}^d(V^\vee)$ are the symmetric powers of the standard representation of $\mathrm{SL}(2,\mathbb{C})$. These are all the irreducible representations of $\mathrm{SL}(2,\mathbb{C})$.

Out of $\mathbb{P}^n_k$ in general, we tease out the representations $\mathrm{Sym}^d(V^\vee)$ for the group $\mathrm{SL}(n+1,\mathbb{C})$, but $\mathbb{P}^n_k$ is only a partial flag variety when $n > 1$ and there are many more irreducible representations (e.g. $\wedge^k V^\vee$) to be found using the full flag variety.

2. Compact Complex Manifolds.

The complex projective spaces $\mathbb{P}^n_{\mathbb{C}}$ are compact complex analytic manifolds with the following remarkable property. If $M \subset \mathbb{P}^n_{\mathbb{C}}$ is an *analytic* embedding of a compact complex manifold M (e.g. a Riemann surface), then M is a complex projective manifold, cut out by (homogeneous) **polynomial** equations (Chow's Lemma).

A first example of this is the torus:

$$\mathbb{C}/\Lambda \text{ for a lattice } \Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$$

which has an analytic embedding:

$$(\mathcal{P},\mathcal{P}') : \mathbb{C}/\Lambda - 0 \to \mathbb{C}^2$$

$$\text{where } \mathcal{P}(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \text{ and } \mathcal{P}'(z) = -2\sum_{\omega\in\Lambda}\frac{1}{(z-\omega)^2}$$

are the Weierstrass $\mathcal{P}$-function and its derivative. The image satisfies the equation:

$$y^2 = x^3 - Ax - B = 0 \text{ where}$$

$$A = 60\sum_{\omega\in\Lambda^*}\frac{1}{\omega^4} \text{ and } B = 140\sum_{\omega\neq 0}\frac{1}{\omega^6}$$

are the first two holomorphic Eisenstein series. The fact that such a polynomial relation exists is "explained" by the extension of this embeddeing across 0 to an embedding:

$$(1 : \mathcal{P} : \mathcal{P}') : \mathbb{C}/\Lambda \to \mathbb{CP}^2$$

taking $0 \in \mathbb{C}/\Lambda$ to the point $(0:0:1)$.

All Riemann surfaces embed in complex projective space, and algebraic geometry is a powerful tool for studying them. Compact complex manifolds of dimension > 1 may or may not embed. A topological obstruction to embedding M in $\mathbb{P}^n_{\mathbb{C}}$ is the fact that all the odd betti numbers $b_{2k+1}(M)$ of a projective manifold are even. (This follows from Hodge Theory).

3. Field Theory. Algebraic Geometry studies the function fields of varieties over k. These are fields K that are *finite* extensions of a purely transcendental field:

$$(*)\ k \subset k(x_1, ..., x_n) \subset K$$

The "birational problem" is to classify all such fields, up to k-isomorphism.

The first example of this are the quadratic "hyperelliptic" field extensions:

$$k(x) \subset K = k(x)(\alpha) \text{ with } \alpha^2 - \phi = 0$$

In general, if the extension $k(x_1, ..., x_n) \subset K$ is separable, then there is a primitive element $\alpha \in K$ with minimal polynomial $\psi(\alpha) = \alpha^d + \phi_{d-1}\alpha^{d-1} + \cdots + \phi_0 = 0$ gives us a **geometric model** for K, namely the hypersurface $X \subset \mathbb{A}^{n+1}$ for:

$$f(x_1, ..., x_{n+1}) = g(x_1, ..., x_n)\psi(x_{n+1}) \in k[x_1, ..., x_{n+1}]$$

where g is chosen to clear denominators minimally, so that f is a prime polynomial.

In the hyperelliptic example, the model looks like:

$$(x - s_1) \cdots (x - s_m)y^2 = c(x - r_1) \cdots (x - r_m)$$

These models have the advantage of living in the smallest possible affine space, but the disadvantage of being highly singular, in general. If the roots of the hyperelliptic example are distinct, then this model is smooth, but even here, the completion to a projective hypersurface $\mathbb{P}^2_k$ is very singular at the point(s) at infinity. Much of 19th and early 20th century algebraic geometry relied on finding the "least singular" hypersurface models of K to study properties of the field.

The existence of a non-singular model for characteristic zero fields was proved by Hironaka in the 60s, and the proof has been streamlined in the past few decades. It is unknown if such a model exists in general for fields k of characteristic $p > 0$. In dimension one, the "integral closure" $\overline{k[x]}$ of the polynomial ring in K finitely generated as a k-algebra and gives smooth affine models:

$$\text{maxspec}(\overline{k[x]})$$

to which points may be attached "at infinity" to obtain a non-singular projective model of K. This gives the **unique** smooth projective model of K, but it doesn't tell you in which projective space to find it.

In higher dimensions the integral closure is not, in general, nonsingular and the non-singular projective models are not, in general, unique. The **minimal model program** seeks nevertheless to find and study the "best" such models. A crucial theorem is the finite generatedness of the graded ring $A_\bullet$ of holomorphic pluri-n-forms associated to the field K, which allows us to define a **canonical model**

$$\text{maxproj}(A_\bullet)$$

for the fields "of general type" which, though singular, is relatively tame and is the springboard for producing minimal non-singular models. The finite-generatedness of this ring was proved by my colleague Chris Hacon (and collaborators).